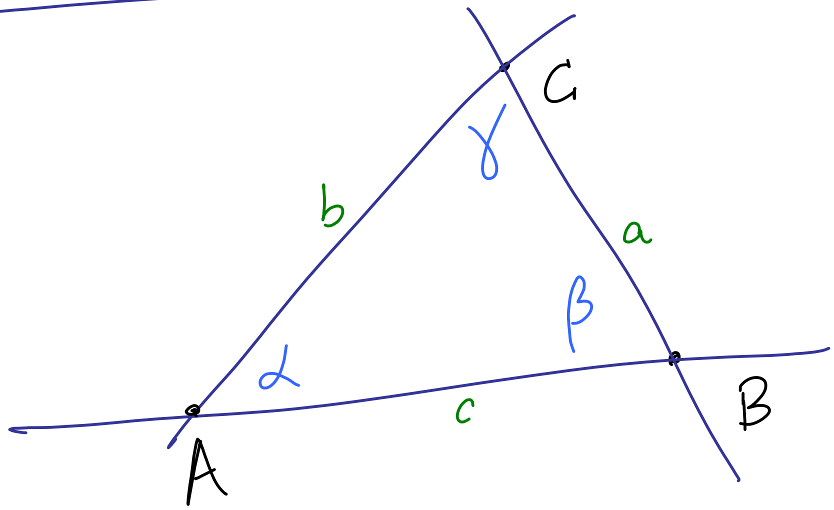
TRIGONOMETRY



COSINE LAW

$$K = +\frac{1}{R^2}$$

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R} + \sin\frac{a}{R}\sin\frac{b}{R}\cos\gamma$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad (K=0)$$

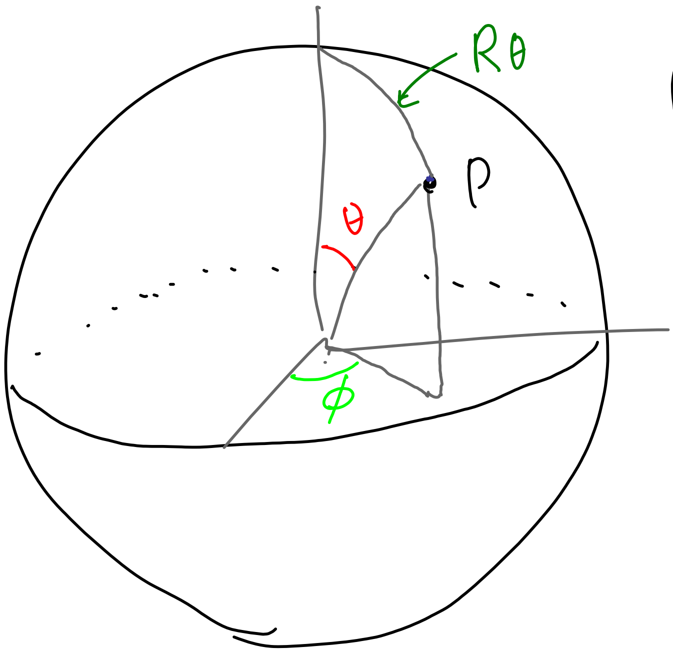
$$\cosh\frac{c}{R} = \cosh\frac{a}{R}\cosh\frac{b}{R} + \sinh\frac{a}{R}\sinh\frac{b}{R}\cos\gamma$$

$$\left(K = -\frac{1}{R^2}\right)$$

$R \to \infty$ $\quad K \to 0$ (limiting flat case)

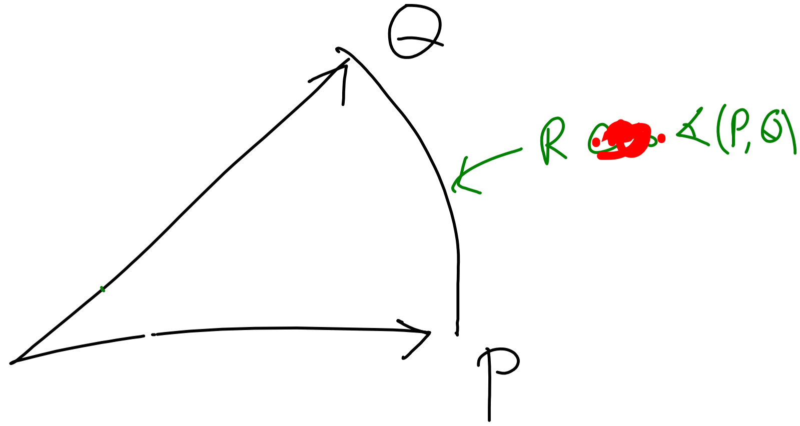
spherical $\quad$ Euclidean $\quad$ hyperbolic

