QUADRATIC FORMS AND THE SECOND DERIVATIVE TEST

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1. Preliminaries on Matrices

Let A be an $n \times n$ real matrix and I be the identity matrix of the same size in this section.

Definition 1.1. (*a*) *A real number λ is called a real eigenvalue of A if it is a root of the equation*

$$\det(A - xI) = 0. \tag{1.2}$$

(*b*) *A nonzero column vector V is a* real eigenvector *of A if $AV = \lambda V$ holds for some real number λ.*

Remark 1.3. In general Eqn.(1.2) may have no real roots. But the fundamental theorem of algebra guarantees that it has complex roots. These are called eigenvalues of A. We will see shortly that when A is *symmetric*, i.e., $A^t = A$, then all eigenvalues are real.

Remark 1.4. Now $AV = \lambda V$ is equivalent to $(A - \lambda I)V = 0$, i.e., V is a nonzero vector in the null space of $A - \lambda I$, so this matrix must have zero determinant. This in turn implies that λ is a root of Eqn. (1.2). Reversing the arguments, a real root of Eqn. (1.2) shows that the real matrix $A - \lambda I$ has determinant zero, and so there is a real vector V which is an eigenvector with eigenvalue λ.

Definition 1.5. (*a*) *The trace of A is the sum of the diagonal entries of A.*

(*b*) *Two square matrices A and B are said to be* conjugate *if there is an invertible matrix P such that $B = P^{-1}AP$. (Some people like to use the term "similar" instead of "conjugate".)*

Remark 1.6. We can define conjugacy between A and B by $B = QAQ^{-1}$ as well—just set $P = Q^{-1}$. Two conjugate matrices have the same trace and determinant. This is because

$$\mathrm{tr}(P^{-1}AP) = \mathrm{tr}((P^{-1}A)P) = \mathrm{tr}(P(P^{-1}A)) = \mathrm{tr}((PP^{-1})A) = \mathrm{tr}A$$

where we have used the important fact that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for any two square matrices X, Y. of the same size. Similarly, since $\det(XY) = \det X \det Y$ (so in particular $1 = \det I = \det P \det(P^{-1})$, we have

$$\det(P^{-1}AP) = \det P \det A \det(P^{-1}) = \det A.$$

Definition 1.7. *A square matrix is diagonal if the only nonzero entries it has lie in the diagonal.*

A square matrix A is called diagonalizable if it is conjugate to a diagonal matrix, i.e., if there is an invertible matrix P such that $P^{-1}AP$ is diagonal.

Suppose that A is diagonalizable. More specifically, let $P^{-1}AP = D$, where D is diagonal with entries $\lambda_1, \cdots, \lambda_n$. If we rewrite the above relation as

$$AP = PD,$$

note that the RHS can be described as the matrix obtained from P by multiplying the kth column by λ_k. So the above equation simply says that the kth column of P is an eigenvector of A with eigenvalue λ_k.

Since P is invertible, the columns of P are linearly independent, so we have a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

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Definition 1.8. *A matrix P is orthogonal if $P^tP = PP^t = I$. Thus P is invertible by assumption and $P^{-1} = P^t$.*

Remark 1.9. The usual dot product in $\mathbb{R}^n$ between two vectors U, V (thought of as column vectors) is given by $U \cdot V = \langle U, V\rangle = U^tV$, where the RHS uses matrix multiplication between the $1 \times n$ matrix U^t and the $n \times 1$ matrix V.

For vectors W, Z in $\mathbb{C}^n$ (thought of again as column vectors but with complex components) their scalar product is defined by

$$W \cdot Z := W^t\bar{Z}$$

where $\bar{Z}$ is the column vector obtained from Z by replacing each component by its complex conjugate.

The defining property of an orthogonal matrix P just says that the columns are an orthonormal basis of $\mathbb{R}^n$ and that the rows are an orthonormal basis as well. Orthonormal means that the vectors have length 1 and are mutually perpendicular.

