Hermite polynomials

$H_n(x)$ is a polynomial of degree n

defined by

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n \left[e^{-x^2}\right]$$

$$= e^{x^2/2} \left(x - \frac{d}{dx}\right)^n \left[e^{-x^2/2}\right]$$

Physicists

$$\tilde{H}_n(x) = (-1)^n e^{-x^2/2} \left(\frac{d}{dx}\right)^n \left[e^{-x^2/2}\right]$$

Probabilists

$$= 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right)$$

examples: $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$

etc.

$H_n(x)$ solves Hermite's diff. equation.

$$y'' - 2xy' + 2ny = 0$$

or in Sturm-Liouville form

$$-(e^{-x^2} y')' = 2n\, e^{-x^2} y$$

$p = e^{-x^2}$, $q = 0$, $\rho = e^{-x^2}$, $\lambda = 2n$

Orthogonality:

$$\int_{-\infty}^{+\infty} H_n(x)\, H_m(x)\, e^{-x^2}\, dx = 0 \qquad n \neq m$$

Normalization

$$\int_{-\infty}^{\infty} H_n^2(x)\, e^{-x^2} dx = 2^n n! \sqrt{\pi}$$

$$\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{H}_n^2(x)\, e^{-x^2/2} dx = n! \right)$$

$\{H_n\}$ form a complete orthogonal basis for functions on $\mathbb{R}$ square integrable w.r.t. $e^{-x^2} dx$

Expansion $f(x) = \sum a_n H_n(x)$

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} f(x)\, H_n(x)\, e^{-x^2} dx$$

Generating Function:

$$e^{2xu - u^2} = \sum_{n=0}^{\infty} H_n(x) \frac{u^n}{n!}$$

$\parallel$ $G(x,u)$

Recursion

$$\frac{\partial G}{\partial u} = 2(x-u)G \quad , \quad \frac{\partial G}{\partial x} = 2uG$$

$$H_n'(x) = 2n\, H_{n-1}(x)$$

$$H_{n+1}(x) - 2x\, H_n(x) + 2n\, H_{n-1}(x) = 0$$

$$\left(H_n'(x) = 2x\, H_n(x) - H_{n+1}(x) \right)$$

$$H_n(x) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k}{k!\,(n-2k)!} (2x)^{n-2k}$$

Formula

The functions

$$\psi_n(x) = (-1)^n \sqrt{\frac{1}{2^n n! \sqrt{\pi}}} \, e^{-x^2/2} H_n(x)$$

are eigen functions of the quantum harmonic oscillator

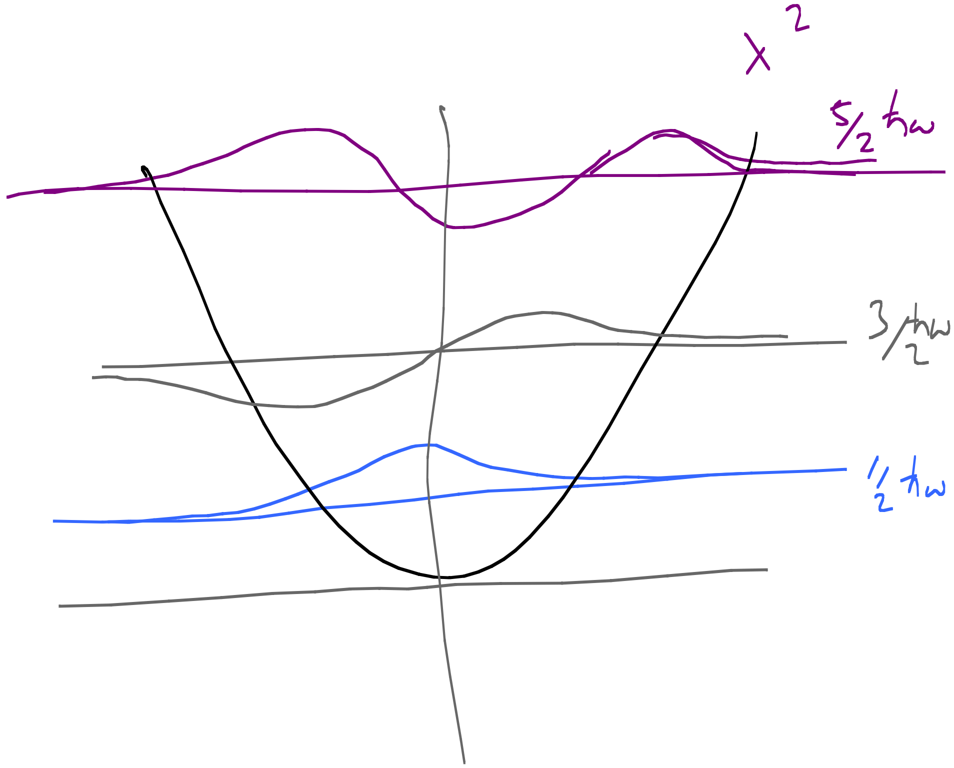
$$H \psi_n = (n + \tfrac{1}{2}) \psi_n$$

where $H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$

In physics $H = \frac{1}{2}\left(-\frac{\hbar^2}{m}\frac{d^2}{dx^2} + m\omega^2 x^2\right)$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$$

$$H\psi_n = \hbar\omega\left(n+\frac{1}{2}\right)$$ ← quantized energy levels



x^2 potential well

$$H_n' = 2x H_n - H_{n+1}$$

$$H_n'' = 2H_n + 2x H_n' - H_{n+1}'$$

$$H_n'' - 2x H_n' = 2H_n - H_{n+1}'$$

$$= 2H_n - 2(n+1) H_n$$

$$= -2n H_n$$

$$\underline{y'' - 2x y' + 2n y = 0}$$