Math 3C03
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Assignment #4

Due: Thursday, November 7th, 2013 in class at the beginning of the lecture

1. Show that

$$\int_0^1 (J_n(\alpha r))^2 \, r \, dr = \frac{1}{2} (J_{n+1}(\alpha))^2$$

where α is any root (zero) of the Bessel function J_n

2. Find the electric potential **outside** a spherical capacitor, consisting of two hemispheres of radius $1\,m$, joined along the equator by a thin insulating strip, if the upper hemisphere is kept at $+110\,V$ and the lower hemisphere at $-110\,V$.

3. Show that

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{y^2 + (x-\xi)^2} f(\xi)\, d\xi \qquad \textit{Poisson Formula}$$

solves Laplace equation $\Delta u = 0$ in the upper half plane $y > 0$ with boundary values $u(x,0) = f(x)$.

4. Find a radially symmetric solution $u(r,t)$ of the two-dimensional wave equation

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

on the unit disk: $r^2 = x^2 + y^2 \le 1$, satisfying the boundary condition: $u(1,t) = 0$ for all $t \ge 0$ and initial conditions:

$$u(r,0) = 1 - r^2, \quad \frac{\partial}{\partial t} u(r,0) = 0$$

5. Do problem 21.18 on page 771 in the textbook.

6. (*bonus question*) Prove the following formulas for Bessel functions (of the first kind):

$$\begin{aligned} \frac{d}{dx} (x^n J_n(x)) &= x^n J_{n-1}(x) \\ \frac{d}{dx} \left(x^{-n} J_n(x)\right) &= -x^{-n} J_{n+1}(x) \end{aligned}$$

and hence show that the zeros of the Bessel functions interlace, i.e. show that between any two consecutive positive zeros of $J_n(x)$, there is exactly one zero of $J_{n+1}(x)$.