Proposition 1.10. (*i*) *If P is an orthogonal matrix then for any column vectors $U, V \in \mathbb{R}^n$ we have*

$$\langle PU, PV\rangle = \langle U, V\rangle.$$

(*ii*) *If A is a symmetric matrix then for any column vectors $U, V \in \mathbb{R}^n$ we have*

$$\langle AU, V\rangle = \langle U, AV\rangle.$$

Proof. For part (i), we have (using $P^tP = I$)

$$\langle PU, PV\rangle = (PU)^t(PV) = U^tP^tPV = U^t(P^tP)V = U^tV = \langle U, V\rangle.$$

For part (ii), using $A^t = A$ we have

$$\langle AU, V\rangle = (AU)^tV = U^tA^tV = U^tAV = \langle U, AV\rangle.$$

□

Theorem 1.11. *The eigenvalues of a real symmetric matrix are real.*

Proof. We start with an eigenvalue λ which is possibly complex, guaranteed by the fundamental theorem of algebra (see Remark 1.3). This means that there is a nonzero complex vector Z—-a column vector with complex entries—such that $AZ = \lambda Z$. We use the usual inner product in $\mathbb{C}^n$ (see Remark 1.9) below, using the fact that A is *real* and *symmetric*.

$$\lambda(Z \cdot Z) = (AZ) \cdot Z = (AZ)^t\bar{Z} = Z^tA^t\bar{Z} = Z^t(A\bar{Z}) = Z^t(\overline{AZ}) = Z^t(\bar{\lambda}\bar{Z}) = \bar{\lambda}Z^t\bar{Z}.$$

Since $Z \neq 0$, $Z \cdot Z := Z^t\bar{Z} \neq 0$, so upon cancelling, we get $\lambda = \bar{\lambda}$. Hence λ is real. □

2. The Real Spectral Theorem

In this section we will use calculus to prove

Theorem 2.1. *Every symmetric real matrix A has a real eigenvalue.*

Consider the function

$$f(X) = \frac{1}{2}\langle AX, X\rangle$$

where X is an arbitrary column vector of unit length.

We view f as a function with domain $\mathbb{R}^n$ but we restrict attention to those column vectors X belonging to the unit sphere $x_1^2 + \cdots + x_n^2 = 1$ inside $\mathbb{R}^n$. f has the explicit formula

$$f(X) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_iA_{ij}x_j$$

and so is clearly infinitely differentiable.

Important Fact: A continuous function on a closed and bounded subset of $\mathbb{R}^n$ always has a minimum and a maximum point.

Apply this fact to f above where the closed and bounded subset is the unit sphere $x_1^2+\cdots+x_n^2=1$. Let V be a minimum point and let the minimum value of f be $f(V)=m$.

Choose any C^1 curve $X(t)$ lying in the unit sphere with the property that $X(0)=V$ and $X'(0)$ is an arbitrary tangent vector W to the unit sphere at V. Note that

$$\langle X(t), X(t)\rangle = 1.$$

Differentiating this equation and setting $t=0$ we get

$$\langle W, V\rangle + \langle V, W\rangle = 2\langle V, W\rangle = 0.$$

Since V is a minimum point of the restriction of f to the unit sphere, we can differentiate

$$f(X(t)) = \frac{1}{2}\langle AX(t), X(t)\rangle$$

with respect to t and set $t=0$. We get (since A does not depend on t)

$$0 = \frac{1}{2}\left(\langle AW, V\rangle + \langle AV, W\rangle\right).$$

Now use part (ii) of Proposition 1.10 to get

$$\langle AV, W\rangle = 0.$$

Since W is an arbitrary vector tangent to the unit sphere at V and such vectors are perpendicular to V, it follows from the last equation that AV is proportional to V. In other words, $AV=\lambda_1 V$ for some real λ_1, i.e., V is an eigenvector of A.