$S^2 \subset \mathbb{R}^3 \qquad x^2 + y^2 + z^2 = R^2$



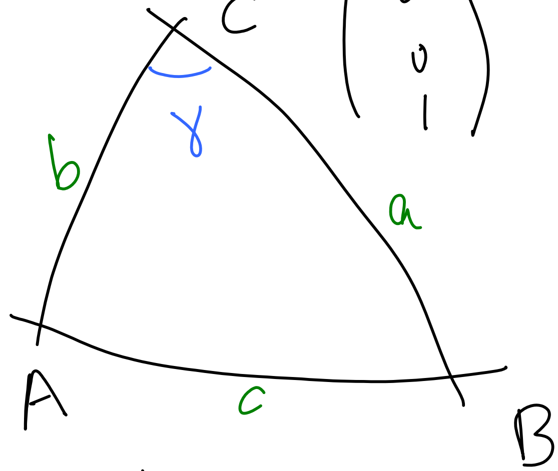
$$P = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}$$

spherical coordinates



$$\cos \measuredangle(P,Q) = \frac{1}{R^2} \langle P, Q \rangle$$

$R=1$



$$\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \theta=0$$

$$\begin{pmatrix}\sin a\cos\gamma\\ \sin a\sin\gamma\\ \cos a\end{pmatrix} \qquad \theta=a,\quad \phi=\gamma$$

$$\begin{pmatrix}\sin b\\ 0\\ \cos b\end{pmatrix}$$

$$\theta=b$$

$$\cos c = \langle A, B\rangle = \sin a \sin b \cos\gamma + \cos a \cos b$$

QED

Do it for any $R>0$ and take the limit $R\to\infty$

$$1-\frac{1}{2}\frac{c^2}{R^2}+\ldots = \left(1-\frac{1}{2}\frac{a^2}{R^2}+\ldots\right)\left(1-\frac{1}{2}\frac{b^2}{R^2}+\ldots\right) + \left(\frac{a}{R}-\ldots\right)\left(\frac{b}{R}-\ldots\right)\cos\gamma$$

$$c^2 = a^2+b^2-2ab\cos\gamma$$

flat case

sum of interior angles

$$\alpha + \beta + \gamma = \pi + \frac{1}{R^2}(\text{area of } \Delta)$$

$$\alpha + \beta + \gamma = \pi \quad (R = \infty!)$$

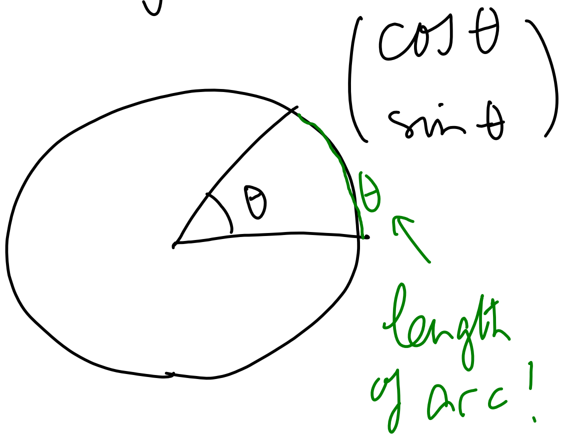
$$\alpha + \beta + \gamma = \pi - \frac{1}{R^2}(\text{area of } \Delta)$$

