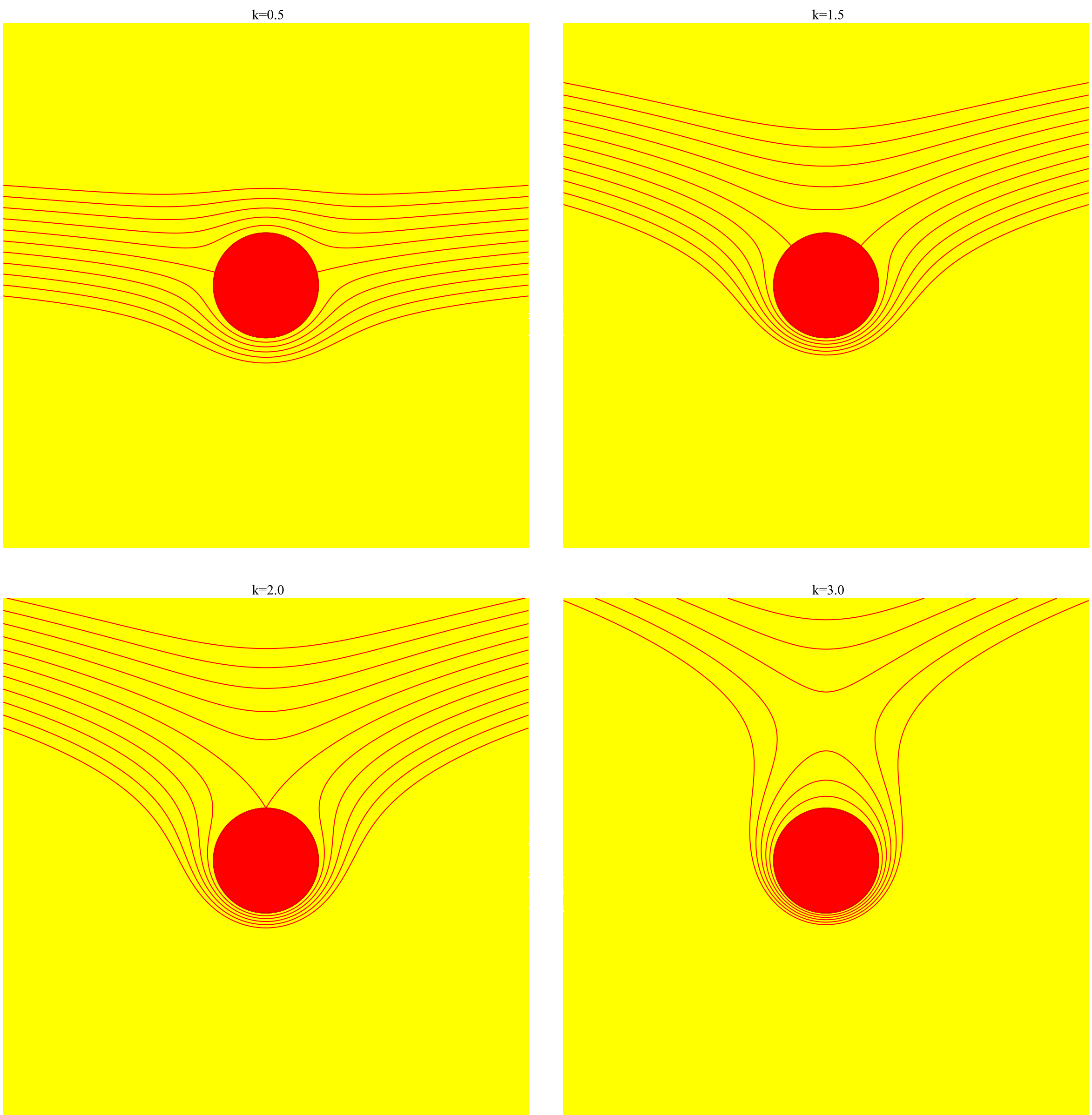
Math 3D03
Short Answers to Assignment #3

#1. The complex potential

$$\Omega(z) = z + \frac{1}{z} - i\,\kappa \log(z)$$

where κ is a positive real number, describes a fluid flow around a cylinder with circulation. Locate the stagnation points (as a function of κ) and sketch the streamlines of the flow, using computer software such as Matlab, for the following κ values: $\kappa = 0.5,\ 1.5,\ 2,\ 3$.

The stagnation points are obtained by solving the quadratic equation $\Omega'(z) = 1 - \frac{1}{z^2} - i\,\kappa\frac{1}{z} = 0$ with roots $\frac{1}{2}\left(i\kappa \pm \sqrt{4-\kappa^2}\right)$. *(I owe the following pictures to L. Ambroszkiewicz)*



2. The WKB approximation for the ODE:

$$\psi''(x) + \frac{\omega^2 m_0}{T} f(x)\psi(x) = 0$$

in the regime where $\lambda^2 = \frac{\omega^2 m_0}{T} >> 1$ is given by:

$$\psi_{WKB}(x) = \frac{C_\pm}{(f(x))^{\frac{1}{4}}} e^{\pm i\lambda \int_0^x \sqrt{f}}$$

For $f(x) = 1 + \epsilon \sin(\frac{2\pi}{L}x)$, with $\epsilon << 1$, we can use the first order approximations:

$(f(x))^{-\frac{1}{4}} \approx 1 - \frac{1}{4}\epsilon \sin(\frac{2\pi}{L}x)$

and

$\int_0^x f^{\frac{1}{2}} \approx \int_0^x (1 + \frac{1}{2}\epsilon \sin(\frac{2\pi}{L}x)) = x + \epsilon \frac{L}{4\pi}(1 - \cos(\frac{2\pi}{L}x)) = x + \epsilon \frac{L}{2\pi} \sin^2(\frac{2\pi}{L}x)$.

