A Short Introduction to Differential Calculus on Manifolds

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October 3, 2014

Abstract

These lecture notes are part (let's say 40%) of the basic material for the course Math 4B03 (Calculus on Differential Manifolds) that I am teaching during the Fall Term 2014 at McMaster University. I am just writing down some preliminary definitions, notations and brief proofs here. I do make a point of emphasising the Lie derivative and giving rather slick unified proofs of both the Poincare Lemma and Stokes' Theorem by integrating Cartan's Formula for the Lie derivative which expresses the invariance of deRham cohomology under infinitesimal diffeomorphisms. The notes are unfortunately still rather stiff, pedantic and formal like most mathematical expositions. The more interesting things (including my jokes) will be done live during the lectures, which I actually shouldn't prepare, since otherwise I would just be copying stuff from my notes or whatever I can find in textbooks and the internet. It's better to be bored than be boring!

1 Differentiable Manifolds

Definition 1.1. *A* topological manifold of dimension *n is a Hausdorff, second countable topological space M for which each point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. If x is such a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^n$, we call x a* coordinate map *and the pair (U, x) is called a* coordinate system *or* chart*. If $p \in U$ and $x(p) = 0$, then the coordinate system is said to be centred at p.*

(if you don't know the words Hausdorff or second countability, don't worry about it too much, it's just some legalistic formality to avoid pathological stuff)

Definition 1.2. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a map. We say that f is* differentiable of class C^∞ *(or simply f is C^∞) if all of the component functions f^i have partial derivatives of all orders.*

Definition 1.3. *A* differentiable structure $\mathcal{A}$ of class C^∞ *on a topological manifold* M *is a collection of coordinate systems* $\{(U_\alpha, x_\alpha)|\alpha \in \mathcal{A}\}$ *satisfying the following three properties:*

- $\bigcup_{\alpha\in\mathcal{A}} = M$
- $x_\alpha \circ x_\beta^{-1}$ *is* C^∞ *wherever it is defined, i.e. for all* $\alpha, \beta \in \mathcal{A}$ *such that* $U_\alpha \cap U_\beta \neq \varnothing$.
- *The collection* $\mathcal{A}$ *is maximal with respect to the previous property, i.e. if* (U, x) *is a coordinate system such that* $x \circ x_\alpha^{-1}$ *and* $x_\alpha^{-1} \circ x$ *are* C^∞ *for all* $\alpha \in \mathcal{A}$ *such that* $U \cap U_\beta \neq \varnothing$, *then* $(U, x) \in \mathcal{A}$.

Remark 1.4. *If* $\mathcal{A}_0$ *is any collection of coordinate systems (called an* atlas*) satisfying the first two properties, then there is a unique differentiable structure* $\mathcal{A}$ *containing* $\mathcal{A}_0$*. Namely,* $\mathcal{A} = \{(U, x)|\ x \circ x_\alpha^{-1}$ *and* $x_\alpha^{-1} \circ x$ *are* C^∞ *for all* $\alpha \in \mathcal{A}_0\}$

Definition 1.5. *An* n-dimensional differentiable manifold of class C^∞ *(or simply a* smooth manifold*) is a pair* $(M, \mathcal{A})$ *consisting of an* n*-dimensional topological manifold* M *together with a differentiable structure* $\mathcal{A}$ *of class* C^∞ *for* M*.*

Example 1.6.

- *The standard differentiable structure on* $\mathbb{R}^n$ *is defined to be the maximal collection containing the single coordinate chart* $id : \mathbb{R}^n \to \mathbb{R}^n$.
- *An open subset* U *of a smooth manifold* M *inherits a canonical smooth structure defined by*

