Math@Mac Online Mathematics Competition

Wednesday, November 28, 2018

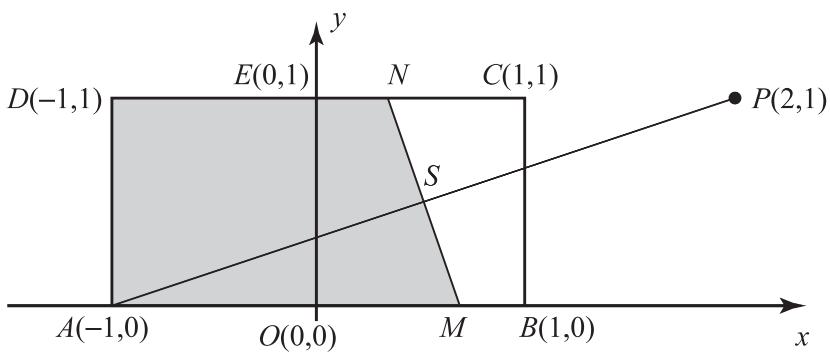
SOLUTIONS

1. Pick a random point (call it T) inside the rectangle with vertices $A(-1,0)$, $B(1,0)$, $C(1,1)$, and $D(-1,1)$. What is the probability that the point T is closer to the vertex A than it is to the point $P(2,1)$?

Write your answer in the form of a fraction a/b.

Answer: 3/4

Solution: The points which are closer to A than to P lie in the half-plane (which contains A) defined by the perpendicular bisector of the segment $\overline{AP}$. Denote the points of intersection of the bisector with the sides $\overline{AB}$ and $\overline{CD}$ by M and N, as shown.



The required probability is the ratio of the area of the trapezoid $AMND$ and the area of the rectangle $ABCD$.

Let $O(0,0)$ and $E(0,1)$.

The segment $\overline{MN}$ passes through the centre $S(1/2,1/2)$ of the square $OBCE$, and thus divides it into two congruent trapezoids. It follows that the area of the trapezoid $OMNE$ is $1/2$, and thus the area of the trapezoid $AMND$ is $3/2$.

Consequently, the required probability is equal to $(3/2)/2 = 3/4$.

2. Find the number of solutions (x, y) of the inequality

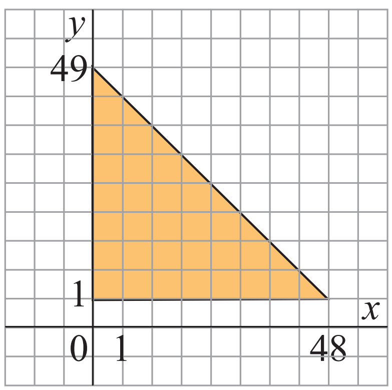
$$|x| + |y| < 50$$

where both x and y are integers.

Answer: 4901

Solution: We start by counting the number of solutions that are in the first quadrant, and including the y-axis (but not including the x-axis).

When $x = 0$, y can be one of $1, 2, \dots 49$ (so there are 49 solutions); see the diagram below. When $x = 1$, y can be one of $1, 2, \dots 48$ (so there are 48 more solutions); when $x = 2$, y can be one of $1, 2, \dots 47$ (so there are 47 more solutions); and so on.
Finally, when $x = 48$, then $y = 1$ (1 more solution).



Thus, the number of solutions is

$$1 + 2 + 3 + \cdots + 49 = \frac{49 \cdot 50}{2} = 1225.$$

These solutions are represented by the points with integer coordinates inside, or on the boundary of the shaded triangle.

Multiplying this number by 4 (to account for all four quadrants), and adding 1 (as the origin is a solution), we obtain $4(1225) + 1 = 4901$ solutions.

3. Find all ordered triples (p, q, r) where p, q, and r are positive integers at least two of which are prime, such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

To answer this question, enter a single number: multiply the number of distinct solutions (i.e., distinct triples) by the largest value of p in all solutions.

Answer: 18 (All solutions are $(2, 2, 1)$, $(3, 6, 2)$, and $(6, 3, 2)$)

Solution: Multiply the given equation by pqr to get

$$qr + pr = pq.$$

We consider three cases.

