Notes on Differentiable Manifolds

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Abstract

These lecture notes are supplementary to the material presented during the course Math 3GP3 in the Fall 2013 at McMaster University. Here we briefly introduce the concepts of Differentiable Manifold, Tangent Space and Vector Fields and their flows.

1 Differentiable Manifolds

Definition 1.1. *A* topological manifold of dimension *n is a Hausdorff, second countable topological space M for which each point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. If x is such a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^n$, we call x a* coordinate map *and the pair (U, x) is called a* coordinate system *or* chart. *If $p \in U$ and $x(p) = 0$, then the coordinate system is said to be centred at p.*

Definition 1.2. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \mapsto \mathbb{R}^n$ be a map. We say that f is* differentiable of class C^∞ *(or simply f is C^∞) if each of the component functions f^i has partial derivatives of all orders.*

Definition 1.3. *A* differentiable structure $\mathcal{A}$ of class C^∞ *on a topological manifold M is a collection of coordinate systems $\{(U_\alpha, x_\alpha)|\alpha \in \mathcal{A}\}$ satisfying the following three properties:*

- $\bigcup_{\alpha \in \mathcal{A}} = M$
- $x_\alpha \circ x_\beta^{-1}$ *is C^∞ for all $\alpha, \beta \in \mathcal{A}$ such that $U_\alpha \cap U_\beta \neq \varnothing$.*
- *The collection $\mathcal{A}$ is maximal with respect to the previous property, i.e. if (U, x) is a coordinate system such that $x \circ x_\alpha^{-1}$ and $x_\alpha^{-1} \circ x$ are C^∞ for all $\alpha \in \mathcal{A}$ such that $U \cap U_\beta \neq \varnothing$, then $(U, x) \in \mathcal{A}$.*

Remark 1.4. *If $\mathcal{A}_0$ is any collection of coordinate systems (called an* atlas*) satisfying the first two properties, then there is a unique differentiable structure $\mathcal{A}$ containing $\mathcal{A}_0$. Namely, $\mathcal{A} = \{(U, x)|x \circ x_\alpha^{-1}$ and $x_\alpha^{-1} \circ x$ are C^∞ for all $\alpha \in \mathcal{A}_0\}$*

Definition 1.5. *An* n-dimensional differentiable manifold of class C^∞ *(or simply a* smooth manifold*) is a pair* $(M, \mathcal{A})$ *consisting of an* n*-dimensional topological manifold* M *together with a differentiable structure* $\mathcal{A}$ *of class* C^∞ *for* M.

Example 1.6.
- *The standard differentiable structure on* $\mathbb{R}^n$ *is defined to be the maximal collection containing the single coordinate chart* $id : \mathbb{R}^n \to \mathbb{R}^n$.

- *An open subset* U *of a smooth manifold* M *inherits a canonical smooth structure defined by*

  $\mathcal{A} := \{(U \cap U_\alpha, x_\alpha|_{U \cap U_\alpha}) \,|\, (U, x_\alpha) \in \mathcal{A}_M\}$.

- $GL(n;\mathbb{R}) := \{A | A is\ a\ matrix\ n \times n, det(A) \neq 0\} \subset \mathbb{R}^{n^2}$ *is an open subset of* $\mathbb{R}^{n^2}$ *and hence inherits a smooth manifolds structure*

- *If* $(M_1, \mathcal{A}_1)$ *and* $(M_2, \mathcal{A}_2)$ *are smooth manifolds of dimension* n_1, n_2*, then the product* $M_1 \times M_2$ *inherits a smooth structure* $\mathcal{A} = \{(U_\alpha \times V_\beta, x_\alpha \times y_\beta : U_\alpha \times V_\beta \mapsto \mathbb{R}^{n_1+n_2}) | (U_\alpha, x_\alpha) \in \mathcal{A}_1, (V_\beta, y_\beta) \in \mathcal{A}_2\}$

- *The* n*-sphere* $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} | |x|^2 = 1\}$ *is a smooth dimensional manifold by taking* $\mathcal{A}$ *to be the maximal atlas containing* $\{(\mathbb{S}^n \setminus \{n\}, sp_n), (\mathbb{S}^s \setminus \{n\}, sp_s)\}$ *where* sp_n *and* sp_s *are stereographic projections from the north pole* $n = (0, \ldots, 1)$ *and south pole* $s = (0, \ldots, -1)$ *respectively.*