Therefore the fluctuation in amplitude is roughly $\pm\frac{1}{4}\epsilon A_0$ of the amplitude A_0 at $x = 0$.

As for the phase, it is shifted roughly by the amount $\epsilon\lambda \frac{L}{2\pi} \sin^2(\frac{2\pi}{L}x) = \epsilon \frac{L}{2\pi}\sqrt{\frac{\omega^2 m_0}{T}} \sin^2(\frac{2\pi}{L}x)$ from the linear term λx.

#3 Let A denote the Ascii's, B denote the Biscii's, T the truth, F a lie and D the event that you get the same answer twice.
Then $Prob(A) = \frac{11}{16}$, $Prob(B) = \frac{5}{16}$, $Prob(D|A) = Prob(TT|A) + Prob(FF|A) = \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2$
$Prob(D|B) = 1$, $Prob(TT|A \cap D) = \frac{3^2}{3^2+1^2} = \frac{9}{10}$
So by Bayes' formula

$$Prob(A|D) = \frac{\frac{5}{8}\frac{11}{16}}{\frac{5}{8}\frac{11}{16} + \frac{5}{16}} = \frac{11}{19}$$

and since $Prob(TT|B) = 0$,

$$Prob(TT|D) = Prob(TT|A)Prob(A|D) = \frac{9}{10}\frac{11}{19} = \frac{99}{190} > \frac{1}{2}$$

so he should go left. Similarly if E is the event that you get the same answer three times, we find:

$$Prob(A|E) = \frac{\left(\frac{3}{4}\right)^3 + \left(\frac{1}{4}\right)^3 \frac{11}{16}}{\left(\frac{3}{4}\right)^3 + \left(\frac{1}{4}\right)^3 \frac{11}{16} + \frac{5}{16}} = \frac{77}{157}$$

and since $Prob(TTT|A \cap E) = \frac{3^3}{3^3+1^3} = \frac{27}{28}$

$$Prob(TTT|E) = Prob(TTT|A)Prob(A|D) = \frac{27}{28}\frac{77}{157} = \frac{297}{628} < \frac{1}{2}$$

so he should go right.

#4 For the classical case we get: $\mu =< x >= \frac{1}{2}a$ and $\sigma^2 =< x^2 > - < x >^2 = \frac{1}{12}a^2$
In the quantum case $\mu =< x >= \frac{1}{2}a$ and

$$\sigma^2 =< x^2 > - < x >^2 = \frac{2}{a}\int_0^a x^2 \sin^2 \frac{n\pi x}{a} - \frac{1}{4}a^2 = a^2\left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right) \rightarrow \frac{a^2}{12} \text{ as } n \rightarrow \infty$$

#5 If 7 indistinguishable marbles are placed at random into 7 boxes, what is the probability that **exactly two** boxes are empty?

There are 7^7 equally probable ways of putting 7 balls into 7 boxes. There are two mutually exclusive possibilities for 2 boxes to be empty: (i) 1 box with 3 balls and four boxes with 1 ball each (ii) 2 boxes with 2 balls each and three boxes with 1 ball each.
Case (i): the number of ways to do this is $\frac{7!}{1!4!2!} \times \frac{7!}{3!1!1!1!1!}$ by the multinomial formula applied to tag the boxes according to the occupancy numbers of the boxes and also to tag the balls according to which type of box they fall into. Case (ii): the number of ways to do this is $\frac{7!}{2!3!2!} \times \frac{7!}{2!2!1!1!1!}$
Altogether the probability of getting exactly two boxes empty is:

$$\left(\frac{(7!)^2}{4!2!3!} + \frac{(7!)^2}{2!3!2!2!2!}\right) 7^{-7} \approx 0.4284$$

It is interesting to note that the probability of getting exactly 1 box empty is ≈ 0.1285 and the probability of filling all boxes is ≈ 0.0061

#6 *bonus question* Let $q = 1 - p$.

(a) $\binom{6}{3}p^3q^3$

(b) $4 \times (p^4 + q^4) + 5 \times \binom{4}{1}(p^4q + q^4p) + 6 \times \binom{5}{2}(p^4q^2 + q^4p^2) + 7 \times \binom{6}{3}p^3q^3$

(c) $4 \times p^4 + 5 \times \binom{4}{1}p^4q + 6 \times \binom{5}{2}p^4q^2 + 7 \times \binom{6}{3}p^4q^3$

(d) p *(the outcome of the last game is independent of what happened before)*