Case 1: Assume that p and q are prime. From $r(p + q) = pq$ we conclude that $r = 1$, $r = p$, $r = q$, or $r = pq$.
If $r = 1$, then the equation reduces to $p + q = pq$; rearranging it, we obtain $p = \frac{q}{q-1}$, and thus the only solution is $p = q = 2$. So, one triple is $(2, 2, 1)$.
If $r = p$, then the equation reduces to $p + q = q$, and thus $p = 0$, which contradicts the assumption that p is positive. Likewise, $r = q$ leads to a contradiction.
If $r = pq$, then $p + q = 1$, and there are no positive integer solutions.

Case 2: Assume that p and r are prime. Rewrite the equation above as $pr = q(p - r)$, and conclude that $q = 1$, $q = p$, $q = r$, or $q = pr$.
If $q = 1$, then $pr = p - r$ and $r = \frac{p}{p+1}$. Since p is prime, the right side is not an integer, so there are no solutions for r.
If $q = p$, the equation reduces to $r = p - r$ and $p = 2r$, which contradicts the fact that p is prime.
If $q = r$, then $r = 0$, again contradiction.
If $q = pr$, then $p - r = 1$ and $p = r + 1$. As both r and $r + 1$ need to be prime, $r = 2$ (as 2 and 3 are the only two consecutive numbers which are prime). Thus, $p = 3$, and thus the triple $(3, 6, 2)$ is a solution.

Case 3: Assume that q and r are prime. This case is symmetric to case 2, and yields the triple $(6, 3, 2)$.

4. Alice and Bob play a game where they take turns removing either 1, or 2, or 3, or 4, or 5 coins from a pile which originally contains 2018 coins. The player who takes the last coin(s) wins the game. (So, for instance, if there are 4 coins left then the player who goes next wins, as they can remove all 4 coins.)

If Alice goes first, there is a strategy that she can use to win. Using this strategy, the number of coins she should take on her first move is

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

Answer: (B) 2

Solution: If a pile has 6 coins and Bob has to go first, then Alice wins: if Bob removes any of $k = 1, 2, 3, 4, 5$ coins, then Alice removes the remaining $6 - k$ coins.

If a pile has 12 coins and Bob has to go first, then Alice wins: if Bob removes any of $k = 1, 2, 3, 4, 5$ coins, then Alice removes $6 - k$ coins, and thus the pile has 6 coins left (and it's Bob's next move).

It follows that Alice's strategy is to keep the number of coins to be a multiple of 6 after her every move.

Note that $2018 = 336 \cdot 6 + 2$. Thus if A starts by removing 2 coins, the pile will have $2016 = 336 \cdot 6$ coins left, with Bob having to make the next move.

5. The wheels of a car have a six-way radial symmetry (see diagram) and measure 0.75 m in diameter. A video camera which captures 12 frames per second creates a video of the car in motion. In the video, the wheels of the car appear to be not turning at all. If it is known that the car is traveling at a speed between 15 m/s and 20 m/s, which of the following is a possible speed for the car, measured in m/s?

(A) 4π

(B) 15

(C) 18

(D) 6π

(E) 24

Answer: (D) 6π

Solution: For the wheels to appear stationary, they must be turning some multiple (call it n) of 60 degrees every one-twelfth of a second, so they are turning at

$$12 \cdot \frac{60n}{360} = 12 \cdot \frac{n}{6} = 2n$$

revolutions per second.

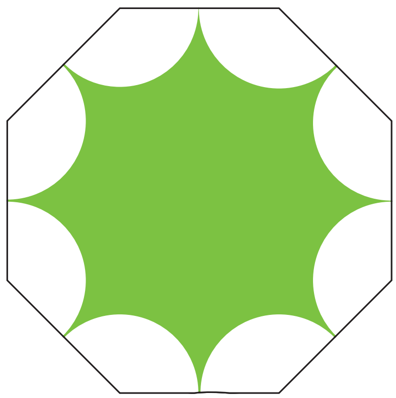
Since the diameter of a wheel is 0.75 m, the car is moving at

$$2n \cdot \text{circumference} = 1.5n\pi \text{ m/s}.$$

Thus is roughly $1.5n\pi \approx 4.7n$, which falls between 15 and 20 if $n = 4$.