- $\mathbb{T}^n := \mathbb{S}^1 \times \mathbb{S}^\times \cdots \times \mathbb{S}^1$ *a product of* n *one dimensional sphere* $\mathbb{S}^1$ *has a canonical smooth differentiable structure.*

Definition 1.7. *A continuous map* $f : M \mapsto N$ *between two differentiable manifolds is said to be* differentiable manifold of class C^∞ *iff* $x \circ f \circ y^{-1}$ *is* C^∞ *for each coordinate map* x *on* M *and* y *on* N.

Since the composition of two smooth maps is again smooth we now have a category of smooth manifolds and smooth maps. The ring of the $\mathbb{R}$-valued smooth function on a manifold M will be denoted by $\mathcal{F}(M)$.

Definition 1.8. *A* diffeomorphism *is a smooth map whose inverse is also smooth.*

A diffeomorphism is therefore an isomorphism in the category smooth manifolds and maps. In general it is not easy to describe whether two differentiable structures in a given topological manifold are diffeomorphic or not.

2 The Tangent Space

Let M be a smooth manifold of dimension n.

Definition 2.1. *Let $p \in M$. Two $\mathbb{R}$-valued functions f, g defined on an open set containing p are said to have the same* germ *at p if they agree on some neighbourhood of p. This defines an equivalence relation of smooth functions defined near p and the equivalence classes are called* germs. *We denote them by $\mathcal{F}_p$. We will denote the germ of f with $\bar{f}$.*

Remark 2.2. *$\mathcal{F}_p$ is an algebra over $\mathcal{R}$.*

Definition 2.3. *The* tangent space *T_pM of a smooth manifold M at a point p is defined to be the space of all derivations of the algebra $\mathcal{F}_p$, i.e. the space of all linear maps $v : \mathcal{F}_p \mapsto \mathbb{R}$ satisfying $v(\bar{f} \cdot \bar{g}) = \bar{f}(p)v(\bar{g}) + \bar{g}(p)v(\bar{f})$ for each $\bar{f}, \bar{g} \in \mathcal{F}_p$.*

Remark 2.4. *T_pM is a vector space over $\mathbb{R}$. Moreover for any $v \in T_pM$ and any constant germ c, $v(c) = 0$.*

For a coordinate system (U, x) around a point $p \in M$ with $x(p) = a$, we will denote by $e_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$ the tangent vector belonging to T_pM defined by: $e_i(\bar{f}) = \frac{\partial}{\partial x^i}|_{x=a}(f \circ \phi)$ where $\phi = x^{-1}, a = x(p)$ and $\bar{f}$ represents the germ of f at p. We then have $e_i(x^j) = \delta_i^j$ where x^j is the germ of the coordinate function x^j at p.

Proposition 2.5. *For any coordinate chart (U, x) around p, the vectors $\{e_i\}_{i=1,\ldots,p}$ form a base for T_pM.*

The proof of the proposition depends on the following lemma.

Lemma 2.6. *Let (U, x) be a coordinate system centred at p and let f be a smooth function defined near p. Then there exists n smooth functions $f_1, \ldots, f_n$ defined near p such that*

1. *$f_i(p) = e_i(\bar{f}_p)$ where $\bar{f}_p \in \mathcal{F}_p$ is the germ of f at p.*
2. *$f = f(p) + \Sigma_{i=1}^n x^i f_i$ in a neighbourhood of p.*

Proof. Let $F = f \circ \phi$ where $\phi = f^{-1}$. F is defined on a small ball B around 0 in $\mathbb{R}^n$. For any $a \in B$ we have $F(a) - F(0) = \int_0^1 \frac{d}{dt} F(ta^1, \ldots, ta^n)dt = \Sigma_{i=1}^n a^i \int_0^1 D_i F(ta^1, \ldots, ta^n)dt$ where D_iF is the i-th partial derivative of F. Now we set $F_i(a) = \int_0^1 D_i F(ta^1, \ldots, ta^n)dt$ and let $f_i = F_i \circ x$. This proves the lemma. □