$$\alpha + \beta + \gamma = \pi + \iint_{\Delta} K \, dA$$

Gauss-Bonnet

Circle in $\mathbb{R}^2$

$x^2 + y^2 = 1$



$\cos^2\theta + \sin^2\theta = 1$

$ds^2 = dx^2 + dy^2$

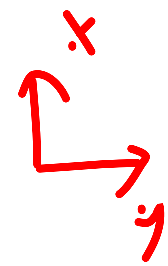
Euclidean $\boxed{\text{Metric}}$

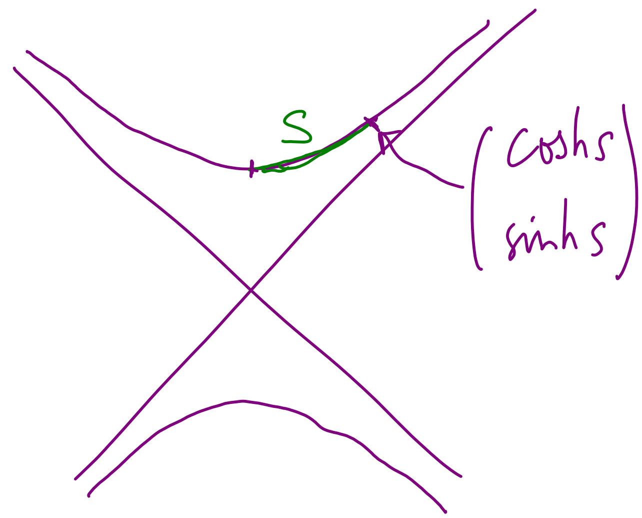
in general

$\mathbb{R}^n$

Hyperbola in $\mathbb{R}^{1,1}$

$x^2 - y^2 = 1$





$\cosh^2 s - \sinh^2 s = 1$

$ds^2 = -dx^2 + dy^2$

Lorentzian

in general

$\mathbb{R}^{1,n-1}$

Minkowski space $\boxed{\mathbb{R}^{1,3}}$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$\boxed{c=1}$

speed of light

Inner (or scalar) products.

$$\langle v, w \rangle = \sum_{i=1}^{n} v^i w^i$$

$$= \sum_{i,j} \delta_{ij} v^i w^j$$

$$= \delta_{ij} v^i w^j \quad = \quad v^T Id\, w$$

Einstein summation convention!

$$\langle\langle v, w \rangle\rangle = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

$$= \eta_{ij} v^i w^j$$

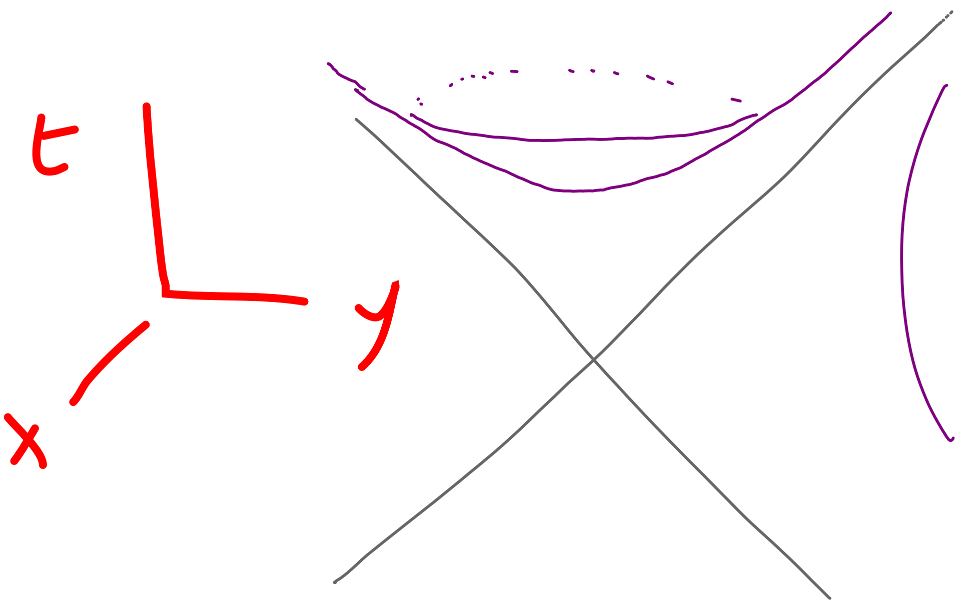
$$= (v^0, \dots, v^3) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w^0 \\ \vdots \\ w^3 \end{pmatrix}$$

Hyperbolic space (in 2 dimensions)

$$H^2 \subset \mathbb{R}^{1,2} \qquad ds^2 = -dt^2 + dx^2 + dy^2$$

hyperboloid

$$t^2 - x^2 - y^2 = 1 \qquad t > 0$$



$$\begin{pmatrix} \cosh s \\ \sinh s \cos\phi \\ \sinh s \sin\phi \end{pmatrix}$$

Changing coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= -dt^2 + dr^2 + r^2 d\theta^2$$

$$= d\rho^2 + \sinh^2\rho \, d\theta^2$$

$t = \cosh\rho$

$r = \sinh\rho$

2-dimensional hyperbolic metric

$\rho > 0$, $\theta \in S^1$ circle

Flat metric on $\mathbb{R}^2$ in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2$$

Spherical metric on $S^2 \subset \mathbb{R}^3$

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

(Just put $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$

in $dx^2 + dy^2 + dz^2$

$$dx = +\cos\theta\cos\phi \, d\theta - \sin\theta\sin\phi \, d\phi$$
$$dy = +\cos\theta\sin\phi \, d\theta + \sin\theta\cos\phi \, d\phi$$
$$dz = -\sin\theta \, d\theta$$

