Math 3C03 28/09/2012

Note Title 28/09/2012

2^{nd} order Linear O.D.E's

$$y'' + py' + qy = f$$ (p, q, f functions - of x only)

$$L(y) = y'' + py' + qy$$ linear operator

$$L(uv) = u''v + 2u'v' + uv'' + p(u'v + uv') + quv$$

$$= u v'' + (2u' + pu) v' + \overbrace{(u'' + pu' + qu)}^{L(u)} v$$

Method 1: Kill the middle term

$$2u' + pu = 0 \qquad u = e^{-\frac{1}{2}\int p}$$

$$v'' + \frac{L(u)}{u} v = \frac{f}{u} = \tilde{f} \qquad \text{(divide by } u\text{)}$$

$$\frac{L(u)}{u} = -\frac{1}{2} p' - \frac{1}{4} p^2 + q$$

$$v'' + \left(q - \frac{1}{4}p^2 - \frac{1}{2}p'\right)v = f\,e^{\frac{1}{2}\int p}$$

Example:

$$y'' - 2xy' + 2ny = 0$$

Hermite's equation

$p = -2x$, $q = 2n$ (n is an integer) $f = 0$

$$\boxed{-v'' + x^2 v = (2n+1)v}$$

Quantum harmonic oscillator

energy levels

$$L(uv) = uv'' + (2u' + pu)v' + L(u)v$$

Method 2 Kill the last term

$Lu = 0$ u is a sol^{n} of the homogeneous eq^{n}.

$$(v')' + \left(2\frac{u'}{u} + p\right)v' = \frac{f}{u} = \tilde{f}$$

linear in v'

Solve for v' and then integrate one more time to get v

A more general method

Variation of Parameters

Suppose you know both (linearly independent) solutions u_1, u_2 of the homogeneous eq^n:

$\{u_1, u_2\}$ base for Kern L

Ansatz ; $y = v_1 u_1 + v_2 u_2$

where v_1, v_2 are <u>not</u> constants but functions of x (the independent variable)

We want to solve for v_1, v_2 so that

$$L(y) = f$$

$$y' = v_1 u_1' + v_2 u_2' + \underbrace{(u_1 v_1' + u_2 v_2')}_{\text{Kill this term}}$$

$$(u_1, u_2)\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = 0$$

$$y'' = v_1 u_1'' + v_2 u_2'' + (u_1' v_1' + u_2' v_2')$$

don't kill this

make it equal to f

$$(u_1', u_2') \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = f$$

Then

$$y'' + py' + qy = f$$

Solve $\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$

and then integrate one more time.

The determinant of this matrix is called the <u>WRONSKIAN</u>

$$W = u_1 u_2' - u_2 u_1'$$

$W \neq 0$ if the 2 solutions $u_1, u_2 \in \operatorname{Ker} L$ are linearly independent

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \frac{1}{W} \begin{pmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$= \frac{1}{W} \begin{pmatrix} -u_2 f \\ u_1 f \end{pmatrix}$$

Example

$$y'' + y = \operatorname{cosec} x = \frac{1}{\sin x}$$

$u_1 = \cos x$

$u_2 = \sin x$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} -1 \\ \cot x \end{pmatrix}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$v_1 = -x$$

$$v_2 = \int \cot x \, dx = \log(\sin x)$$

$$= 1$$

General sol^{n}:

$$y = c_1 \cos x + c_2 \sin x - x + \log(\sin(x))$$

About the Wronskian

$$(u_1 u_2' - u_2 u_1')' = u_1 u_2'' - u_2 u_1''$$

$$= u_1(-p u_2' - q u_2) - u_2(-p u_1' - q u_1)$$

$$= -p(u_1 u_2' - u_2 u_1') = -pW$$

$$\boxed{W' + pW = 0} \qquad W = e^{-\int p}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \frac{1}{W}\begin{pmatrix} -u_2 f \\ u_1 f \end{pmatrix}$$

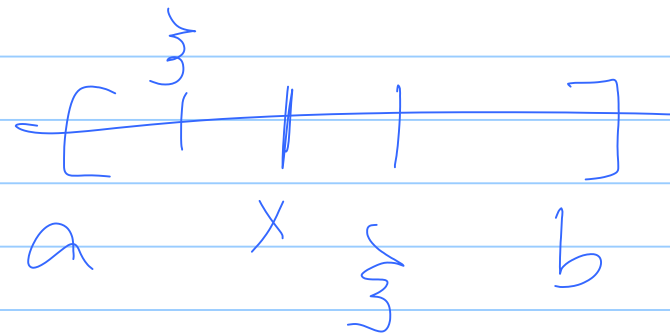
$$v_1 = -\int \frac{u_2 f}{W} \quad , \quad v_2 = \int \frac{u_1 f}{W}$$

$$y = -\int_a^x \frac{u_2(\xi) f(\xi)}{W(\xi)} d\xi \quad u_1(x)$$

$$+ \int_a^x \frac{u_1(\xi) f(\xi)}{W(\xi)} d\xi \quad u_2(x)$$

This leads to formulas for the GREEN'S FUNCTION

Boundary conditions $y(a) = y(b) = 0$



$$G(x,\xi) = \begin{cases} u_1(\xi)\, u_2(x) / W(\xi) & a \le \xi \le x \le b \\ u_2(\xi)\, u_1(x) / W(\xi) & a \le x \le \xi \le b \end{cases}$$

$u_1(a) = 0 \quad , \quad u_2(b) = 0 \qquad L(u_1) = 0$

$L(u_2) = 0$

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$$

satisfies $Ly = f$

and the ∂-conditions $\quad y(a) = 0$

boundary $\quad y(b) = 0$