Proof. (of Proposition 2.5) We first show that any $v \in T_pM$ can be represented as $v = \Sigma_{i=1}^n v(\bar{x}^i)e_i$ where $\bar{x}^i \in \mathcal{F}_p$ is the germ of x^i at p. If $x(p) \neq 0$ we change coordinates to

$y = x - x(p)$ which is centred at p. We now apply Lemma 2.5 and write any smooth function f defined near p as $f = f(p) + \Sigma_{i=1}^n x^i f_i$ with $f_i \in C^\infty$. Thus

$$v(\bar{f}) = v(\bar{f}_i(p)) + \Sigma v(\bar{y}^i)\bar{f}_i(p) + \Sigma \bar{y}^i(p) v(\bar{f}_i) =$$
$$0 + \Sigma v(\bar{x}^i - \bar{x}^i(p))\bar{f}_i(p) + 0 = \Sigma v(\bar{x}^i)\bar{f}_i(p) = \Sigma v(\bar{x}^i) e_i(\bar{f}),$$

since $\bar{f}_i(p) = \frac{\partial}{\partial y^i}\bar{f}_p = \frac{\partial}{\partial x^i}\bar{f}_p = \bar{e}_i \bar{f}_p$. If $w = \Sigma \lambda^i e_i = 0$, then $0 = w(\bar{x}^j) = \Sigma \lambda^i \delta_i^j = \lambda^j$ for all j. Hence, $\{e_i\}_{i=1,\ldots,n}$ are linearly independent and form a basis of T_pM. $\square$

Definition 2.7. *The* differential df_p at a point p *of a smooth map* $f : M \mapsto N$ *is defined to be the linear map* $df_p : T_pM \to T_qN$*, defined by* $df_p(v) = v \circ f^*$ *where* $q = f(p)$ *and* $f^* : \mathcal{F}_q \to \mathcal{F}_p$ *is defined by* $f^*(\bar{g}) = \overline{g \circ f}$ *for* $\bar{g} \in \mathcal{F}_q$.

Remark 2.8. *If* x *and* y *are local coordinates about* p *and* q*, then the matrix representing* df_p *with respect to the basis* $\{\frac{\partial}{\partial x^i}\}$ *and* $\{\frac{\partial}{\partial y^i}\}$ *is the* Jacobian Matrix *of the function* $F = y \circ f \circ x^{-1}$*, i.e.* $df_p(\frac{\partial}{\partial x^i}) = \Sigma \frac{\partial F^j}{\partial x^i}\frac{\partial}{\partial y^j}$

Remark 2.9. *If* $c : [a, b] \mapsto M$ *is a smooth curve on* M*,* $[a, b] \subset \mathbb{R}$*, then we denote by* $\dot{c}(t)$ *the tangent vector* $dc_t(\frac{\partial}{\partial t}) \in T_{c(t)M}$ *where* t *is the standard coordinate of* $\mathbb{R}$*. For a curve on* M *with* $c(0) = p$*, it is easily seen that* $\dot{c}(0)$ *is the tangent vector given by* $\mathcal{F}_p \to \mathbb{R}$*,* $\bar{f} \mapsto \frac{d}{dt}|_{t=0} f(c(t))$*. This leads to the following geometric interpretation of* T_pM*. Each* $v \in T_pM$ *is equal to some* $\dot{c}(0)$ *for some curve* $c(t)$ *with* $c(0) = p$ *and two curves* c_1 *and* c_2 *with* $c_1(0) = c_2(0) = p$ *define the same tangent vector, i.e.* $\dot{c}_1(0) = \dot{c}_2(0)$ *iff for any coordinate system* x *about* p*, we have* $\frac{d}{dt}|_{t=0} x(c_1(t)) = \frac{d}{dt}|_{t=0} x(c_2(t))$*. So we may think of a tangent vector* v_p *as an equivalence class of curves through* p*, which have the same "tangent" at* p*.*

Definition 2.10. *The* tangent bundle of a differentiable manifold M *is the disjoint union of all the tangent spaces of* M*. That is,* $TM := \bigcup_{p\in M} T_pM$*. There is a natural projection that maps each tangent space* T_pM *to the single point* p *defined by* $\pi` : TM \to M$ *where* $\pi : v \mapsto \pi(v) = p$ *if* $v \in T_pM$*.*