  $\mathcal{A} := \{(U \cap U_\alpha, x_\alpha|_{U\cap U_\alpha}) \,|\, (U, x_\alpha) \in \mathcal{A}_M\}$.
- $GL(n; \mathbb{R}) := \{A|\ A\, is\ a\ matrix\ n \times n, det(A) \neq 0\} \subset \mathbb{R}^{n^2}$ *is an open subset of* $\mathbb{R}^{n^2}$ *and hence inherits a smooth manifolds structure*
- *If* $(M_1, \mathcal{A}_1)$ *and* $(M_2, \mathcal{A}_2)$ *are smooth manifolds of dimension* n_1, n_2*, then the product* $M_1 \times M_2$ *inherits a smooth structure* $\mathcal{A} = \{(U_\alpha \times V_\beta, x_\alpha \times y_\beta : U_\alpha \times V_\beta \mapsto \mathbb{R}^{n_1+n_2})|(U_\alpha, x_\alpha) \in \mathcal{A}_1, (V_\beta, y_\beta) \in \mathcal{A}_2\}$
- *The* n*-sphere* $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1}||x|^2 = 1\}$ *is a smooth dimensional manifold by taking* $\mathcal{A}$ *to be the maximal atlas containing* $\{(\mathbb{S}^n \setminus \{n\}, sp_n), (\mathbb{S}^s \setminus \{n\}, sp_s)\}$ *where* sp_n *and* sp_s *are stereographic projections from the north pole* $n = (0, \ldots, 1)$ *and south pole* $s = (0, \ldots, -1)$ *respectively.*
- $\mathbb{T}^n := \mathbb{S}^1 \times \mathbb{S}^\times \cdots \times \mathbb{S}^1$ *a product of* n *one dimensional sphere* $\mathbb{S}^1$ *has a canonical smooth differentiable structure.*
- $\mathbb{R}P^n$ *the real projective space of all lines through the origin in* $\mathbb{R}^{n+1}$

- $\mathbb{C}P^n$ *the complex projective space of all complex lines through the origin in* $\mathbb{C}^{n+1}$
- $\mathbb{H}P^n$ *the complex projective space of all quaternionic lines through the origin in* $\mathbb{H}^{n+1}$
- $Gr_{\mathbb{R}}(k, N)$ *the Grassmannian of all* $k-$*dimensional linear subspaces of* $\mathbb{R}^N$
- $Gr_{\mathbb{C}}(k, N)$ *the Grassmannian of all* $k-$*dimensional linear subspaces of* $\mathbb{C}^N$
- $SO(n)$ *the group of all* $n \times n$ *orthogonal matrices of determinant* $= 1$.
- $SU(n)$ *the group of all* $n \times n$ *unitary matrices of determinant* $= 1$.
- *level surfaces* $F^{-1}(c)$ *of a differentiable map* $F : \mathbb{R}^m \to \mathbb{R}^n \quad m \geq n$, *provided* c *is a regular value.*
- *level surfaces* $P^{-1}(c)$ *of a polynomial map* $P : \mathbb{C}^m \to \mathbb{C}^n \quad m \geq n$, *provided* c *is a regular value.*

et cetera et cetera!

Definition 1.7. *A continuous map* $f : M \to N$ *between two differentiable manifolds is said to be* differentiable manifold of class C^∞ *iff* $x \circ f \circ y^{-1}$ *is* C^∞ *(whenever it is defined!) for each coordinate map* x *on* M *and* y *on* N.

Since the composition of two smooth maps is again smooth we now have a category of smooth manifolds and smooth maps. The ring of the $\mathbb{R}$-valued smooth function on a manifold M will be denoted by $\mathcal{F}(M)$.

Definition 1.8. *A* diffeomorphism *is a smooth map whose inverse is also smooth.*

A diffeomorphism is therefore an isomorphism in the category smooth manifolds and maps. In general it is not easy to describe whether two differentiable structures on a given topological manifold are diffeomorphic or not.

2 The Tangent Space

Let M be a smooth manifold of dimension n.

Definition 2.1. *Let* $p \in M$. *Two* $\mathbb{R}$*-valued functions* f, g *defined on an open set containing* p *are said to have the same* germ *at* p *if they agree on some neighbourhood of* p. *This defines an equivalence relation of smooth functions defined near* p *and the equivalence classes are called* germs. *We denote them by* $\mathcal{F}_p$. *We will denote the germ of* f *with* $\bar{f}$.

