Math 3C03 Oct. 10

Sturm-Liouville equations:

$$Ly = \lambda \rho y$$

$$Ly = -\left((py')' + q\right)$$

$y, p, q, \rho$ functions of $x$

$p(x) > 0$ , $\rho(x) > 0$

metric

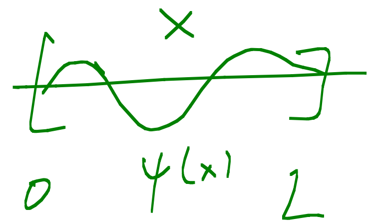
weight function

$$p y'' + p' y' + q y + \lambda \rho y = 0$$

Examples: 0. $y'' + \omega^2 y = 0$

$p \equiv 1$ , $\rho \equiv 1$ , $q \equiv 0$ , $\lambda = \omega^2$

$$-\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$$




$$\psi(0) = \psi(L) = 0$$

eigenvalues $E_n = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}n^2$

eigenvectors $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$

periodic boundary conditions $\psi(0) = \psi(L)$

$\psi'(0) = \psi(L)$

$$E_n = \frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}n^2$$

eigenfunctions: $\cos\left(\frac{2\pi}{L}nx\right)$ $\sin\left(\frac{2\pi}{L}nx\right)$

1. Legendre:

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

$p(x) = 1 - x^2 > 0$ in $(-1, 1)$

$\rho(x) \equiv 1$ , $\lambda = \ell(\ell+1)$

$q(x) \equiv 0$

2. Hermite: $y'' - 2xy + 2\nu y = 0$

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2\nu e^{-x^2}y = 0$$

$$(e^{-x^2}y')' + 2\nu e^{-x^2}y = 0$$

$p(x) = e^{-x^2} > 0$ , $\rho(x) = e^{-x^2} > 0$ , $q \equiv 0$

$\lambda = 2\nu$

$\mathbb{R}$

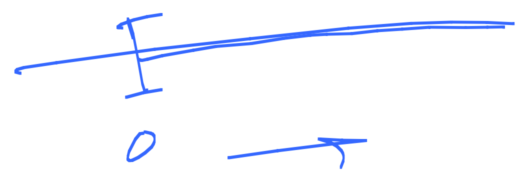
3. Laguerre $\quad xy'' + (1-x)y' + \nu y = 0$

multiply by $e^{-x}$

$$(xe^{-x}y')' + \nu e^{-x} y = 0$$

$\rho(x) = e^{-x} > 0$, $\quad p(x) = xe^{-x} > 0$

on $(0,\infty)$




$$x \in [a,b] \subset \mathbb{R}$$

$a, b$ could be $\pm\infty$

$\mathcal{H}$ = vector space of functions (real or complex valued) defined on $[a,b]$

with scalar product (positive-definite)

$$\langle f | g \rangle = \int_a^b \underbrace{\bar{f}(x)\, g(x)\, \rho(x)\, dx}$$

weighted measure

$L$ is obviously a linear operator

but we want $L$ to be self-adjoint or Hermitian w.r.t. the scalar product

$$\langle f | L g \rangle = \langle L f | g \rangle$$

If this is true, then all the eigenvalues of $L$ will be real (real spectrum)

and eigenvectors corresponding to different eigenvalues will be orthogonal

$$Lf = \lambda f \qquad Lg = \mu g \qquad \lambda \neq \mu$$

$$\lambda \langle f | g \rangle = \langle Lf | g \rangle$$

$$= \langle f | Lg \rangle = \mu \langle f | g \rangle$$

$$\Rightarrow \langle f | g \rangle = 0 \quad \text{if } \lambda \neq \mu$$

In order to make $L$ Hermitian we have to impose <u>boundary conditions</u>

Dirichlet

$$y(a) = \alpha$$
$$y(b) = \beta$$

Neumann

$$y'(a) = \tilde{\alpha}$$
$$y'(b) = \tilde{\beta}$$

<u>Mixed</u>

$$c_1 y(a) + d_1 y'(a) = \alpha$$
$$c_2 y(b) + d_2 y'(b) = \beta$$

If $a, b$ are at infinity we impose

decay conditions at $\infty$.

