Math 2X03 Supplement

Maung Min-Oo

November 22, 2009

1 Kepler's Laws

First Law
The orbit of a Planet is an ellipse with the Sun at one focus.

Second Law
The line joining the Sun to the Planet sweeps out equal area in equal times.

Third Law
The square of the period of revolution is proportional to the cube of the length of the major axis.

1.1 Derivation (by Newton)

Let the Sun of mass M be at the origin O and let the Planet P of mass $m = 1$, be at the point $\vec{r}(t)$ at time t. The velocity of P is $\vec{v}(t) = \vec{r}\,'(t)$ and its acceleration is $\vec{a}(t) = \vec{r}\,''(t)$. By Newton's Laws: $\vec{F} = m\vec{a}$, and the gravitational force acting on P is $\vec{F} = -\frac{GM}{r^3}\,\vec{r} = -\vec{\nabla}V$, where G is the gravitational constant, $r = \|\vec{r}\|$ and $V(r) = -\frac{GM}{r}$ is the potential energy. To determine the orbit of the planet all we have to do is solve the ODE:

$$\vec{r}\,''(t) = -\frac{GM}{r^3}\,\vec{r}(t)$$

We derive first the two obvious conserved quantities (constants of motion).

1. The energy

$$E = \frac{1}{2}\|\vec{v}\|^2 + V(r)$$

The force is conservative and so: $E' = \vec{a}\cdot\vec{v} + \vec{\nabla}V\cdot\vec{v} = 0$.

2. The angular momentum

$$\vec{h} = \vec{r} \times \vec{v}$$

The force is central and so: $\vec{h}' = \vec{r} \times \vec{a} = \vec{0}$.

Let $\vec{u} = \frac{\vec{r}}{r}$ be the unit vector in the direction of $\vec{r}$. Then $\vec{v} = r'\vec{u} + r\vec{u}'$ and so $\vec{h} = r^2\vec{u} \times \vec{u}'$. This proves Kepler's Second Law, since $\frac{1}{2}r^2\|\vec{u} \times \vec{u}'\|$ is the rate at which P sweeps out area. In polar coordinates this would be $\frac{1}{2}r^2\frac{d\theta}{dt}$.

From now on we will assume that $\vec{h} \neq \vec{0}$. Orbits with zero angular momentum are radial (they will just fall into the sun or escape radially!). Then $\vec{r}$ lies in the fixed plane perpendicular to $\vec{h}$ and hence the orbit is planar.

Since $\vec{u}$ is a unit vector, $\vec{u} \cdot \vec{u}' = 0$ and $\vec{u} \times (\vec{u} \times \vec{u}') = -\vec{u}'$. Therefore $\vec{a} \times \vec{h} = -\frac{GM}{r^2}\vec{u} \times (r^2(\vec{u} \times \vec{u}')) = GM\vec{u}'$. This means that $\frac{d}{dt}((\vec{v} \times \vec{h}) - GM\,\vec{u}) = \vec{0}$. By integrating this we get a third conserved quantity:

3. The Laplace-Runge-Lenz vector:

$$\vec{C} = (\vec{v} \times \vec{h}) - GM\,\vec{u}$$

The conservation of this vector is not totally trivial, since it is not a consequence of an obvious symmetry. It depends crucially on the fact that the gravitational force obeys the inverse square law (and not just any central force law!). It will imply that the orbits are conic sections. We will concentrate on elliptical orbits with negative energy $E < 0$. These are bound states. Orbits with $E = 0$ are parabolas and orbits with $E > 0$ are hyperbolas (objects that don't get captured by the Sun's pull).To prove the orbit is an ellipse we claim that the point $A = \frac{\vec{C}}{E}$ is the other focus by showing that $\|\vec{OP}\| + \|\vec{AP}\|$ is a constant of motion. (This is the classical definition of an ellipse!). Now let $Q = -\frac{GM}{E}\vec{u}$. Then Q moves around a circle of fixed radius around the Sun. We will show that $\|\vec{AP}\| = \|\vec{PQ}\|$, which will imply that $\|\vec{OP}\| + \|\vec{AP}\| = \|\vec{OP}\| + \|\vec{PQ}\| = -\frac{GM}{E}$, a constant of motion.

First we note that $\vec{QA} = \frac{1}{E}(GM\vec{u} + \vec{C}) = \frac{1}{E}(\vec{v} \times \vec{h})$ and hence $\vec{QA} \cdot \vec{v} = 0$.

Let $L^2 = \|\vec{h}\|^2 = (\vec{r} \times \vec{v}) \cdot \vec{h} = (\vec{v} \times \vec{h}) \cdot \vec{r}$. Then $\|\vec{v} \times \vec{h}\|^2 = \|\vec{v}\|^2\|\vec{h}\|^2 = 2(E + \frac{GM}{r})L^2$ and so $\vec{QA} \cdot (\vec{v} \times \vec{h}) = 2L^2(1 + \frac{GM}{Er})$.

On the other hand: $\vec{QP} \cdot (\vec{v} \times \vec{h}) = (1 + \frac{GM}{Er})\vec{r} \cdot (\vec{v} \times \vec{h}) = L^2(1 + \frac{GM}{Er})$, which is exactly $\frac{1}{2}$ of $\vec{QA} \cdot (\vec{v} \times \vec{h})$. This proves that $\|\vec{AP}\| = \|\vec{PQ}\|$ (draw a picture!) and hence the orbit is an ellipse, which is Kepler's Second Law.

To prove the Third Law, we first compute: $\|\vec{C}\|^2 = \|\vec{v} \times \vec{h} - \frac{GM}{r}\vec{r}\|^2 = \|\vec{v} \times \vec{h}\|^2 - 2\frac{GM}{r}(\vec{v} \times \vec{h}) \cdot \vec{r} + G^2M^2 = 2EL^2 + G^2M^2$. If a is the semi-major axis then $\|\vec{OP}\| + \|\vec{AP}\| = 2a = -\frac{GM}{E}$ (when P is at the perihelion) and if b is the semi-minor axis, then $b^2 = a^2 - \|\frac{\vec{C}}{2E}\|^2 = -\frac{L^2}{2E}$ (by Pythagoras!) and so $\frac{b^2}{a} = \frac{L^2}{GM}$. The area of the ellipse is πab and the rate which the Planet sweeps area is $\frac{1}{2}\|\vec{h}\| = \frac{L}{2}$, so the period is $T = \frac{2\pi ab}{L}$. Therefore $T^2 = \frac{4\pi^2 a^2 b^2}{L^2} = \frac{4\pi^2}{GM}a^3$, which is Kepler's Third Law:

$$T^2 = \frac{4\pi^2}{GM}a^3$$

QED (lol)