**Math 3C03**
**Short Answers to Assignment #1**

# 1. The positive symmetric matrix $R = A^T A = \begin{pmatrix} 882 & 504 & -558 \\ 504 & 936 & -504 \\ -558 & -504 & 882 \end{pmatrix}$

has eigenvalues $\lambda_1 = 1944, \lambda_2 = 432, \lambda_3 = 324$ (in descending order) with corresponding unit eigenvectors:

$u_1 = \frac{1}{\sqrt{3}}(1, 1, -1)^T \qquad u_2 = \frac{1}{\sqrt{6}}(1, -2, -1)^T \qquad u_3 = \frac{1}{\sqrt{2}}(1, 0, 1)^T$

*You can do this "by hand" but it is less labour-intensive if you use Matlab to compute*

So $R = U\Lambda U^T$ with $U = (u_1, u_2, u_3)$ and $\Lambda$ is the diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \lambda_3$.

Similarly the positive symmetric matrix $S = AA^T$ has the spectral decomposition: $S = V\tilde{\Lambda}V^T$ with the same non-zero eigenvalues as $R$: $\lambda_1 = 1944, \lambda_2 = 432, \lambda_3 = 324$ plus an additional zero eigenvalue $\lambda_4 = 0$ with corresponding eigenvectors: (which are the columns of $V$):
$v_1 = \frac{1}{\sqrt{18}}(4, 1, 1, 0)^T$ $v_2 = \frac{1}{\sqrt{18}}(-1, 2, 2, -3)^T$ $v_3 = \frac{1}{\sqrt{2}}(0, -1, 1, 0)^T$ $v_4 = \frac{1}{\sqrt{18}}(-1, 2, 2, 3)^T$
The SVD of $A$ is therefore:

$$\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix} = \frac{1}{3\sqrt{2}} \begin{pmatrix} 4 & -1 & 0 & -1 \\ 1 & 2 & -3 & 2 \\ 1 & 2 & 3 & 2 \\ 0 & -3 & 0 & 3 \end{pmatrix} \begin{pmatrix} \sqrt{1944} & 0 & 0 \\ 0 & \sqrt{432} & 0 \\ 0 & 0 & \sqrt{324} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Let $B = U\hat{\Lambda}V^T$, where $\hat{\Lambda} = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_3}} & 0 \end{pmatrix}$.

The "best" solution (in the sense of minimising the squared length of the error) of $Ax = b$ is given by $x = Bb$. $B$ is known as the "pseudo-inverse" (B= pinv(A) in Matlab)

For $b = (6, -39, 15, 18)^T$, we get $x = (1, 1, 2)^T$, which is an exact solution.

For $b = (9, -42, 15, 15)^T$ , we get $x = \frac{1}{36}(40, 37, 74)$ with error vector $(-1, 2, 2, 3)^T$, so the residual (length of the error vector) is $= \sqrt{18}$

# 2. From the given equations: $-A\,\ddot{x} = B\,x$, with

$$A = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} cm+d & -d & 0 \\ -d & cM+2d & -d \\ 0 & -d & cm+d \end{pmatrix}$$

we solve for $\omega^2$ in the characteristic equation:

$$|B - \omega^2 A| = \begin{vmatrix} cm+d-m\omega^2 & -d & 0 \\ -d & cM+2d-M\omega^2 & -d \\ 0 & -d & cm+d-m\omega^2 \end{vmatrix} = 0$$

and obtain the three eigenvalues:

$$\omega_1^2 = c, \qquad \omega_2^2 = c + \frac{d}{m}, \qquad \omega_3^2 = c + \frac{d}{m} + \frac{2d}{M}$$

with corresponding eigenvectors (normal modes):

$$e_1 = (1,1,1)^T \qquad e_2 = (1,0,-1)^T \qquad e_3 = (1, -\frac{2m}{M}, 1)^T$$

When $d \to 0$, the system decouples into three independent harmonic oscillators with the same frequency $\sqrt{c}$.

When $d \to \infty$, the system is rigidly coupled and only the first mode $e_1$ with frequency $\sqrt{c}$ survives.

#3. $$\begin{vmatrix} x & a & b \\ x^2 & a^2 & b^2 \\ a+b & x+b & x+a \end{vmatrix} = \begin{vmatrix} x-a & a & b-a \\ x^2-a^2 & a^2 & b^2-a^2 \\ a-x & x+b & a-b \end{vmatrix} = (b-a)(x-a) \begin{vmatrix} 1 & a & 1 \\ x+a & a^2 & b+a \\ -1 & x+b & -1 \end{vmatrix}$$
$$= (b-a)(x-a) \begin{vmatrix} 1 & a & 1 \\ x+a & a^2 & b+a \\ 0 & x+b+a & 0 \end{vmatrix} = (b-a)(x-a)(x-b)(x+a+b)$$

# 4. $< 1, 1 >= \int_{-1}^{1} 1^2\,dt = 2$, so $\mathbf{e_0} = \frac{1}{\sqrt{2}}$.

$< t, 1 >= \int_{-1}^{1} t\,dt = 0$ and $< t, t >= \int_{-1}^{1} t^2\,dt = \frac{2}{3}$, so $\mathbf{e_1} = \sqrt{\frac{3}{2}}\,t$

