MATH 3C03
TEST # 2
Short Answers

1. (*6 marks*) Find the **normalized** eigenstates and eigenvalues of the 2-dimensional time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

in a square $[0,\pi]\times[0,\pi]\subset\mathbb{R}^2$ with boundary conditions that ψ vanishes on all sides of the square.

Eigenvalues $= \frac{\hbar^2}{2m}(n_1^2+n_2^2)$, *where* n_1, n_2 *are integers with normalised eigenstates* $\psi = \frac{2}{\pi}\sin(n_1 x) sin(n_2 y)$

2 (*12 marks*).

(i) Expand $f(x) = x(1-x);\ \ -1 \le x \le +1$ in terms of Legendre polynomials.
Hint: $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1)$

(ii) Solve Laplace's equation:

$$\Delta u(x,y,z) = -\nabla^2 u(x,y,z) = 0$$

in the **interior** of the unit ball $x^2+y^2+z^2 \le 1$ in $\mathbb{R}^3$, with boundary conditions:
$u(x,y,z) = z(1-z)$ on the unit sphere $x^2+y^2+z^2 = 1$.

(i) $x(1-x) = -\frac{1}{3}P_0(x) + P_1(x) - \frac{2}{3}P_2(x)$ *(ii)* $u(r,\theta,\phi) = -\frac{1}{3} + rP_1(\cos\theta) - \frac{2}{3}r^2P_2(\cos\theta)$

3 (*6 marks*). Use the recursion formula:

$$(n+1)P_{n+1}(x) = (2n+1)\,x\,P_n(x) - nP_{n-1}(x)$$

and the normalization $\int_{-1}^{+1}(P_n(x))^2dx = \frac{2}{2n+1}$ for Legendre polynomials to compute the following integral:

$$\int_{-1}^{+1} x\,P_4(x)P_5(x)dx$$

Answer $\frac{10}{99}$

4 (*6 marks*). Use the generating function

$$exp\left(\frac{x}{2}(h-h^{-1})\right) = \sum_{n=-\infty}^{\infty} J_n(x)\,h^n$$

to show the following addition formula for integral Bessel functions:

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y)$$

$\sum_{n=-\infty}^{\infty} J_n(x+y)\,h^n = exp\left(\frac{x+y}{2}(h-h^{-1})\right) = exp\left(\frac{x}{2}(h-h^{-1})\right) exp\left(\frac{y}{2}(h-h^{-1})\right)$
$= \left(\sum_{n=-\infty}^{\infty} J_k(x)\,h^k\right)\left(\sum_{n=-\infty}^{\infty} J_l(y)\,h^l\right)$. *Now equate coefficients.*

5 (*3 marks bonus*). Find the **normalized** wave function with the **lowest non-zero** energy of the two-dimensional time-independent Schrödinger operator

$$-\frac{\hbar^2}{2m}\nabla^2\psi$$

acting on functions defined on the unit disk $x^2 + y^2 \leq 1$ satisfying Dirichlet boundary conditions: $\psi = 0$ on the unit circle.

Answer: $\psi(r,\theta) = \frac{1}{\sqrt{\pi}}\frac{1}{J_1(\alpha_1)}J_0(\alpha_1 r)$ *where* α_0 *is the first zero of the Bessel function* J_0