Remark 2.2. $\mathcal{F}_p$ *is an algebra over* $\mathcal{R}$.

Definition 2.3. *The* tangent space T_pM *of a smooth manifold* M *at a point* p *is defined to be the space of all derivations of the algebra* $\mathcal{F}_p$*, i.e. the space of all linear maps* $v : \mathcal{F}_p \mapsto \mathbb{R}$ *satisfying* $v(\bar{f} \cdot \bar{g}) = \bar{f}(p)v(\bar{g}) + \bar{g}(p)v(\bar{f})$ *for each* $\bar{f}, \bar{g} \in \mathcal{F}_p$*.*

Remark 2.4. T_pM *is a vector space over* $\mathbb{R}$*. Moreover for any* $v \in T_pM$ *and any constant germ* c*,* $v(c) = 0$*.*

For a coordinate system (U, x) around a point $p \in M$ with $x(p) = a$, we will denote by $e_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$ the tangent vector belonging to T_pM defined by: $e_i(\bar{f}) = \frac{\partial}{\partial x^i}|_{x=a}(f \circ \phi)$ where $\phi = x^{-1}, a = x(p)$ and $\bar{f}$ represents the germ of f at p. We then have $e_i(x^j) = \delta_i^j$ where x^j is the germ of the coordinate function x^j at p.

Proposition 2.5. *For any coordinate chart* (U, x) *around* p*, the vectors* $\{e_i\}_{i=1,\ldots,p}$ *form a base for* T_pM*.*

The proof of the proposition depends on the following lemma.

Lemma 2.6. *Let* (U, x) *be a coordinate system centred at* p *and let* f *be a smooth function defined near* p*. Then there exists* n *smooth functions* $f_1, \ldots, f_n$ *defined near* p *such that*

1. $f_i(p) = e_i(\bar{f}_p)$ *where* $\bar{f}_p \in \mathcal{F}_p$ *is the germ of* f *at* p*.*
2. $f = f(p) + \Sigma_{i=1}^n x^i f_i$ *in a neighbourhood of* p*.*

Proof. Let $F = f \circ \phi$ where $\phi = f^{-1}$. F is defined on a small ball B around 0 in $\mathbb{R}^n$. For any $a \in B$ we have $F(a) - F(0) = \int_0^1 \frac{d}{dt} F(ta^1, \ldots, ta^n)dt = \Sigma_{i=1}^n a^i \int_0^1 D_i F(ta^1, \ldots, ta^n)dt$ where D_iF is the i-th partial derivative of F. Now we set $F_i(a) = \int_0^1 D_i F(ta^1, \ldots, ta^n)dt$ and let $f_i = F_i \circ x$. This proves the lemma. □

Proof. (of Proposition 2.5) We first show that any $v \in T_pM$ can be represented as $v = \Sigma_{i=1}^n v(\bar{x}^i)e_i$ where $\bar{x}^i \in \mathcal{F}_p$ is the germ of x^i at p. If $x(p) \neq 0$ we change coordinates to $y = x - x(p)$ which is centred at p. We now apply Lemma 2.5 and write any smooth function f defined near p as $f = f(p) + \Sigma_{i=1}^n y^i f_i$ with $f_i \in C^\infty$. Thus

$$v(\bar{f}) = v(\bar{f}(p)) + \Sigma v(\bar{y}^i)\bar{f}_i(p) + \Sigma \bar{y}^i(p)v(\bar{f}_i) =$$
$$0 + \Sigma v(\bar{x}^i - \bar{x}^i(p))\bar{f}_i(p) + 0 = \Sigma v(\bar{x}^i)\bar{f}_i(p) = \Sigma v(\bar{x}^i)e_i(\bar{f}),$$

since $\bar{f}_i(p) = \frac{\partial}{\partial y^i}\bar{f}_p = \frac{\partial}{\partial x^i}\bar{f}_p = \bar{e}_i\bar{f}_p$. If $w = \Sigma\lambda^i e_i = 0$, then $0 = w(\bar{x}^j) = \Sigma\lambda^i\delta_i^j = \lambda^j$ for all j. Hence, $\{e_i\}_{i=1,\ldots,n}$ are linearly independent and form a basis of T_pM. □