$< t^2, \mathbf{e_0} > \mathbf{e_0} = \frac{1}{3}$, $< t^2, \mathbf{e_1} >= 0$ and $|t^2 - \frac{1}{3}|^2 = \int_{-1}^{1}(t^4 - \frac{2}{3}t^2 + \frac{1}{9})dt = \frac{8}{45}$ so $\mathbf{e_2} = \frac{3\sqrt{5}}{2\sqrt{2}}\,(t^2 - \frac{1}{3})$

$< t^3, 1 >= 0\,,\ < t^3, \mathbf{e_1} > \mathbf{e_1} = \frac{3}{5}t\,,\ < t^3, t^2 - \frac{1}{3} >= 0$ and $|t^3 - \frac{3}{5}t|^2 = \int_{-1}^{1}(t^6 - \frac{6}{5}t^4 + \frac{9}{25}t^2)\,dt = \frac{8}{175}$

so $\mathbf{e_3} = \frac{5\sqrt{7}}{2\sqrt{2}}\,(t^3 - \frac{3}{5}\,t)$

Answer: $\{\frac{1}{\sqrt{2}}\,,\ \sqrt{\frac{3}{2}}\,t\,,\ \frac{3\sqrt{5}}{2\sqrt{2}}\,(t^2 - \frac{1}{3})\,,\ \frac{5\sqrt{7}}{2\sqrt{2}}\,(t^3 - \frac{3}{5}\,t)\,\}$

(*up to normalisation these are the first four Legendre polynomials*)

# 5. (in Matlab code)

$A = [\,0.11\,0.19\,0.10\,;\ 0.49\ -0.31\,0.21\,;\ 1.55\ -0.70\,0.70\,];$
$B = [\,0.11\,0.19\,0.10\,;\ 0.49\ -0.31\,0.21\,;\ 1.55\ -0.70\,0.71\,];$
$e = [\,1\,1\,1\,]';$
$x = inv(A) * e;$
$y = inv(B) * e;$
$x = [\,-74.1651\ \ -25.5413\ \ 140.1101\,]'$
$y = [\,-142.5327\ \ -50.5162\ \ 262.7667\,]'$
so the two solutions are quite different!

$det(A) = 0.0027, det(B) = 0.0015$ (relatively small)
A and B are ill-conditioned matrices with condition numbers:
$cond(A) = 342.1096$ and $cond(B) = 644.6448$ (pretty large)

#6 $f(x) = x^3 - 4x$ for $0 \leq x \leq 2$, extended periodically as an odd function with period 4.

$$b_n = \int_0^{+2} (x^3 - 4x)\, \sin(\frac{1}{2} n\pi x)\, dx = (-1)^n \frac{96}{n^3\pi^3}$$

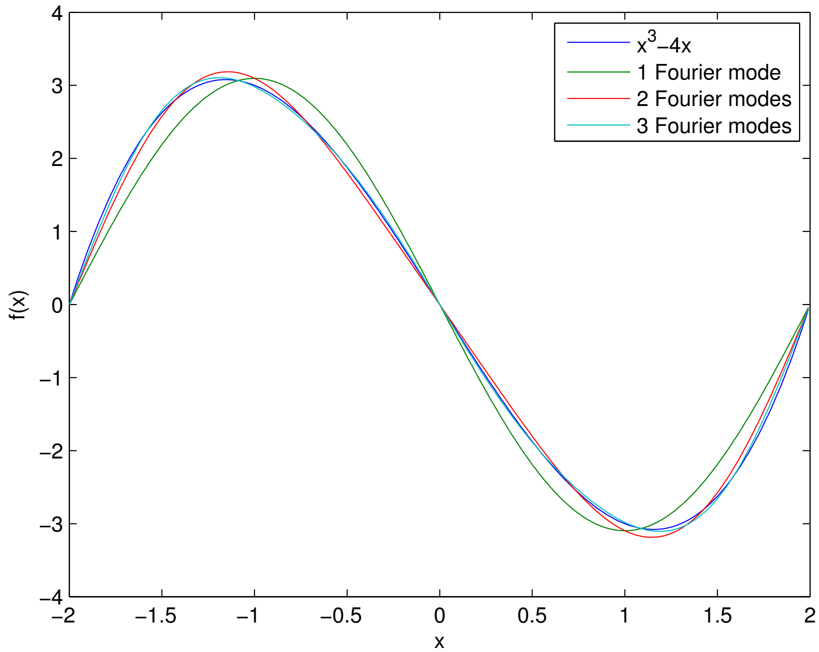
so by Parseval's identity

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{96^2}{n^6\pi^6} = \frac{1}{4} \int_{-2}^{+2} (x^3 - 4x)^2\, dx = \frac{512}{105}$$

from which it follows that

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

The approximation is already very good after summing up three terms of the Fourier series as you can see here:


4
3
2
1
0
−1
−2
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f(x)
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
x
x^3−4x
1 Fourier mode
2 Fourier modes
3 Fourier modes