Remark 2.11. *If* U *is an open subset of* M*,* $\pi^{-1}(U) = TU$ *by definition of* π *since* $T_pU = T_pM$ *for* $p \in U$*.*

If (U, x) is a coordinate system for M, then we have the following *trivializing map* for TU: $(x, \frac{\partial}{\partial x}) : TU \to U \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ with $v_p \mapsto (p, v^1, \ldots, v^n) \mapsto (x(p), v^1, \ldots, v^n)$. Here $p = \pi(v)$ and $v = \Sigma v^i \frac{\partial}{\partial x^i}$, $v^i = v(\bar{x}^i)$. The collection of these charts

$$\{(\pi^{-1}(U), (x, \frac{\partial}{\partial x}) | (U, x) \text{ a coordinate system of } M)\}$$

cover TM and we define a C^∞ differentiable structure on TM by taking a maximal atlas compatible with this collection. The topology of TM is also generated by the sets TU. The transition function between two such charts $(x, \frac{\partial}{\partial x}) \mapsto (y, \frac{\partial}{\partial y})$ is given by

$$(y \circ x^{-1}, \frac{\partial y^i}{\partial x^j}) : x(U \cap V) \times \mathbb{R}^n \to y(U \cap V) \times \mathbb{R}^n$$

with $x(p, a) \mapsto (y(p), d(y \circ x^{-1})(a))$. Hence TM becomes a smooth manifold of dimension $2n$ and $\pi : TM \to M$ is a smooth map. In addition since TM is a disjoint union of vector bundles locally trivialized by the above maps $\pi^{-1} = TU \simeq U \times \mathbb{R}^n$, it is a vector bundle in the following sense.

Definition 2.12. *A* An n-dimensional real vector bundle *is a continuous surjective map* $\pi : E \to X$ *between two topological spaces with the following properties:*

1. *Each* fibre $E_x = \pi^{-1} \subset E$ *carries the structure of a vector space over* $\mathbb{R}$, *such that the vector space operations are continuous.*

2. $\forall x \in X$, $\exists$ *a neighbourhood* U *of* x *and a homeomorphism* ϕ

$$\begin{tikzcd} \pi^{-1}(U) \arrow[d, dashed, "\simeq"'] \arrow[dr, "\pi"] & \\ U \times \mathbb{R}^n \arrow[r, "pr_1"'] & U \end{tikzcd}$$

   such that the diagram is commutative and such that $\phi|_{E_x} : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^n$ *is an homeomorphism of vector spaces for each* $x \in U$

E *is called the* total space *and* X *the* base space *of the bundle. A map* ϕ *as in Proposition (??) is called a* bundle chart.

Definition 2.13. *A* section *of a vector bundle* $TM \to^\pi M$ *is a continuous map* $s : X \to E$ *such that* $\pi \circ s = id_X$

Definition 2.14. *The vector bundle* $TM \to^\pi M$ *is called the* tangent bundle. *If* $M \to^f N$ *is a smooth map, then its* differential df *is the map* $df : TM \to TN$, *with* $v_p \mapsto df_p(v_p) \in T_{f(p)N}$. df *is* C^∞ *and* π *linear on each fibre.*

3 Vector Fields and Their Flows

Definition 3.1. *A* smooth vector field *on a smooth manifold M is a smooth section $X : M \to TM$ of the tangent bundle*

A vector field X associates to each point $p \in M$ a vector $X_p \in T_pM$ in a smooth manner. If (U, x) is a coordinate system on M, then $X|_U$ can be written as $X = \Sigma_{i=1}^n X^i \frac{\partial}{\partial x^i}$ where $X^i = X(x^i)$ are smooth functions. Therefore locally with respect to a coordinate system a vector field is represented by a smooth vector valued function: $X : U \to \mathbb{R}^n$, $p \mapsto (X^1(p), \dots, X^n(p))$

Definition 3.2. *A curve $c : [a, b] \to M$ is called an* integral curve *of a vector field X on M if $\dot{c}(t) = X_{c(t)}$ for all $t \in (a, b)$.*