(e.g. on $\mathbb{R}$ $\int_{-\infty}^{\infty} |y|^2 \, dx < \infty$

$L^2$-function

Let's change $L$ a bit

$$\tilde{L} = \frac{1}{\rho} L$$

$\rho$ got cancelled

$$\langle f \mid \tilde{L} g \rangle = \int_a^b \bar{f} \left( -(p g')' - q g \right) dx$$

$$= \int_a^b \left( \bar{f}' p g' - q \bar{f} g \right) - p \bar{f} g' \Big|_{x=a}^{x=b}$$

integration by parts

$$\langle \tilde{L} f | g \rangle = \int_a^b \left(-(p \bar{f}')' - q \bar{f}\right) g \, dx$$

$$= \int_a^b (p \bar{f}' g' - q \bar{f} g) \quad - \quad p \bar{f}' g \Big]_{x=a}^{x=b}$$

$$\langle \tilde{L} f | g \rangle - \langle f | \tilde{L} g \rangle$$

$$= (p \bar{f} g' - p \bar{f}' g) \Big|_{x=a}^{x=b}$$

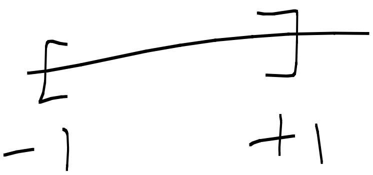
Kill this boundary term

by imposing the right boundary conditions

If you're lucky and $p(a) = p(b) = 0$

then you don't even need boundary conditions e.g. Legendre equation

$p(x) = 1 - x^2$ vanishes at $x = \pm 1$




The Legendre operator ($\rho = 1$, $p = 1 - x^2$, $q = 0$)

$$Ly = -\left((1-x^2)y'\right)'$$

acting on functions defined on $[-1, 1]$

has eigenvalues $\lambda_\ell = \ell(\ell+1)$

with polynomial eigenfunctions $P_\ell$

$$LP_\ell = \lambda_\ell P_\ell$$

($P_\ell$ a degree $\ell$ polynomial bounded at $\pm 1$)

$$\langle P_\ell | P_k \rangle = \int_{-1}^{1} P_\ell(x)\, P_k(x)\, dx = 0$$

$$\ell \neq k$$

$$|P_\ell|^2 = \int_{-1}^{1} P_\ell^2(x)\, dx = \frac{2}{\ell+1}$$

normalization

$$P_\ell(1) = 1 \quad , \quad P_\ell(-1) = (-1)^\ell$$

$$P_0(x) = 1 \;, \quad P_1(x) = x \;, \quad P_2(x) = \frac{3}{2}\left(x^2 - \frac{1}{3}\right), \;\dots$$

ORTHOGONAL POLYNOMIALS

Hermite polynomials

Laguerre polynomials etc.

Let $e_1(x), \dots, e_n(x), \dots$ be an orthonormal complete set of eigenfunctions for $L$ corresponding to eigenvalues $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le \dots$

Any function $f(x)$ can be expanded as

$$f(x) = \sum f_n e_n(x)$$

converges in the $L_2$-norm

where $f_n = \langle e_n | f \rangle$

components (coefficients)

$|e_n|^2 = 1$

To solve $Ly = f$

just put $y = \sum \frac{1}{\lambda_n} f_n e_n(x)$

0 is a "bad" eigenvalue

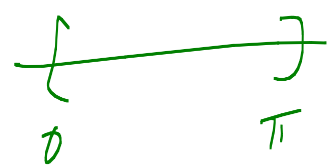
or

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$$

where $G(x,\xi) = \sum \frac{1}{\lambda_n} \bar{e}_n(\xi)\, e_n(x)$

Green's Function $\quad \lambda_n \neq 0$

Example $Ly = y'' + \omega^2 y$




$$G(x,\xi) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin nx \sin n\xi}{\omega^2 - n^2}$$

$y(0) = y(\pi) = 0$

Fourier Series