Thus the car is traveling at 6π m/s.

6. Consider a regular octagon of side length 6. Draw arcs of radius 3 centred at each of the vertices of the octagon, thus creating circular sectors. The region inside the octagon but outside of the sectors is shaded. The area of the shaded region is:



(A) $9\left(12+8\sqrt{2}-2\pi\right)$

(B) $8\left(12+9\sqrt{2}-2\pi\right)$

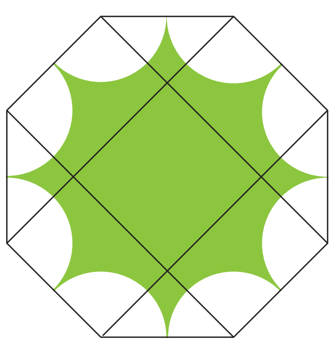
(C) $9\left(8+9\sqrt{2}-2\pi\right)$

(D) $8\left(9+9\sqrt{2}-3\pi\right)$

(E) $9\left(8+8\sqrt{2}-3\pi\right)$

Answer: (E) $9\left(8+8\sqrt{2}-3\pi\right)$

Solution: Each internal angle in the octagon is $3\pi/4$ radians.



Thus, the area of each circular sector is

$$\frac{1}{2}r^2\theta = \frac{1}{2}\left(3^2\right)\left(\frac{3\pi}{4}\right) = \frac{27\pi}{8}$$

and the area of all eight is 27π.

Next, we find the area of the regular octagon of side length 6.

The four triangles are right-angle triangles of hypothenuse 6 (and thus have the legs of length $3\sqrt{2}$). Their total area is

$$4 \cdot \frac{1}{2} \cdot \left(3\sqrt{2}\right)^2 = 36$$

Each of the four rectangles has the sides $3\sqrt{2}$ and 6, and so their total area is

$$4 \cdot 3\sqrt{2} \cdot 6 = 72\sqrt{2}$$

Adding the central square of side 6, we compute the area of the octagon to be

$$36 + 36 + 72\sqrt{2} = 72\left(1 + \sqrt{2}\right)$$

Thus, the area of the shaded region is

$$72\left(1 + \sqrt{2}\right) - 27\pi = 9\left(8 + 8\sqrt{2} - 3\pi\right).$$

7. Given that $f(x) = 3x^2 - x + 4$, find all polynomials $g(x)$ so that $f(g(x)) = 3x^4 + 18x^3 + 50x^2 + 69x + 48$. The sum of all coefficients of all solutions $g(x)$ is

(A) 6

(B) 1/3

(C) 69

(D) 48

(E) None of the above

Answer: (B) 1/3

Solution: The function $g(x)$ must be a quadratic polynomial, $g(x) = ax^2 + bx + c$. Then from

$$\begin{aligned} f(g(x)) &= 3(ax^2 + bx + c)^2 - (ax^2 + bx + c) + 4 \\ &= 3a^2x^4 + 6abx^3 + (6ac + 3b^2 - a)x^2 + (6bc - b)x + (3c^2 - c + 4) \\ &= 3x^4 + 18x^3 + 50x^2 + 69x + 48 \end{aligned}$$

it follows that $3a^2 = 3$, and so $a = \pm 1$.

From $6ab = 18$ we find that $b = \pm 3$.

If $a = 1$, then $b = 3$, and from $6ac + 3b^2 - a = 50$ we get $6c + 26 = 50$ and $c = 4$.

If $a = -1$, then $b = -3$, and from $6ac + 3b^2 - a = 50$ we get $-6c + 28 = 50$ and $c = -11/3$.

As both $(a, b, c) = (1, 3, 4)$ and $(a, b, c) = (-1, -3, -11/3)$ satisfy $6bc - b = 69$, and $3c^2 - c + 4 = 48$, there are two solutions: $g(x) = x^2 + 3x + 4$, and $g(x) = -x^2 + -3x - 11/3$.

Thus, the sum of the coefficients is $1 + 3 + 4 - 1 - 3 - 11/3 = 1/3$.

8. Four players are left in a game of paintball. Each player randomly selects one of the remaining three players. At the same time, they fire, and each player hits the player they targeted. What is the chance that no player remains "alive," i.e., that all four players are hit?

