Math 3C ϕ3 Oct. 2nd

Chapter 16

Series solutions of O.D.E.'s

2nd order linear O.D.E. $Ly = f$

$$Ly = a_2(x)\,y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x)$$

$$= y'' + p y' + q y$$

$p = \dfrac{a_1}{a_2}$

$q = a_0/a_2$

(sometimes we write z for x
since x could be a complex variable)

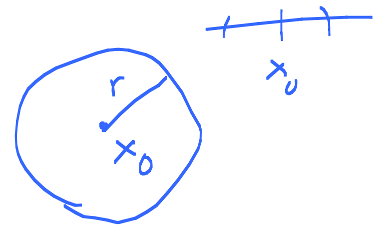
A function $f(x)$ is said to be ANALYTIC (or holomorphic) at a point x_0 iff

$f(x)$ can be represented as a sum of a convergent power series near x_0, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with positive radius of convergence

(f is said to be singular at x_0 otherwise)



x_0 is said to be an ordinary point of the operator $Ly = y'' + py' + qy$ if both p and q are analytic at x_0.

otherwise L is said to be singular at x_0.

A singular point x_0 is called a regular singular point iff $(x - x_0)\, p(x)$

and $(x - x_0)^2 q(x)$ are both analytic at x_0

To study the point $x_0 = \infty$, we make the change of variables $w = \frac{1}{x}$ and study the behaviour at $w = 0$.

(you have to change the derivatives as well!
e.g. $dx = -\frac{dw}{w^2}$ etc.)

Otherwise the singular point is called <u>irregular</u>

<u>EXAMPLES:</u>

LEGENDRE

$$(1-x^2)y''(x) - 2xy'(x) + \lambda y = 0$$

(λ is a constant)

regular singular points at $-1, +1, \infty$

same for the associated Legendre

$$(1-x^2)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right)y$$

λ constant $\rightarrow l(l+1)$

m constant

Chebyshev $\quad (1-x^2)y'' - xy' + \nu^2 y = 0$

ν a constant

regular singular points: $-1, +1, \infty$

Hermite $\quad y'' - 2xy' + 2\nu y = 0$

no singular points except at ∞ (irregular)

Laguerre $\quad xy'' + (1-x)y' + \nu y$

associated Laguerre $\quad xy'' + (m+1-x)y' + (\nu-m)y$

0 is a regular singular point
∞ is an irregular singular point

BESSEL $\quad x^2 y'' + xy' + (x^2-\nu^2)y$

0 is a regular singular point $\quad \infty$ irregular singular point

How to solve $Ly = 0$

Simple case: x_0 is an ordinary point.

You can always find an analytic function

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$ defined near x_0

that solves the equation $Ly = 0$

(Just plug in the power series into the equation and write down the algebraic equations for the coefficients a_n)

In fact you can always find two independent solutions $y_1(x)$ and $y_2(x)$.

Sometimes (depending on the coefficients of L), we can even find polynomial solutions.

Hermite's equation has polynomial solutions when ν is a positive integer. (Hermite polynomials)

The more interesting case: Regular singularity at x_0

We can assume $x_0 = 0$ (by a translation)

There is at least one solution of the form

$$y(x) = x^{\sigma} \sum_{n=0}^{\infty} a_n x^n \qquad (\text{with } a_0 \neq 0)$$

where the index σ and the coefficients a_n

are determined by "plugging into" the O.D.E.

σ satisfies a quadratic equation (indicial eqⁿ)

$$\sigma(\sigma - 1) + \tilde{p}(0)\sigma + \tilde{q}(0) = 0$$

where $\tilde{p}(x) = x\, p(x)$, $\tilde{q}(x) = x^2 q(x)$

($\tilde{p}$ and $\tilde{q}$ are analytic at 0)

The two roots σ_1 and σ_2 are in general complex numbers.

Case 1 (the easier case)

If $\sigma_1 - \sigma_2$ is not an integer, we get two independent solutions y_1 and y_2

$$y_1(x) = x^{\sigma_1} \sum a_n x^n \quad , \quad y_2(x) = x^{\sigma_2} \sum b_n x^n$$

Case 2 $\sigma_1 - \sigma_2$ is an integer (including 0)

One solution: $y_1(x) = x^{\sigma_1} \sum a_n x^n$

(we normally use the root σ_1 with the larger real part)

For the second solution we cannot use σ_2

We have to come up with a different technique

(i) use the Wronskian

$$W = e^{-\int p}$$

$$y_2(x) = y_1(x) \int^x \frac{W(\xi)}{y_1^2(\xi)} \, d\xi$$

(because $\left(\frac{y_2}{y_1}\right)' = \frac{W}{y_1^2}$)

(ii) differentiate the index σ

Put $y(x,\sigma) = x^\sigma \sum a_n(\sigma) x^n$

into the equation $Ly = 0$

Solve for the $a_n(\sigma)$ using σ as a parameter (don't solve for σ) such that all the higher order terms are zero except for the lowest order term involving x^σ.

Then $$\left[\frac{\partial}{\partial \sigma} y(x,\sigma)\right]\bigg|_{\sigma=\sigma_1}$$

is the second solution y_2 of $Ly=0$

in case $\sigma_1 = \sigma_2$

$\sigma_1 - \sigma_2$ is an integer $n \neq 0$

$$y_2(x) = \left[\frac{\partial}{\partial \sigma}\left((\sigma - \sigma_2)\, y(x,\sigma)\right)\right]_{\sigma=\sigma_2}$$

is the second solution.

Examples will make this clearer.