Math 3C03 Oct. 31, 2012

Some basic facts about Bessel functions of the first kind: $J_\nu(x)$

Differential equation:

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

Formula:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}$$

[If $\nu = n$ an integer, then $J_{-n}(x) = (-1)^n J_n(x)$]

Orthogonality:

For any 2 distinct zeros α and β of $J_\nu(x)$

$J_\nu(\alpha) = J_\nu(\beta) = 0$, $\alpha \neq \beta$

$$\boxed{\int_0^1 J_\nu(\alpha x)\, J_\nu(\beta x)\, x\, dx = 0}$$

and

Normalisation

$$\boxed{\int_0^1 J_\nu(\alpha x)^2\, x\, dx = \frac{1}{2}\left(J_{\nu+1}(\alpha)\right)^2}$$

α is a zero of $J_\nu(x)$ $\quad J_\nu(\alpha) = 0$

Fourier - Bessel expansion

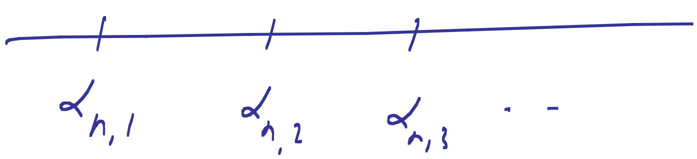
$f(r)$ radial function $0 \le r \le R$

$$\boxed{f(r) = \sum_{k=1}^{\infty} c_k \, J_n\left(\alpha_{n,k} \frac{r}{R}\right)}$$

n fixed

where $\alpha_{n,1} < \alpha_{n,2} < \cdots < \alpha_{n,k} < \cdots$

are the distinct zeros of $J_n(x)$



and the coefficients c_k are given by

$$\boxed{c_k = \frac{2}{\left(R \, J_{n+1}(\alpha_{n,k})\right)^2} \int_0^R f(r) \, J_n\left(\alpha_{n,k} \frac{r}{R}\right) r \, dr}$$

Proofs:

For any $a > 0$ $J_\nu(ax)$ satisfies:

$$x^2 y'' + x y' + (a^2 x^2 - \nu^2) y = 0$$

i.e. $$Ly = (xy')' - \frac{\nu^2}{x} y = -a^2 x y$$

(in Sturm-Liouville form: $p(x) = x$, $q(x) = -\frac{\nu^2}{x}$, $r(x) = x$)

Let $y_1(x) = J_\nu(\alpha x)$ and $y_2(x) = J_\nu(\beta x)$

By Sturm-Liouville theory (see my lecture notes from Oct. 10th)

$$(\alpha^2 - \beta^2) \langle y_1 | y_2 \rangle =$$

$$= \langle \tilde{L} y_1 \mid y_2 \rangle - \langle y_1 \mid \tilde{L} y_2 \rangle$$

$$= \Big[x (y_1 y_2' - y_2 y_1') \Big]_{x=0}^{x=1} = J_\nu(\alpha) J_\nu'(\beta) - J_\nu(\beta) J_\nu'(\alpha)$$

where $\tilde{L} = \frac{1}{x} L$ and the scalar product is defined by $\langle f \mid g \rangle = \int_0^1 f g \, x \, dx$

weight ρ

So if α, β are 2 distinct roots of J_ν, then

$$\int_0^1 J_\nu(\alpha x) J_\nu(\beta x) \, x \, dx = 0$$

$$\frac{1}{2}\frac{d}{dx}\left[(\alpha^2x^2-\nu^2)y_1^2+x^2(y_1')^2\right]$$

$$=\alpha^2 x y_1^2+\left[x^2y_1''+xy_1'+(\alpha^2x^2-\nu^2)y_1\right]y_1'$$

$$=\alpha^2 x y_1^2$$

$$\int_0^1 y_1^2\, x\,dx=\frac{1}{2\alpha^2}\left[(\alpha^2x^2-\nu^2)y_1^2+x^2(y_1')^2\right]_{x=0}^{x=1}$$

$$=\frac{1}{2\alpha^2}(y'(1))^2=\frac{1}{2}\left(J_\nu'(\alpha)\right)^2=\frac{1}{2}\left(J_{\nu+1}(\alpha)\right)^2$$

if α is a root of J_ν

$y(1)=J_\nu(\alpha)=0$

$y(0)=0$

We are using here the recurrence relation

$$xJ_\nu'(x)-\nu J_\nu(x)=-xJ_{\nu+1}(x)$$

at $x=\alpha$

Some Recurrence Relations:

$$\frac{d}{dx}\left(x^{\nu} J_{\nu}(x)\right) = x^{\nu} J_{\nu-1}(x)$$

$$\frac{d}{dx}\left(x^{-\nu} J_{\nu}(x)\right) = -x^{-\nu} J_{\nu+1}(x)$$

$$x J_{\nu}'(x) + \nu J_{\nu}(x) = x J_{\nu-1}(x)$$

$$x J_{\nu}'(x) - \nu J_{\nu}(x) = -x J_{\nu+1}(x)$$

$$2 J_{\nu}'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$$

$$2\nu J_{\nu}(x) = x\left(J_{\nu-1}(x) + J_{\nu-1}(x)\right)$$

For integral $\nu = n$, we have a generating function

$$G(x,u) = \exp\left(\frac{x}{2}\left(u - \frac{1}{u}\right)\right) = \sum_{k=-\infty}^{+\infty} J_k(x)\, u^k$$

(The recurrence relations for integral $\nu = n$ can also be derived using the generating function)

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

$$\frac{1}{2} e^{ix\sin\theta} = \frac{1}{2} J_0(x) + \sum_{k=1}^{\infty} J_{2k}(x)\cos(2k\theta)$$

$$+ \sum_{k=1}^{\infty} J_{2k-1}(x)\sin((2k-1)\theta)$$

(as you saw in Ass. # 3)

etc.!