But

$$m = f(V) = \frac{1}{2}\langle AV, V\rangle = \frac{\lambda_1}{2}\langle V, V\rangle$$

and since V is a unit vector, $\lambda_1 = 2m$. In other words, twice the minimum value of f on the unit sphere is an eigenvalue of A.

We have therefore proved the above theorem.

Remark 2.2. In algebra courses, the above theorem is proved using the fundamental theorem of algebra. However, you may ask: how does one prove the fundamental theorem of algebra ?

It turns out that most proofs actually use the Important Fact above. So unless one avoids using the Important Fact, a proof of Theorem 2.1 using the fundamental theorem of algebra is no better than the calculus proof above. Furthermore, the calculus proof can be generalized to the context of Hilbert spaces, which is important in quantum mechanics for describing atomic spectra. Essentially, the quantized energy levels are just the eigenvalues of certain self-adjoint operators on the vector space of wave functions.

A corollary of Theorem 2.1 is the real spectral theorem for symmetric matrices.

Corollary 2.3. *For any real symmetric matrix A one can find an orthogonal matrix P such that P^tAP is a diagonal matrix.*

Proof. The proof uses induction on n, the size of the matrix. The assertion is clear if $n=1$ as A is just a 1×1 matrix.

By Theorem 2.1, we have an eigenvector V with eigenvalue λ_1. By Gram-Schmidt, set $V=U_1$ and construct an orthonormal basis of R^n of the form $\{U_1,\cdots,U_n\}$. Express the matrix A using this new basis. Now for $j>1$ we have

$$\langle AU_j, U_1\rangle = \langle U_j, AU_1\rangle = \lambda_1\langle U_j, U_1\rangle = 0.$$

This shows that A maps the span of $U_2, \cdots, U_n$ to itself while mapping U_1 to $\lambda_1 U_1$. Thus, if we use the above basis, the matrix A is transformed to a block matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & B \end{pmatrix}.$$

In other words, there is an orthogonal matrix Q such that $Q^t AQ$ gives the above block matrix. Since A is symmetric and Q is orthogonal, the matrix B is an $(n-1) \times (n-1)$ symmetric real matrix. One can then use the inductive hypothesis on B to deduce the corollary.

□

3. **Diagonalising Quadratic Forms**

Suppose we have a quadratic form

$$q(x) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}(x_i - a_i)(x_j - a_j)$$

where the matrix A is symmetric. (For example, this may come from the degree two Taylor polynomial of a C^2 function f at a critical point $a = (a_1, \cdots, a_n)$, in which case $A_{ij} = f_{x_i x_j}(a)$.)

Set $V = x - a$. Then q can be expressed in terms of V succintly as

$$q(V) = \frac{1}{2}\langle AV, V\rangle.$$

Let P be an orthogonal matrix such that $P^t AP = D$ where D is diagonal. Then if we set $W := P^t V$, we have

$$\langle AV, V\rangle = V^t AV = (V^t P)D(P^t V) = W^t DW = \sum_{i=1}^{n} \lambda_i w_i^2.$$

Because V and W determine each other, we can now express q as a function of W:

$$q(W) = \frac{1}{2} \sum_{i=1}^{n} \lambda_i w_i^2.$$

By permuting the entries of W, we may further assume that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Thus if all eigenvalues of A are positive, then (recalling that $x - a = V = PW$),

$$f(x) - f(a) = \frac{1}{2} \sum_{i=1}^{n} \lambda_i w_i^2 \geq 0$$

so that near a we have $f(x) \geq f(a)$ with equality iff $x = a$ (i.e., a is a strict local minimum.) Likewise, if all eigenvalues of A are negative then a is a strict local maximum.

If all eigenvalues of A are ≥ 0 then a is a local minimum but not a strict one, i.e., there may be points nearby a at which $f(x) = f(a)$.

A *saddle point* of f is a critical point where all eigenvalues are nonzero and they take on both signs. When $n = 2$ there is only one type of saddle point (one positive and one negative eigenvalue), but when $n > 2$, saddles points are further distinguished by the number of negative eigenvalues—called the *index* of the critical point.

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