Write your answer in the form of a fraction a/b.

Answer: 1/9

Solution: Label the four players by A, B, C, and D. By a shooting arrangement (or just *arrangement*) ABCD $\to$ WXYZ we mean that A aims at W, B aims at X, C aims at Y, and D aims at Z.

For instance, one possible arrangement is ABCD $\to$ BDAA (in which case C is not hit). The arrangement ABCD $\to$ BDCC is not possible, as no player aims at themselves.

To compute the total number of possible shooting arrangements, note that in

$$\text{ABCD} \to \text{WXYZ}$$

there are three options for each of W, X, Y, and Z. Thus, there is a total of 3^4 shooting arrangements.

Which arrangements ABCD $\to$ WXYZ will result in all four players being hit?

Clearly, the string WXYZ must contain all of A, B, C, D, and must be such that W is not A, X is not B, Y is not C, and Z is not D.

The total number of permutations of the letters A, B, C, and D is $4! = 24$.
From this number we must subtract those arrangements which have exactly one "fixed point" (i.e., one player aims at themselves), exactly two "fixed points" (i.e., two players aim at themselves), exactly three "fixed points," and all four "fixed points."

Exactly one "fixed point:" there are 4 ways of picking 1 point to be fixed (i.e., the player who aims at themselves), and then there are only two arrangements with no further fixed points. This gives $4 \cdot 2 = 8$ arrangements.

Exactly two "fixed points:" there are $\binom{4}{2}$ ways of picking exactly 2 fixed points; after these are picked, there is only one option left which leaves no further fixed points. This gives 6 arrangements.

Exactly three "fixed points:" impossible, as the fourth point would have been fixed as well.

All four "fixed points:" one arrangement, ABCD $\to$ ABCD.

Since $24 - 8 - 6 - 0 - 1 = 9$, the desired probability is $9/3^4 = 1/9$.

9. You are afraid that you will forget your PIN number, but also do not want to write it down as you are afraid that someone will see it.

Your PIN number is a four-digit number $\overline{abcd}$, where a, b, c, and d are its digits.

Instead of writing down your PIN, you compute the sum $\overline{abcd}+\overline{abc}+\overline{ab}+a$, and write that number on a piece of paper. (Notation: if $a=4$, $b=3$, $c=8$ and $d=0$, then $\overline{abcd}=4380$, $\overline{abc}=438$, and so on.)

Few days later, you need your PIN, and indeed, you forgot what it is. However, you look at the piece of paper, and see the number 2018 written there. What is your PIN?

Answer: 1818

Solution: From

$$\overline{abcd}+\overline{abc}+\overline{ab}+a=2018,$$

we obtain

$$(1000a+100b+10c+d)+(100a+10b+c)+(10a+b)+a=2018$$
$$1111a+111b+11c+d=2018$$

where a, b, c, and d can be any of the numbers $0,1,2,3,\ldots,9$.

Note that if $a=0$, then the largest value of $1111a+111b+11c+d$ is $999+99+9<2018$. If $a=2$, then $1111a=2222>2018$. Thus, a must be equal to 1.

Substituting $a=1$ into $1111a+111b+11c+d=2018$ we obtain

$$111b+11c+d=2018-1111=907.$$

Now b cannot be 9, since $111b+11c+d\geq 999>907$. If $b=7$, then $11c+d=907-777=130$ has no solutions for c and d.

Thus, $b=8$ and the above equation simplifies to

$$11c+d=907-888=19.$$

It follows that $c=1$ and $d=8$, and so the PIN number is 1818.

10. How many non-negative integers n are there such that $n+2$ divides $(n+18)^2$?

(A) 1

(B) 3

(C) 8

(D) More than 12

Answer: (C) 8

Solution: Perform long division to show that

$$(n+18)^2 = (n+2)(n+34) + 256$$

So, for $(n+2)$ to divide $(n+18)^2$, it must be that $(n+2)$ divides 256. But the only divisors of 256 that are at least 2 are just the eight powers of 2: 2^1 (then $n=0$), 2^2 (then $n=2$), 2^3 (then $n=6$), $\ldots$, 2^8 (then $n=254$).