Let $(-\varepsilon, \varepsilon) \to M$ be an integral curve of the vector field X and let (U, x) be a coordinate system about $p = c(0)$. Then if we let $c^i = x^i \circ c$ and $F^i = X^i \circ x^{-1}$ where the X^i's are the component functions of X with respect to the base $\frac{\partial}{\partial x^i}$, i.e. $X = \Sigma_{i=1}^n X^i \frac{\partial}{\partial x^i}$ on U, we obtain the following system of ODEs for the c^i's

$$\frac{dc^i}{dt} = F^i(c^1(t), \dots, c^n(t)),$$

for small $t \in c^{-1}(U)$. Suppose now that $x(p) = 0$ (by a translation if necessary). Then the F^i's are smooth functions defined on an open ball around 0 and hence by the fundamental existence and uniqueness theorem for systems of ODEs, there exists a unique set of solutions $c_k^i(t)$ satisfying the initial condition $c_u^i(0) = u \in \mathbb{R}^n$ (and depending smoothly on the initial condition) for $|t| < \varepsilon$ and $|u| < a$ where ε and a are small positive numbers depending only on F. Set $\phi_t(q) := X^{-1}(c_u(t))$ where $u = x(q)$, $q \in x^{-1}(B_a(0)) \subset U$ and $|t| < \varepsilon$. If $|t| < \varepsilon$, $|s| < \varepsilon$ and $|s + t| < \varepsilon$, and both $|u| = |x(q)| < a$ and $|x(\phi_s(q))| < a$, then the functions $\gamma^i(t) = c_u^i(t + s)$ are solutions of the ODE system with initial conditions $\gamma^i(0) = c_u^i(s)$. Therefore by uniqueness $\gamma^i(t) = c_{\tilde{u}}^i$ where $\tilde{u} = (c_u^1(s), \dots, c_u^n(s))$. This proves $\phi_t(\phi_s(q)) = \phi_{t+s}(q)$. Since $\phi_0 = id$ and $\phi_t(\phi_{-t}(q)) = \phi_0(q) = q$, ϕ_t is a diffeomorphism of a small neighbourhood of p for t small enough.

Definition 3.3. *A* local 1-parameter group of local diffeomorphisms *or simply a* local flow *on a manifold M is a mapping $\phi : (-\varepsilon, \varepsilon) \times U \to M$ where U is an open neighbourhood in M with the following two properties:*

1. *$\forall t \in (-\varepsilon, \varepsilon)$, $\phi_t : pmapsto\phi(t, p)$ is a diffeomorphism of U onto its image $\phi_t(U) \subset M$.*

2. *if $t, s, t + s \in (-\varepsilon, \varepsilon)$ and if $p, \phi_s(p) \in U$, then $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.*

A local flow defines a vector field $X_p = \frac{d}{dt}|_{t=0}\phi_t(p)$. We can therefore paraphrase our above discussion about the existence of local integral curves of a vector field as follows.

Proposition 3.4. *Let X be a smooth vector field on M. Then $\forall p \in M$, $\exists$ a neighbourhood U of p, $\varepsilon > 0$ and a local flow $\phi : (-\varepsilon, \varepsilon) \times U \to M$, of X.*

Definition 3.5. *A* global flow *on M is a map $\phi : \mathbb{R} \times M \to M$, satisfying*

1. *$\forall t \in \mathbb{R}$, $\phi_t : p \mapsto \phi(t, p)$ is a diffeomorphism;*
2. *$\forall t, s \in \mathbb{R}\phi_{t+s} = \phi_t \circ \phi_s$*

Definition 3.6. *If X generates a global flow, then X is said to be complete.*

Proposition 3.7. *On a compact manifold every vector field is complete.*

Proof. By Proposition 3.4 and compactness, there exists a finite set of local flows $\phi^i : (-\varepsilon_i, \varepsilon_i) \times U_i \to M$, for $i = 1, \dots, N$ with $\bigcup U_i = M$. Set $\varepsilon = \min_{i=1,\dots,N}\{\varepsilon_i\}$. Then we have a flow $\phi : (-\varepsilon, \varepsilon) \times M \to M$ and hence a global flow $\phi : \mathbb{R} \times M \to M$, by iterating the flow. □

We denote by $\mathcal{F}(M)$, the algebra of all smooth real valued functions on M and by $\mathcal{X}$ the vector space of all vector fields on M.

