Math 3D03
Short solutions to assignment #1

1. Compute the Taylor, respectively Laurent series expansion and determine the region of convergence of the following functions around the point $z = 0$:

$$(a)\ \ f(z) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) \qquad\qquad (b)\ \ f(z) = \frac{e^{\frac{1}{z}}}{1-z}$$

(a) $\frac{1}{2i}\left(\log(1+iz) - \log(1-iz)\right) = \frac{1}{2i}\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}(iz)^k}{k} + \frac{(iz)^k}{k}\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}z^{2k-1} (= \arctan(z))$.
This Taylor series converges for $|z| < 1$.

(b) $\frac{e^{\frac{1}{z}}}{1-z} = \left(\sum_{k=0}^{\infty}\frac{z^{-k}}{k!}\right)\left(\sum_{l=0}^{\infty}z^l\right) = \sum_{n=1}^{\infty}a_{-n}z^{-n} + e\sum_{n=0}^{\infty}z^n$, where $a_{-n} = \sum_{j=n}^{\infty}\frac{1}{j!}$.
(by multiplying the two series and collecting terms). This Laurent series converges for $0 < |z| < 1$.

2. Classify all the singular points and compute the residues at the poles of the following functions:

$$(a)\ \ f(z) = \frac{\pi z}{\sin(\pi z)} \qquad (b)\ \ f(z) = \frac{z}{1-z^2}\sinh\frac{1}{1-z} \qquad (c)\ \ f(z) = \frac{z}{1-e^{-z}}$$

(a) The singular point at $z = 0$ is a removable singularity since $\lim_{z\to 0}\frac{\pi z}{\sin \pi z} = 1$.
The other singular points $z_k = k$, where $k \neq 0$ is an integer, are all simple poles with residue given by $\frac{k\pi}{\pi\cos(k\pi)} = (-1)^k$

(b) $z = -1$ is a simple poles with residue $= -\sinh(\frac{1}{2})$. $z = 1$ is an essential singularity with residue $= +\sinh(\frac{1}{2})$
Laurent series around the point $z = 1$:
$\frac{1}{1-z}\sinh(\frac{1}{1-z}) = \frac{1}{z-1}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}(z-1)^{-(2k+1)}$ and $\frac{z}{1+z} = \frac{1}{2}\left(1 + \sum_{l=1}^{\infty}(-1)^{l-1}\frac{1}{(2(z-1))^l}\right)$ and so the coefficient of $\frac{1}{z-1}$ in the Laurent expansion is $\sum_{n=1}^{\infty}\frac{1}{2^{2n+1}(2n+1)!} = \sinh(\frac{1}{2})$
Another way to compute the residue at $z = 1$ is to compute the residue at ∞ which happens to be 0

(c) $z = 0$ is a removable singularity and $z_k = ik\pi$ for $k \in \mathbb{Z},\ k \neq 0$ are all simple poles with residues $= ik\pi$.

3. Evaluate the following complex contour integrals:

$$(a)\ \oint_C \frac{dz}{1+z^4} \qquad (b)\ \oint_C \frac{e^{iz}dz}{1-z^2} \qquad (c)\ \oint_C \frac{z^3\,dz}{(z+1)^2(z^2+4)}$$

where C is the ellipse defined by: $3x^2 + 4y^2 = 10^{10}$

(a) 0
The residues at the four poles: $\pm\exp(\pm\frac{i\pi}{4})$ cancel in pairs. The residue at $z = \exp(\frac{i\pi}{4})$ is $\frac{1}{4}\exp(\frac{-3i\pi}{4})$ at $z = -\exp(\frac{i\pi}{4})$ is $-\frac{1}{4}\exp(\frac{-3i\pi}{4})$ at $\exp(\frac{-i\pi}{4})$ is $\frac{1}{4}\exp(\frac{3i\pi}{4})$ at $z = -\exp(\frac{-i\pi}{4})$ is $-\frac{1}{4}\exp(\frac{3i\pi}{4})$
an easier way to see this is to compute the residue at ∞ which happens to be 0

