Math 3C03

M. Min-Oo

Short Answers to Assignment #4

1. Show that

$$\int_0^1 (J_n(\alpha r))^2 \; r\,dr = \frac{1}{2}\,(J_{n+1}(\alpha))^2$$

where α is any root (zero) of the Bessel function J_n

I did that in class and you can find the notes on the course web page. Besides you can find a more general formula on page 610 in the textbook

2. Find the electric potential **outside** a spherical capacitor, consisting of two hemispheres of radius $1\,m$, joined along the equator by a thin insulating strip, if the upper hemisphere is kept at $+110\,V$ and the lower hemisphere at $-110\,V$.

The potential in the exterior is given by:

$$u(r, z = \cos\theta) = \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(z)$$

The Dirichlet boundary conditions $u(1,z) = +110$ for $0 < z \leq 1$ and $u(1,z) = -110$ for $-1 \leq z < 0$ are satisfied if we choose

$$B_l = 110\,\frac{2l+1}{2}\left(\int_0^1 P_l(z)\,dz - \int_{-1}^0 P_l(z)\,dz\right)$$

Obviously all the even B_{2k}'s are zero and for odd l we can use the formula that I derived in class:

$$\int_0^1 P_{2k-1}(x)\,dx = \binom{\frac{1}{2}}{k} \qquad \text{to get} \qquad B_{2k-1} = 110(4k-1)\binom{\frac{1}{2}}{k}$$

The first few B_l's are given by: $B_1 = 165$, $B_3 = -\frac{385}{4}$, etc.

3. Show that

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{y}{y^2 + (x-\xi)^2}\, f(\xi)\,d\xi \qquad\qquad \textit{Poisson Formula}$$

solves Laplace equation $\Delta u = 0$ in the upper half plane $y > 0$ with boundary values $u(x,0) = f(x)$.

The Green's function vanishing on the boundary for the upper half-plane in $\mathbb{R}^2$ is given by

$$G(p,q) = \frac{1}{2\pi}\left(\log(|p-q|) - \log(|p+\tilde{q}|)\right)$$

where for $q = (x,y) \mapsto \tilde{q} = (x,-y)$ is the reflection across the boundary. With $\nu = (0,-1)^T, q = (x,y)$ and $p = (\xi, 0)$ (on the boundary) $\frac{\partial G}{\partial \nu}$ is computed to be:

$$< \nabla G, \nu > = \frac{1}{2\pi}\left(\frac{< (p-q)\,,\,\nu >}{|p-q|^2} - \frac{< (p-\tilde{q})\,,\,\nu >}{|p-\tilde{q}|^2}\right) = \frac{1}{\pi}\left(\frac{y}{y^2+(x-\xi)^2}\right)$$

Now apply Green's formula.

4. Find a radially symmetric solution $u(r,t)$ of the two-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

on the unit disk: $r^2 = x^2 + y^2 \leq 1$, satisfying the boundary condition: $u(1,t) = 0$ for all $t \geq 0$ and initial conditions:

$$u(r,0) = 1 - r^2, \quad \frac{\partial}{\partial t} u(r,0) = 0$$

We are looking for a function $u(r,t)$ solving the equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

Separation of variables: $u(r,t) = y(r)h(t)$ gives rise to the two equations:

$$\ddot{h}(t) = -\omega^2 h(t) \qquad \text{and} \qquad y''(r) + \frac{1}{r}y'(r) = -\frac{\omega^2}{c^2}y(r)$$

where ω is a constant to be determined by the boundary values. The first equation is a simple harmonic oscillator and if we change the independent variable in the second equation from r to $x = \frac{\omega}{c}\,r$, then we obtain Bessel's equation with $\nu = 0$:

$$y''(x) + \frac{1}{x}y'(x) + y(r) = 0$$

whose solution is the Bessel function $J_0(x) = J_0(\frac{\omega}{c}\,r)$. In order to satisfy the boundary condition $u(1,t) = 0$ for all t, we require that $\omega_k = c\,\alpha_k$, where $\alpha_1, \alpha_2, \ldots,$ are the positive zeros of J_0.

Hence the general solution of the wave equation on a circular drum is a linear combination of the normal modes:

$$\sum_{k=1}^{\infty}(a_k \cos c\alpha_k t + b_k \sin c\alpha_k t)\, J_0(\alpha_k r)$$

The initial condition $\frac{\partial}{\partial t}u(r,0) = 0$ forces all the b_k's to vanish. The other initial condition $u(r,0) = 1 - r^2$ fixes the coefficients a_k by the Fourier-Bessel series: $1 - r^2 \sim \sum_{k=1}^{\infty} a_k\, J_0(\alpha_k r)$. a_k is given by:

$$a_k = \frac{2}{J_1^2(\alpha_k)}\int_0^1 (1-r^2)\, J_0(\alpha_k r)\, r\, dr$$

Using integration by parts and well-known formulas for Bessel functions (or more conveniently by using Wolfram alpha), we can evaluate the integral and finally get the explicit formula: $a_k = \frac{8}{\alpha_k^3\, J_1(\alpha_k)}$ and hence the solution is:

$$u(r,t) = 8\sum_{k=1}^{\infty}\frac{J_0(\alpha_k r)}{\alpha_k^3\, J_1(\alpha_k)}\cos(c\alpha_k t)$$

5. Do problem 21.18 on page 771 in the textbook.

The interior and exterior temperatures are given respectively by:

$$T_1(r,\theta) = \sum_{l=0}^{\infty} A_l\, r^l P_l(\cos\theta) \qquad \text{and} \qquad T_2(r,\theta) = T_\infty + \sum_{l=0}^{\infty} B_l\, r^{-l-1} P_l(\cos\theta)$$

The boundary conditions on the sphere at $r = a$:

$$T_1(a,\theta) = T_2(a,\theta) \qquad \text{and} \qquad k_1 \frac{\partial T_1}{\partial r} - k_2 \frac{\partial T_2}{\partial r} = \frac{1}{a} \sum_{l=0}^{\infty} q_l\, P_l(\cos\theta)$$

imposes the following equations for the coefficients:

$$A_0 = \frac{B_0}{a} + T_\infty \qquad A_l a^l = B_l a^{-l-1}$$

and

$$k_1 l A_l a^l + k_2(l+1) B_l a^{-i-1} = q_l$$

which can now be solved to yield the solutions:

$$T_1(r,\theta) = T_\infty + \sum_{l=0}^{\infty} \frac{q_l}{k_1 l + k_2(l+1)} \left(\frac{r}{a}\right)^l P_l(\cos\theta)$$

and

$$T_2(r,\theta) = T_\infty + \sum_{l=0}^{\infty} \frac{q_l}{k_1 l + k_2(l+1)} \left(\frac{a}{r}\right)^l P_l(\cos\theta)$$

The temperature at the centre of the sphere is $T_\infty + \frac{q_0}{k_2}$

6. (*bonus question*) Prove the following formulas for Bessel functions (of the first kind):

$$\begin{aligned} \frac{d}{dx}\left(x^n J_n(x)\right) &= x^n J_{n-1}(x) \\ \frac{d}{dx}\left(x^{-n} J_n(x)\right) &= -x^{-n} J_{n+1}(x) \end{aligned}$$

and hence show that the zeros of the Bessel functions interlace, i.e. show that between any two consecutive positive zeros of $J_n(x)$, there is exactly one zero of $J_{n+1}(x)$.

The formulas are proved in the textbook (page 611). To prove the interlacing properties of the zeros, use Rolle's theorem which says that between any two zeros of a function there is at least one zero of the derivative.