Math 3D03
Short answers to assignment #2

1. (*8 marks*) Evaluate the following definite (real-valued) integrals:

$$(i)\ \int_0^\infty \frac{(\log(x))^2}{1+x^2}\,dx \qquad (ii)\ \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx \quad \text{for } 0<a<1$$

$$(iii)\ \int_0^\infty \frac{\log(x)}{x^{\frac{3}{4}}(1+x)}\,dx \qquad (iv)\ \int_0^\infty \frac{dx}{1+x^n} \quad \text{where } n\geq 2 \text{ is an integer}$$

(i) This is problem 24.21 in the textbook. Use a large semicircle of radius R in the upper half plane and the real line with a small semicircular dent of radius ϵ around the origin. Log is well-defined there. There is a simple pole at $z=i$ inside ythe contour with residue $\frac{(\log(i))^2}{2i}=i\frac{\pi^2}{8}$. The integrals on the semicircular pieces go to zero when $R\to\infty$ and $\epsilon\to 0$. On the left part of the real axis, the log is phase-shifted: $(\log(xe^{i\pi}))^2=(\log x)^2+2i\pi\log x-\pi^2$, so we get:

$$2\int_0^\infty \frac{(\log(x))^2}{1+x^2}\,dx+2\pi i\int_0^\infty \frac{\log(x)}{1+x^2}\,dx-\pi^2\int_0^\infty \frac{dx}{1+x^2}=-\frac{\pi^3}{4}$$

Equating real and imaginary parts we get:

$$\int_0^\infty \frac{(\log(x))^2}{1+x^2}\,dx=\frac{\pi^3}{8} \qquad \int_0^\infty \frac{\log(x)}{1+x^2}\,dx=0$$

(ii) Use a long horizontal strip $[-R,+R]\times[0,2\pi]$ above the $x-$axis as your contour The integral on the two vertical lines $\to 0$, as $R\to\infty$ since the integrand is bounded from above in amplitude by $\frac{e^{aR}}{e^R}$ (on the right vertical line) and by e^{-aR} (on the left vertical) and $0<a<1$. On the upper horizontal line the integral is a phase shift by $e^{ia2\pi}$ of the integral on the $x-$ axis (in the opposite direction). There is exactly one simple pole at $i\pi$ within the strip with residue $=\frac{e^{ia\pi}}{e^{i\pi}}=-e^{ia\pi}$. Therefore $(1-e^{i2a\pi})\int_{-\infty}^\infty \frac{e^{ax}dx}{1+e^x}=-2\pi i e^{ia\pi}$ and hence the answer is

$$\int_{-\infty}^\infty \frac{e^{ax}dx}{1+e^x}=\frac{\pi}{\sin(a\pi)}$$

(iii) This is problem 24.20 in the textbook. Use a key hole contour around the origin with a cut along the positive real axis.
There is exactly one simple pole at $z=\exp(i\pi)$ with residue $=i\pi\ \exp(-i\frac{3\pi}{4})=$.
Both circular integrals (around the little circle around zero and the big circle around ∞) go to 0 when you let the radii go to zero and ∞ respectively. The integral along the cut (the positive real axis) undergoes a phase shift when it comes back from ∞:

$$\int_\infty^0 \frac{log(x)+2\pi i}{\exp(i\frac{3\pi}{2})\,x^{\frac{3}{4}}(1+x)}dx=-i\int_0^\infty \frac{log(x)}{x^{\frac{3}{4}}(1+x)}dx+2\pi\int_0^\infty \frac{dx}{x^{\frac{3}{4}}(1+x)}$$

Hence

$$(1-i)\int_0^\infty \frac{log(x)}{x^{\frac{3}{4}}(1+x)}dx + 2\pi\int_0^\infty \frac{dx}{x^{\frac{3}{4}}(1+x)} = -2\pi^2\,\exp(-i\frac{3\pi}{4}) = \pi^2\sqrt{2}(1+i)$$

Therefore:

$$\int_0^\infty \frac{log(x)}{x^{\frac{3}{4}}(1+x)}dx = -\pi^2\sqrt{2} \qquad \int_0^\infty \frac{dx}{x^{\frac{3}{4}}(1+x)} = \pi\sqrt{2}$$

(iv) This is problem 24.18 in the textbook but here is how you can do it: $\oint_C \frac{1}{1+z^n}dz$, where the contour C is the $\frac{2\pi}{n}$−sector (of radius $R \to \infty$) in the first quadrant. There is a single simple pole at $e^{i\frac{\pi}{n}}$ inside C with residue $= \frac{1}{n}e^{-i\frac{(n-1)\pi}{n}} = -\frac{1}{n}e^{i\frac{\pi}{n}}$. The integral along the ray $z = re^{i\frac{2\pi}{n}}$ is a phase shift by $e^{i\frac{2\pi}{n}}$ of the integral on the x−axis (in the opposite direction). The integral on the circular arc tends to zero as $R \to \infty$, since $n \geq 2$. Therefore $(1 - e^{i\frac{2\pi}{n}})\int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$ and hence

$$\int_0^\infty \frac{dx}{1+x^n}dx = \frac{\pi}{n}\csc(\frac{\pi}{n})$$

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2. (*2 marks*) How many zeros of the polynomial $z^4 - 5z + 1$ lie in the annulus $1 \leq |z| \leq 2$?