(b) $2\pi \sin(1)$
Simple poles at $z = \pm 1$ with residues $-\frac{1}{2}e^{i}$ and $+\frac{1}{2}e^{-i}$ respectively

(c) $2\pi i$
Simple poles at $z = \pm 2i$ with residues $\frac{1}{25}(6 \pm 8i)$ and a double pole at $z = -1$ with residue $= \frac{13}{25}$
you can instead compute the residue at ∞ *which happens to be* -1

4. Let a be a positive real number. Compute (using an appropriate contour)

$$\int_0^\infty \cos(a\,x^2)\,dx$$

This is Exercise 24.10 on page 868 in the text book. You can just follow the hints given there. We evaluate $\oint_C e^{iz^2}dz$, where the contour C is a $\frac{\pi}{4}$−sector (of radius $R \to \infty$) in the first quadrant. There are no poles, since e^{iz^2} is analytic everywhere on $\mathbb{C}$. The integral along the ray $z = re^{i\frac{\pi}{4}}$ is a phase-shifted Gaussian integral given by $-e^{i\frac{\pi}{4}}\int_0^\infty e^{-x^2}dx = -e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2}$. The integral on the circular arc $\int_0^{\frac{\pi}{4}} e^{iR^2(\cos 2\theta + i\sin 2\theta)}d\theta$ tends to zero as $R \to \infty$, since $\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}d\theta \le \int_0^{\frac{\pi}{4}} e^{-R^2(\frac{2}{\pi}2\theta)}d\theta = \frac{\pi}{4R^2}(1 - e^{-R^2}) \to 0$, using the elementary inequality $\sin x \ge \frac{2}{\pi}x$ for $0 \le x \le \frac{\pi}{2}$ (*just look at the graph of the sine function!*). Therefore

$$\int_0^\infty e^{ix^2}dx = e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2}$$

and hence

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \sqrt{\frac{\pi}{8}}$$

A simple scaling gives:

$$\int_0^\infty \sin(ax^2)dx = \int_0^\infty \cos(ax^2)dx = \sqrt{\frac{\pi}{8a}}$$

5. Compute

$$\int_0^\pi \sin^n\theta\,d\theta$$

What happens when $n \to \infty$?

The integral is obviously zero for odd n, since $\sin(\theta) = -\sin(2\pi - \theta)$. For even n, we have, using the binomial formula:

$$\int_0^{2\pi}(\sin\theta)^{2n}\,d\theta = \oint_{|z|=1}\left(\frac{z - z^{-1}}{2i}\right)^{2n}\frac{dz}{iz} = \frac{1}{(2i)^n}\oint_{|z|=1}\frac{z^{2n}}{iz}(1 - z^{-2})^{2n}dz$$

$$= \frac{1}{2^{2n}i^{2n+1}}\sum_{k=0}^{2n}\oint_{|z|=1}(-1)^k\binom{2n}{k}z^{2n-2k-1}\,dz = \frac{2\pi}{2^{2n}}\binom{2n}{n}$$

As $n \to \infty$, the integral goes to zero. This can be seen, for example, by using Stirling's formula: $\lim_{N\to\infty}\sqrt{2\pi N}\left(\frac{N}{e}\right)^N\frac{1}{N!} = 1$ or by looking at the graph of the function $(\sin\theta)^{2n}$ for large n .

6. (*bonus question*) Consider the $n-1$ diagonals connecting one fixed vertex to all the other vertices of a regular n-gon inscribed in a unit circle. Prove that the products of their lengths is equal to n.

The n^{th} roots of unity $z_1, z_2, \ldots, z_{n-1}$ which are $\neq 1$ satisfy the equation

$$\prod_{k=1}^{n-1}(z - z_k) = \frac{z^n - 1}{z - 1} = z^{n-1} + \cdots + z + 1$$

Now put $z = 1$ and take the modulus (absolute value).

LOL