Math 3GP3
Assignment #3

Due: Tuesday, October 22nd, 2013

1. Express the Schwarzschild metric $ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ in terms of the Eddington-Finkelstein coordinates (v, r, θ, ϕ), where the new null coordinate v is given by $v = t + r + 2m\log(r - 2m)$

2. Show that the metric $ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2$ can be written as

$$ds^2 = \frac{32m^3}{r}\, e^{-\frac{r}{2m}} \left(-\,dT^2 + \,dX^2\right)$$

where the two coordinate systems are related by:

$$X^2 - T^2 = \left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}}\,, \qquad \frac{T}{X} = \tanh\left(\frac{t}{4m}\right)$$

3. The geometry of a spherically symmetric black hole of mass m and charge q, (in units $c = G = 1$) is described by the Reissner-Nordström metric:

$$ds^2 = -A(r)\, dt^2 + A(r)^{-1} dr^2 + r^2 d\sigma^2$$

where

$$A(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$$

Show that the equations of motion of a freely-falling (neutral) particle (i.e. a time-like geodesic) are given by:

$$A(r)\frac{dt}{d\tau} = E\,, \qquad r^2\frac{d\phi}{d\tau} = L\,, \qquad \left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}} = E^2$$

where τ is the proper time of the particle, E, L are constants (the particle's energy and angular momentum per unit mass) and the effective potential is:

$$V_{\text{eff}} = A(r)\left(1 + \frac{L^2}{r^2}\right)$$

4. Compute the Ricci tensor of the 3−dimensional metric:

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2 d\theta^2$$

where $\theta \in S^1$

5. The metric $ds^2 = -dt^2 + dz^2 + dr^2 + r^2(1-4\mu)^2 d\phi^2$ represents a simple model of a straight infinite cosmic string lying along the z-axis, of mass μ per unit length. Show that in the (r, ϕ)-plane, the metric is given by the surface of a cone of semi-angle α, with $\sin\alpha = 1 - 4\mu$.
(*This causes a splitting of geodesics and can perhaps be observed?*)

6. (*bonus question*)

(i) What is a *Killing* vector field on a manifold M with metric g?

(ii) Show that if K is a Killing field, then $\nabla_a K_b + \nabla_b K_a = 0$

(iii) Show that if K is a Killing vector field and $c(t)$ a geodesic, then $g(K, \dot{c})$ is constant along the geodesic.

(iv) Show that the Lie bracket of two Killing vector fields is a Killing field

(v) Find all Killing vector fields of flat Minkowski space-time.

(vi) Show that if K is a Killing field, then $K_{c\,;\,ab} = R_{bcad}K^d$

(vii) What is a Jacobi vector field along a geodesic? Show that a Killing vector field restricted to a geodesic is a Jacobi field along that geodesic.