Asymptotics of the Airy function

1. **Definition**:

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(i(\frac{k^3}{3} + z\,k))\,dk = \frac{1}{\pi}\int_{0}^{+\infty} \cos(\frac{k^3}{3} + z\,k)\,dk$$

Changing variables: $s = ik$ and choosing the right contour C to integrate, we can also write this as:

$$Ai(z) = \frac{1}{2\pi i}\int_C \exp(-\frac{s^3}{3} + z\,s))\,ds$$

The contour C is chosen to lie in the left half of the complex plane and approach the two rays $\pm\frac{2\pi}{3}$ asymptotically (*see textbook, p. 892*)

2. **Asymptotics**:

(i) For $z \to -\infty$ on the real line we have:

$$Ai(z) \approx \frac{1}{\sqrt{\pi}}\frac{1}{(-z)^{\frac{1}{4}}}\cos\left(\frac{2}{3}(-z)^{\frac{3}{2}} - \frac{\pi}{4}\right) = \frac{1}{\sqrt{\pi}}\frac{1}{(-z)^{\frac{1}{4}}}\sin\left(\frac{2}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right)$$

(ii) For $z \to +\infty$ on the real line we have:

$$Ai(z) \approx \frac{1}{2\sqrt{\pi}}\frac{1}{z^{\frac{1}{4}}}e^{-\frac{2}{3}z^{\frac{3}{2}}}$$

Proofs:

Case (i) $z \to -\infty$
Let $\phi(k) = (\frac{k^3}{3} + z\,k)$. Then $\phi'(k) = k^2 + z = 0$ iff $k = \pm k_0$ where $k_0 = \sqrt{-z}$ where $z < 0$ and $\phi''(k_0) = 2k_0$ and hence the Taylor expansions of ϕ near the two critical points are given by

$$\phi(k) \approx \left(\frac{k_0^3}{3} - (-z)k_0\right) + k_0(k-k_0)^2 = -\frac{2}{3}(-z)^{\frac{2}{3}} + k_0(k-k_0)^2$$

and

$$\phi(k) \approx +\frac{2}{3}(-z)^{\frac{2}{3}} - k_0(k-k_0)^2$$

respectively.
By stationary phase approximation:

$$\int e^{i\phi(k)}\,dk \approx e^{-i\frac{2}{3}(-z)^{\frac{3}{2}}}\int_{-\infty}^{+\infty} e^{ik_0(k-k_0)^2}\,dk = e^{-i\frac{2}{3}(-z)^{\frac{3}{2}}}\sqrt{\frac{\pi}{-ik_0}} = \frac{\sqrt{\pi}}{(-z)^{\frac{1}{4}}}e^{-i(\frac{2}{3}(-z)^{\frac{3}{2}} - \frac{\pi}{4})}$$

near the critical point $k = +k_0$ and similarly near the critical point $k = -k_0$, we get the contribution:

$$\int e^{i\phi(k)}\,dk \approx \frac{\sqrt{\pi}}{(-z)^{\frac{1}{4}}}e^{i(\frac{2}{3}(-z)^{\frac{3}{2}} - \frac{\pi}{4})}$$

Adding up the two contributions we finally get:

$$Ai(z) = \frac{1}{2\pi}\int e^{i\phi(k)}\,dk \approx \frac{1}{\sqrt{\pi}}\frac{1}{(-z)^{\frac{1}{4}}}\cos\left(\frac{2}{3}(-z)^{\frac{3}{2}} - \frac{\pi}{4}\right)$$

as $z \to -\infty$

Case (ii) $z \to +\infty$

We use the contour integral representation and the "saddle point method". Choose the contour C so that it passes through the point $-\sqrt{z}$ and is vertical (purely imaginary) The function ϕ is now $\phi(s) = zs - \frac{s^3}{3}$ and the critical points of ϕ are at $s = \pm\sqrt{z} = \pm s_0$ with $\phi''(s) = -2s$, which is positive at $-s_0 = -\sqrt{z}$ through which the contour runs. Near this "saddle point" the Taylor approximation of ϕ is:

$$\phi(s) \approx \phi(-\sqrt{z}) + \sqrt{z}(s+\sqrt{z})^2 = -\frac{2}{3}z^{\frac{3}{2}} + \sqrt{z}(s+\sqrt{z})^2$$

So we have:

$$2\pi i(Ai(z)) = \int_C e^{\phi(s)}ds \approx e^{-\frac{2}{3}z^{\frac{3}{2}}}\int_C e^{\sqrt{z}(s-s_0)^2}ds$$

Evaluating the Gaussian integral (with $s - s_0 = it$):

$$\int_C e^{\sqrt{z}(s-s_0)^2}ds \approx i\int_{-\infty}^{+\infty} e^{-\sqrt{z}\,t^2}dt = i\sqrt{\frac{\pi}{\sqrt{z}}}$$

we finally obtain:

$$Ai(z) \approx \frac{1}{2\sqrt{\pi}}\frac{1}{z^{\frac{1}{4}}}e^{-\frac{2}{3}z^{\frac{3}{2}}}$$

as $z \to +\infty$