Answer: 3 zeros by Rouche's theorem:
On the outer circle $|z| = 2$, $|z^4| > |-5z+1|$ so there are exactly 4 roots in $|z| \leq 2$. On the other hand, $|z^4| < |-5z+1|$ so there is exactly one root in $|z| \leq 1$

3. (*6 marks*) Sum the following infinite series:

$$(a)\ \sum_{n=1}^\infty \frac{1}{n^2+9} \qquad (b)\ \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^4} \qquad (c)\ \sum_{n=-\infty}^\infty \frac{n^2}{n^4-\pi^4}$$

(a) The two residues of $\frac{\pi\cot(\pi z)}{z^2+9}dz$ at the two poles $z = \pm 3i$ add up to

$$\sum_{n=1}^\infty \frac{1}{n^2+9} = -\frac{1}{2}\left(\frac{\pi\cot(3\pi i)}{6i} + \frac{\pi\cot(-3\pi i)}{-6i}\right) = \frac{\pi}{6}\coth(3\pi) - \frac{1}{18}$$

(b)

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^4} = \frac{1}{2}\text{Res}\left(\frac{\pi\csc(\pi z)dz}{z^4};0\right) = \frac{1}{2}\frac{(-1)^3 2(2^3-1)\pi^4}{4!}B_4 = \frac{7\pi^4}{720}$$

(c)

$$\sum_{n=-\infty}^\infty \frac{n^2}{n^4-\pi^4} = \text{sum of residues}\left(\frac{\pi\cot(\pi z)z^2dz}{z^4-\pi^4};\pm\pi,\pm i\pi\right) = \frac{1}{2}(\coth(\pi^2) - \cot(\pi^2))$$

4. (*4 marks*) Do problem 25.14 on page 922 - 923 in the text book.
Here you use a Bromwich contour with a branch cut from$+i$ to $-i$. Since there are no poles outside a closed contour containing that cut we are left with integrating around the "double key hole". The function $\log(z+i)-\log(z-i)$ is well defined around that contour and measures the difference of the arguments around the two pints $\pm i$.There is a phase shift of 2π for the angle difference after going around $+i$. The upshot is that the inverse Laplace transform is the sinc function $f(t)=\frac{\sin t}{t}$. You can check that also by differentiation since $F'(s)=-\frac{1}{s^2+1}$.

5. (*5 marks*) Show that the map

$$w=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

maps circles centered at the origin in the z-plane to ellipses in the w-plane. Draw some images. What happens to other circles? Find the image of the circle centered at the point $z_0=-\frac{1}{5}(1-i)$ with radius $\frac{1}{5}\sqrt{37}$ (*Use Matlab or some other software to plot the graphs*)
$w=u+iv=re^{i\theta}+\frac{1}{r}e^{-i\theta}=(r+\frac{1}{r})\cos\theta+(r-\frac{1}{r})\sin\theta$ when $z=re^{i\theta}$ $\quad r\neq 1$. So the image of a circle centered at the origin is the curve in the w plane defined by:

$$\frac{u^2}{(r+r^{-1})^2}+\frac{v^2}{(r-r^{-1})^2}=1$$

This is an ellipse for $r>1$ and is a hyperbola for $r<1$. For $r=1$ we get $w=2\cos\theta$, so the image is the line segment $[-2,+2]$
For the pictures see Diego's lecture on Tuesday (posted on the course home page).

6. *(bonus question)*

(i) Suppose that $f(z)$ is a non-constant analytic function defined for all $z\in\mathbb{C}$. Show that for every $R>0$ and for every $M>0$ there exists a z such that $|z|>R$ and $|f(z)|>M$.

(ii) Suppose that $f(z)$ is a non-constant polynomial. Show that for every $M>0$ there exists an $R>0$, such that $|f(z)|>M$ for all $|z|>R$.

(iii) Show that there exists an $M>0$, such that for every $R>0$, there exists a z satisfying $|z|>R$ and $|e^z|\leq M$.

(i) Arguing by contradiction, let us assume that there exists $R>0$ and $M>0$ such that $f(z)\leq M$ for every $|z|>R$. Let $a\in\mathbb{C}$. By the Cauchy integral formula : $f'(a)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-a)^2}dz$, where we choose C to be a circle of very large radius r (say $r=100(R+|a|)$). The integral is bounded from above in absolute value by the product of the length of that circle and the maximum absolute value of the integrand which is $\leq 2\pi r\frac{M}{r^2}=\frac{2\pi M}{r}$ and so $|f'(a)|\leq\frac{M}{r}$ for any r sufficiently large. This proves that $f'(a)=0$ for any a and hence f is a constant function.

(ii) By factoring out the top coefficient, we may assume that $p(z)=z^n+q(z)$, where $q(z)$ is a polynomial of degree $\leq n-1$ $(n\geq 1)$. Since $\lim_{z\to\infty}\frac{q(z)}{z^n}=0$, we see that for $|z|$ sufficiently large $|q(z)|<0.1|z^n|$ and so $|p(z)|\geq 0.9|z|^n\geq 0.9R^n$ for $|z|\geq R$ So for any given $M>0$, we can find R such that $|p(z)|>M$ for every z with $|z|>R$.

(iii) For every $R>0$, $z=2iR$ satisfies $|z|>R$ and $|e^z|\leq 1$.