# Legendre polynomials

Math 3cp3, Oct. 16th

$P_\ell(x)$ polynomial of degree $\ell$ satisfying

Legendre's equation: $Ly = \lambda y$

with $Ly = -((1-x^2)y')'$, $\lambda = \ell(\ell+1)$

normalized so that $P(1) = 1$, $P(-1) = (-1)^\ell$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

etc. General Formula:

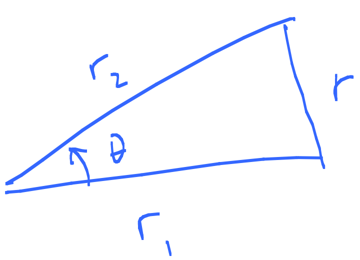
$$P_\ell(x) = \frac{1}{2^\ell} \sum_{k=0}^{[\frac{\ell}{2}]} (-1)^k \frac{(2\ell-2k)!}{k!\,(\ell-k)!\,(\ell-2k)!} x^{\ell-2k}$$

Rodrigues' Formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx}\right)^\ell \left((x^2-1)^\ell\right)$$

Generating function

$$\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{\ell=0}^{\infty} P_\ell(x)\, u^\ell$$




$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 - 2r_1 r_2 \cos\theta + r_2^2}}$$

$$= \frac{1}{r_2} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left(\frac{r_1}{r_2}\right)^\ell$$

This was how Legendre thought about them.

Orthogonality

$$\int_{-1}^{1} P_\ell(x)\, P_k(x)\, dx = 0 \qquad \text{if } k \neq \ell$$

(follows from Sturm-Liouville theory)

normalization:

$$\int_{-1}^{1} (P_\ell(x))^2\, dx = \frac{2}{2\ell+1}$$

One can expand functions on $[-1, 1]$ in terms of Legendre polynomials:

$$f(x) = \sum c_\ell P_\ell(x)$$

where

$$c_\ell = \left(\ell + \frac{1}{2}\right) \int_{-1}^{1} f(x)\, P_\ell(x)\, dx$$

like Fourier Series

# Recurrence Relations. (some of them!)

3 term:

$$x P_\ell'(x) - P_{\ell-1}'(x) = \ell P_\ell(x)$$

differential relation

$$x P_\ell'(x) - P_{\ell+1}'(x) = -(\ell+1) P_\ell(x)$$

one step

$$(1-x^2) P_\ell'(x) = \ell \left( P_{\ell-1}(x) - x P_\ell(x) \right)$$

$$(\ell+1) P_{\ell+1}(x) - (2\ell+1) x P_\ell(x) + \ell P_{\ell-1}(x) = 0$$

algebraic

2 step

$$P_\ell(x) = P_{\ell+1}'(x) - 2x P_\ell'(x) + P_{\ell-1}'(x)$$

differential

2 step

etc.

# Associated Legendre functions

(actually polynomials in $\cos\theta = x$ and $\sin\theta$)

$m, \ell$ integers, $-\ell \leq m \leq +\ell$

$$P_\ell^m(x) = \frac{(1-x^2)^{m/2}}{2^\ell \, \ell!} \left(\frac{d}{dx}\right)^{\ell+m} \left[(x^2-1)^\ell\right]$$

for $0 \leq m \leq \ell$

and $$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

examples: $P_3^2(x) = 15x(1-x^2)$, $P_3^3(x) = 15(1-x^2)^{3/2}$

$P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$ etc.

$P_\ell^m(x)$ solves the associated Legendre equation:

$$(1-x^2)\,y'' - 2x\,y' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] y = 0$$

which in Sturm-Liouville form is:

$$-(py')' - qy = \lambda \rho y$$

with $p = 1-x^2$, $q = \frac{m^2}{1-x^2}$, $\rho = 1$, $\lambda = \ell(\ell+1)$

(or $p = 1-x^2$, $q = \ell(\ell+1)$, $\lambda = -m^2$, $\rho = \frac{1}{1-x^2}$)

## Orthogonality:

$$\int_{-1}^{1} P_\ell^m P_k^m = 0 \qquad \ell \neq k \qquad \text{same } m$$

$$\int_{-1}^{1} P_\ell^m P_\ell^n (1-x^2)^{-1} dx = 0 \qquad m \neq n \qquad \text{same } \ell$$

## Normalization

$$\int_{-1}^{1} (P_\ell^m)^2 = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}$$

## Expanding functions:

$$f(x) = \sum_{k=0}^{\infty} a_{m+k} \, P_{m+k}^m(x)$$ $m$ fixed

where $$a_{m+k} = \left[(m+k)+\frac{1}{2}\right] \frac{k!}{(2m+k)!} \int_{-1}^{1} f \, P_{m+k}^m$$

## Generating function:

$$\frac{(2m)!}{2^m \, m!} \frac{(1-x^2)^{m/2}}{(1-2xu+u^2)^{m+1/2}} = \sum_{k=0}^{\infty} P_{m+k}^m(x) \, u^k$$

# Spherical harmonics

$$Y_\ell^m(\theta,\phi) = (-1)^m \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)\, e^{im\phi}$$

$c_{\ell,m}$ $\qquad 0 \leq m \leq \ell$

$$Y_\ell^{-m} = (-1)^m \overline{Y_\ell^m}$$

examples: $Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

$$Y_2^{-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\phi}$$

etc

( some calculations were done on the blackboard )