Definition 3.8. *For $X, Y \in \mathcal{X}(M)$ define $[X, Y] \in \mathcal{X}(M)$ by setting $[X, Y]_p(\bar{f}) := X_p(Y(f)) - Y_p(X(f))$ for $f \in \mathcal{F}(M)$. $[X, Y]$ is called the* Lie Bracket *of X and Y.*

Proposition 3.9.

1. *$[X, Y]$ is indeed a smooth vector field;*
2. *If $f, g \in \mathcal{F}(M)$, then $[fX, gY] = fg[X, Y] - f\,X(g)\,Y - g\,Y(f)\,X$;*
3. *$[X, Y] = -[Y, X]$;* skew-symmetric
4. *$\forall X, Y, Z \in \mathcal{X}(M)$ we have $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$* Jacobi Identity.

Proof. A soft exercise for the reader. □

Definition 3.10. *A vector space with a skew-symmetric bilinear operator $[,]$ satisfying the Jacobu identity is called a* Lie-Algebra

From 2. we get the following local expression for $[X, Y]$. If $X = \Sigma X^i \frac{\partial}{\partial x^i}$ and $Y = \Sigma Y^j \frac{\partial}{\partial y^j}$, then

$$[X, Y] = \Sigma_{i,j}\Big(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\Big)\frac{\partial}{\partial x^j}.$$

Let X be a vector field of f with associated local flow ϕ_t.

Definition 3.11. *For $f \in \mathcal{F}(M)$, $\mathcal{L}_X f := \lim_{t\to 0} \frac{f\circ\phi_t - f}{t}$ is called the* Lie derivative of f *w.r.t. X.*

Proposition 3.12. *$\mathcal{L}_X f = X(f) \in \mathcal{F}(M)$.*

Definition 3.13. *For $Y \in \mathcal{X}(M)$, $\mathcal{L}_X Y := \lim_{t\to 0} \frac{d\phi_{-t}(Y\circ\phi_t)\circ\phi_t - Y}{t}$ or more precisely*

$$(\mathcal{L}_X Y)_p := \lim_{t\to 0} \frac{d\phi_{-t}(Y\circ\phi_t(p))\circ\phi_t - Y_p}{t}$$

$\mathcal{L}_X Y$ is called the Lie derivative of Y w.r.t X*.*

Proposition 3.14. *$\mathcal{L}_X Y = [X,Y] \in \mathcal{X}(M)$*

Proof. Let $f \in \mathcal{F}(M)$, $p \in M$ and define $F(t,r,s) = f(\phi_s(\psi_r(\phi_t(p))))$ for small t,r,s such that the flows ϕ and ψ associated to the vector fields X, Y respectively are defined. Let $G(t,r) = F(t, r-t)$. Then $G(t,r) = (f\circ\phi_{-t})(\psi_r(\phi_t(p))) = g_{-t}(\psi_r(p_t))$ where $f\circ\phi_{-t} = g_{-t}$, $p_t = \phi_t(p)$. For fixed t,

$$D_2G(t,0) = \frac{d}{dr}|_{r=0} g_{-t}(\psi_r(p_t)) = Y_{p_t}(g_{-t}) = Y_{\phi_t(p)}(f\circ\phi_{-t}) = (d\phi_{-t}(Y_{\phi_t(p)}))\cdot f$$

$$D_1D_2G(0,0) = \frac{d}{dt}|_{t=0}(d\phi_{-t}(Y_{\phi_t(p)}))\cdot f = \lim_{t\to 0}\frac{d\phi_{-t}(Y\circ\phi_t(p))\circ\phi_t - d\phi_0 Y_p}{t} f = (\mathcal{L}_X Y)_p \cdot f$$

by definition of $\mathcal{L}_X Y$ since $\phi_0 = id$.
Now $D_2F(t,0,0) = \frac{d}{dr}|_{r=0}(f\circ\psi_r)(\phi_t(p)) = Y_{\phi_t(p)} f$ and $D_1D_2F(0,0,0) = \frac{d}{dt}|_{t=0}(Y_{\phi_t(p)}f) = \frac{d}{dt}|_{t=0}(Yf(\phi_t(p))) = X_p(Yf)$. Similarly $D_2D_3F(0,0,0) = Y_p(Xf)$. On the other hand, since $G(t,r) = F(t,r,-t)$ we have by the chain rule $D_1G(0,0) = D_1F(0,0,0) - D_3F(0,0,0)$. Therefore $(\mathcal{L}_X Y)_p f = D_1D_2G(0,0) = D_2D_1G(0,0) = D_2D_1F(0,0,0) - D_2D_3F(0,0,0) = D_1D_2F(0,0,0) - D_2D_3F(0,0,0) = X_p(Yf) - Y_p(Xf) = [X,Y]_p f$. And so the thesis $\mathcal{L}_X Y = [X,Y]$. $\square$

