Math 4B03
Assignment #1

DUE: MONDAY, SEPTEMBER 22ND, 2014

1. Derive the classical formulas: $curl(grad\, f) = 0$ and $div(curl\, F) = 0$, where f is a smooth function and F is a smooth vector field in $\mathbb{R}^3$, from the fundamental property $d^2 = 0$ of the exterior derivative d.

2. Let ω be the 2-form in $\mathbb{R}^{2n}$ defined by: $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \ldots + dx^{2n-1} \wedge dx^{2n}$

(a) Compute the exterior product $\omega^n = \omega \wedge \cdots \wedge \omega$ of n copies of ω.

(b) Let $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a smooth map. Compute $\phi^* \omega^n$ (in terms of $d\phi$).

3. Let A be a real $n \times n$ matrix and let X be the linear vector field on $\mathbb{R}^n$ defined by: $X(p) = A\, p$ for $p \in \mathbb{R}^n$, and let ν be the volume form: $\nu = dx^1 \wedge \ldots \wedge dx^n$. Compute the Lie derivative $\mathcal{L}_X\, \nu$.

4. Show that $\mathbb{C}P^1$the set of all complex lines through the origin in $\mathbb{C}^2$ is a differential manifold diffeomorphic to S^2.

5. (i) Show that the map

$$(x, y, z) \mapsto (x^2 - y^2, xy, yz, zx)$$

defines an imbedding of the real projective plane $\mathbb{R}P^2$ into $\mathbb{R}^4$.
(ii) Find an immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$.

6. (*bonus question*) Show that $SU(n)$, the set of all $n \times n$ unitary matrices with determinant $= 1$, form a submanifold in $\mathbb{C}^{n^2}$. What is the dimension of $SU(n)$?