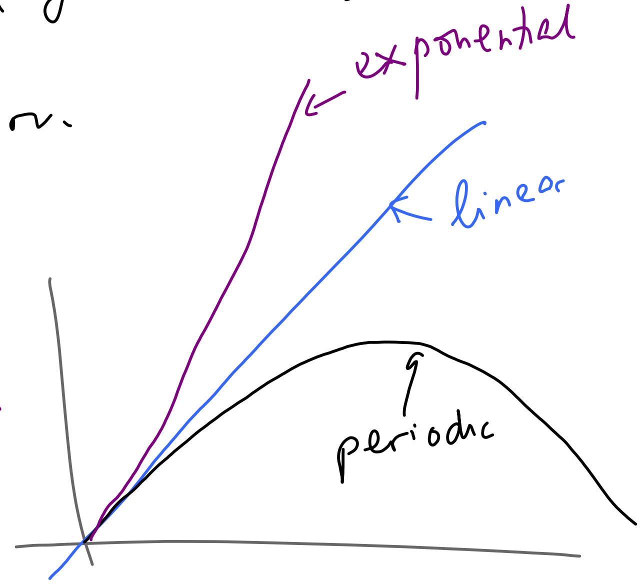
$$ds^2 = dr^2 + \sin^2 r \, d\theta^2$$

Lengths of circles (of radius r)

$2\pi \sin r$ + curv.

$2\pi r$ 0 curv.

$2\pi \sinh r$ − curv.



Area of disks (of radius r)

$2\pi(1-\cos r)$ S^2

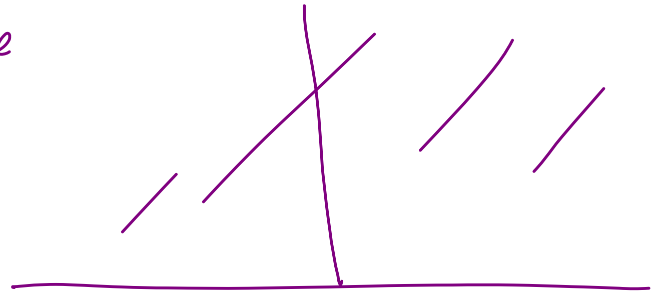
πr^2 $\mathbb{R}^2$

$2\pi(\cosh r - 1)$ $\mathbb{H}^2$

Other models of H^2

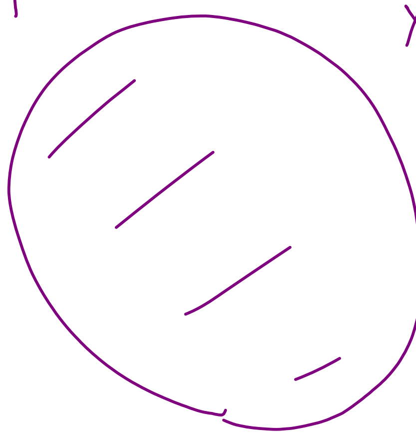
Upper half plane

$$ds^2 = \frac{du^2 + dv^2}{v^2}$$



(u,v) $v>0$

unit disk $|z|<1$ $x^2+y^2<1$



$$ds^2 = 4\frac{dx^2+dy^2}{(1-(x^2+y^2))^2}$$

$$= 4\frac{dz\,d\bar{z}}{(1-|z|^2)^2}$$

Conformal models

Change of variables (transformations)

$$d\rho^2 + (\sinh\rho)^2\, d\theta^2$$

$$z = re^{i\theta}$$

$$r = \tanh\frac{\rho}{2}$$

$$\frac{1}{2}\rho = \operatorname{artanh} r$$

$$d\rho = \frac{2}{1-r^2}$$

$$\sinh\rho = \frac{2r}{1-r^2}$$

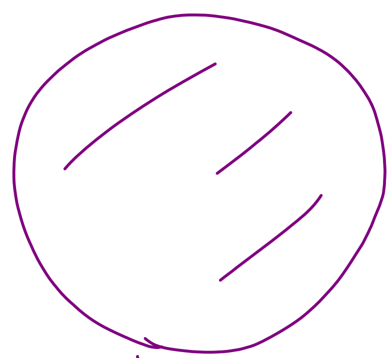
$$\downarrow$$

$$4\,\frac{dz\, d\bar{z}}{(1-|z|^2)^2}$$

$$= 4\,\frac{dx^2+dy^2}{(1-(x^2+y^2))^2}$$

$$\downarrow$$

$$\frac{dw\, d\bar{w}}{(\operatorname{Im}(w))^2} = \frac{du^2+dv^2}{v^2}$$



$$z \downarrow \quad w = i\,\frac{1+z}{1-z}$$

$$z = \frac{w-i}{w+i} \quad \uparrow w$$

(Möbius transform)