Math 4B03
Assignment #4

DUE: WEDNESDAY, NOVEMBER 5TH, 2014

1. What is the "Five Lemma". State it and prove it.

2. Let A be the vector field on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ defined by by $A(\vec{z}) = i\vec{z}$.
Find a function $f : \mathbb{C}^n \to \mathbb{R}$, such that $\omega(A, V) = df(V)$ for all vector fields V, where $\omega = \sum_{k=1}^{n} dx^k \wedge dy^k$ ($z^k = x^k + iy^k$) is the standard symplectic form.

3.

(i) Compute $L(r)$ = length of a circle of radius r in the hyperbolic space

(ii) Compute $A(r)$ = the area of a disk of radius r in hyperbolic space

(iii) What happens to $\frac{L(r)^2}{A(r)}$ as $r \to \infty$. Compare with Euclidean space.

4. Find an identification of Minkowski space-time with the vector space of 2×2 complex Hermitian matrices $\{X \mid X^* = \bar{X}^t = X\}$ such that negative the determinant of a Hermitian matrix corresponds to the squared length of a vector in space-time.
Show that the action of $g \in SL(2;\mathbb{C})$ = complex 2×2 matrices of determinant 1, on Hermitian matrices given by $X \mapsto g\,X\,g^*$ preserves the Lorentzian metric. Find the Lorentzian transformations corresponding to the following elements of $SL(2;\mathbb{C})$

$$\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}, \qquad \begin{pmatrix} e^{\frac{\beta}{2}} & 0 \\ 0 & e^{-\frac{\beta}{2}} \end{pmatrix}, \qquad \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

5. (*bonus question*) Solve the eigenvalue problem:

$$\Delta f = \lambda f$$

where f is a function, on the two dimensional round sphere of curvature 1, i.e., find all the eigenvalues and the eigenfunctions (*spherical harmonics!*)
What about the Laplacian on 1-forms?