Definition 2.7. *The* differential df_p at a point p *of a smooth map* $f : M \to N$ *is defined to be the linear map* $df_p : T_pM \to T_qN$*, defined by* $df_p(v) = v \circ f^*$ *where* $q = f(p)$ *and* $f^* : \mathcal{F}_q \to \mathcal{F}_p$ *is defined by* $f^*(\bar{g}) = \overline{g \circ f}$ *for* $\bar{g} \in \mathcal{F}_q$*.*

Remark 2.8. *If x and y are local coordinates about p and q, then the matrix representing df_p with respect to the basis $\{\frac{\partial}{\partial x^i}\}$ and $\{\frac{\partial}{\partial y^i}\}$ is the* Jacobian Matrix *of the function* $F = y \circ f \circ x^{-1}$*, i.e.* $df_p(\frac{\partial}{\partial x^i}) = \Sigma \frac{\partial F^j}{\partial x^i} \frac{\partial}{\partial y^j}$

Remark 2.9. *If $c : [a, b] \mapsto M$ is a smooth curve on M, $[a, b] \subset \mathbb{R}$, then we denote by $\dot{c}(t)$ the tangent vector $dc_t(\frac{\partial}{\partial t}) \in T_{c(t)M}$ where t is the standard coordinate of $\mathbb{R}$. For a curve on M with $c(0) = p$, it is easily seen that $\dot{c}(0)$ is the tangent vector given by $\mathcal{F}_p \to \mathbb{R}$, $\bar{f} \mapsto \frac{d}{dt}|_{t=0} f(c(t))$. This leads to the following geometric interpretation of T_pM. Each $v \in T_pM$ is equal to some $\dot{c}(0)$ for some curve $c(t)$ with $c(0) = p$ and two curves c_1 and c_2 with $c_1(0) = c_2(0) = p$ define the same tangent vector, i.e. $\dot{c}_1(0) = \dot{c}_2(0)$ iff for any coordinate system x about p, we have $\frac{d}{dt}|_{t=0} x(c_1(t)) = \frac{d}{dt}|_{t=0} x(c_2(t))$. So we may think of a tangent vector v_p as an equivalence class of curves through p, which have the same "tangent" at p.*

Definition 2.10. *The* tangent bundle of a differentiable manifold M *is the disjoint union of all the tangent spaces of M. That is, $TM := \bigcup_{p \in M} T_pM$. There is a natural projection that maps each tangent space T_pM to the single point p defined by $\pi' : TM \to M$ where $\pi : v \mapsto \pi(v) = p$ if $v \in T_pM$.*

Remark 2.11. *If U is an open subset of M, $\pi^{-1}(U) = TU$ by definition of π since $T_pU = T_pM$ for $p \in U$.*

If (U, x) is a coordinate system for M, then we have the following *trivializing map* for TU: $(x, \frac{\partial}{\partial x}) : TU \to U \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ with $v_p \mapsto (p, v^1, \ldots, v^n) \mapsto (x(p), v^1, \ldots, v^n)$. Here $p = \pi(v)$ and $v = \Sigma v^i \frac{\partial}{\partial x^i}$, $v^i = v(\bar{x}^i)$. The collection of these charts

$$\{(\pi^{-1}(U), (x, \frac{\partial}{\partial x}) | (U, x) \text{ a coordinate system of } M)\}$$

cover TM and we define a C^∞ differentiable structure on TM by taking a maximal atlas compatible with this collection. The topology of TM is also generated by the sets TU. The transition function between two such charts $(x, \frac{\partial}{\partial x}) \mapsto (y, \frac{\partial}{\partial y})$ is given by