Definition 3.15. *Let ψ be a diffeomorphism of M and $X \in \mathcal{X}(M)$. Then we set $d\psi(X)$ to be the vector field $d\psi(X)_p = d\psi_q(X_q)$ where $p = \psi(q)$, in other words $d\psi(X)\circ\psi = d\psi\circ X$.*

Proposition 3.16. *$d\psi([X,Y]) = [d\psi(X), d\psi(Y)]$.*

Proof. We have to show that the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "d\psi"] & TM \\ M \arrow[u, "{[X,Y]}"'] \arrow[r, "\psi"'] & M \arrow[u, "{[d\psi(X), d\psi(Y)]}"'] \end{tikzcd}$$

Let $q \in M$, $p = \psi(q)$ and $f \in \mathcal{F}(M)$ We have:

$$d\psi([X,Y]_q)f = [X,Y]_q(f \circ \psi) = X_q(Y(f \circ \psi)) - Y_q(X(f \circ \psi)) = X_q((d\psi \circ Y)f) - Y_q((d\psi \circ X)f)$$

$$= X_q((d\psi(Y)f) \circ \psi) - Y_q((d\psi(X)f) \circ \psi) = d\psi(X)_p((d\psi(Y)f) - d\psi(Y)_p((d\psi(X)f) = [d\psi(X), d\psi(Y)]_p f$$

$\square$

If ϕ_t is the local flow of X and ψ is any diffeomorphism then $d\psi X)$ generates the flow $\psi \circ \phi_t \circ \psi^{-1}$. Because if $p \in M$ then

$$\frac{d}{dt|_{t=0}}(\psi \circ (\phi_t(\psi^{-1}(p)))) = d\psi|_{t=0}(\phi_t(q)) = d\psi_q(X_q) = d\psi(X)_p.$$

If $d\psi(X) = X$, i.e. if X is invariant under ψ, then ψ commutes with the flow ϕ_t, i.e. $\psi \circ \phi_t = \phi_t \circ \psi$ for all t in a small interval $(-\varepsilon, \varepsilon)$, then $d\psi(X) = X$ (just differentiate $\psi \circ \phi_t \circ \psi^{-1}$ at $t = 0$). This implies that $d\phi_t(X) = X$ for all t.

Proposition 3.17. *Suppose $X, Y \in \mathcal{X}(M)$ generate local flows ϕ_t and ψ_s respectively. Then $\phi_t \circ \psi_s = \psi_s \circ \phi_t$ for every s, t (small enough) iff $[X,Y] = 0$.*

Proof. If $\phi_t \circ \psi_s = \psi_s \circ \phi_t$ then $d\psi(X) = X$ for all s and hence $[X,Y] = \mathcal{L}_X Y = \lim_{s \to 0} \frac{d\psi_s(X) - X}{s} = 0$.
Conversely if $[X,Y] = 0$, then

$$d\phi_t([X,Y]) = [d\phi_t(X), d\phi_t(Y)] = [X, d\phi_t(Y)] = \mathcal{L}(d\phi_t(Y)) = 0$$

by previous propositions since $d\phi_t(X) = X$. Now

$$\begin{aligned} \mathcal{L}(d\phi_t(Y)) \qquad &= \lim_{s \to 0} \frac{d\phi_s(d\phi_t(Y)) - d\phi_t(Y)}{s} \\ &= \lim_{s \to 0} \frac{d\phi_{t+s}(Y) - d\phi_t(Y)}{s} = \frac{d}{dt}(d\phi_t(Y)). \end{aligned}$$

Therefore $d\phi_t(Y) = d\phi_0(Y) = Y$ for all t and so ϕ_t commutes with the flow ψ_t of Y.

$\square$