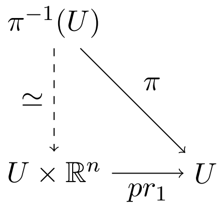
$$(y \circ x^{-1}, \frac{\partial y^i}{\partial x^j}) : x(U \cap V) \times \mathbb{R}^n \to y(U \cap V) \times \mathbb{R}^n$$

with $x(p, a) \mapsto (y(p), d(y \circ x^{-1})(a))$. Hence TM becomes a smooth manifold of dimension $2n$ and $\pi : TM \to M$ is a smooth map. In addition since TM is a disjoint union of vector bundles locally trivialized by the above maps $\pi^{-1} = TU \simeq U \times \mathbb{R}^n$, it is a vector bundle in the following sense.

Definition 2.12. *A* An n-dimensional real vector bundle *is a continuous surjective map* $\pi : E \to X$ *between two topological spaces with the following properties:*

1. *Each* fibre $E_x = \pi^{-1} \subset E$ *carries the structure of a vector space over* $\mathbb{R}$, *such that the vector space operations are continuous.*

2. $\forall x \in X$, $\exists$ *a neighbourhood* U *of* x *and a homeomorphism* ϕ

   such that the diagram is commutative and such that $\phi|_{E_x} : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^n$ *is an homeomorphism of vector spaces for each* $x \in U$

E *is called the* total space *and* X *the* base space *of the bundle. A map* ϕ *as in Proposition (?2?) is called a* bundle chart.

Definition 2.13. *A* section *of a vector bundle* $TM \xrightarrow{\pi} M$ *is a continuous map* $s : X \to E$ *such that* $\pi \circ s = id_X$

Definition 2.14. *The vector bundle* $TM \xrightarrow{\pi} M$ *is called the* **tangent bundle**.
If $M \xrightarrow{f} N$ *is a smooth map, then its* **differential** *is the map* $df : TM \to TN$, *with* $v_p \mapsto df_p(v_p) \in T_{f(p)N}$. df *is* C^∞, *commutes with the bundle projections and is linear on each fibre.*
$f : M \to N$ *is called an* **immersion** *if* df_p *is injective for all* $p \in M$
An immersion is called a **imbedding** *if* f *is injective and* M *is diffeomorphic to the image* $f(M)$, *considered as a subspace of* N.
$f : M \to N$ *is called a* **submersion** *if* f *is surjective and* df_p *is surjective for all* $p \in M$

Definition 2.15. *A subspace* $M \subset N$ *where* N *is a manifold is a submanifold if the inclusion map is an imbedding. In other words if for each* $p \in M \subset N$, *there is a coordinate chart* (U, x) *of the manifold* N *containing* p *such that* x *maps* $U \cap M$ *into the linear subspace* $\mathbb{R}^m \subset \mathbb{R}^n$. *where* m, n *are the dimensions of* M *and* N *respectively. So,* x *restricted to* $U \cap M$ *is a coordinate chart for the submanifold.*

There is a fundamental theorem which says that every manifold M^n can be imbedded into $\mathbb{R}^N$ for a sufficiently large N. $N = 2n + 1$ is good enough, I think.

Since the condition to be a submanifold is local, i.e., can be checked in a coordinate chart, one can use the standard Implicit (or the Inverse) Function Theorem to of Vector Calculus to check it, especially for submanifolds defined as level sets of functions. For example, if $F : N \to L$ is differentaible then $M = F^{-1}(q)$, for $q \in L$ is a submanifold of N if dF_p is of maximal rank at all points $p \in M$.

So what is the dimension of M? What if we replace a point by a submanifold $Q \subset L$

3 Vector Fields and Their Flows

Definition 3.1. *A* smooth vector field *on a smooth manifold M is a smooth section $X : M \to TM$ of the tangent bundle*

A vector field X associates to each point $p \in M$ a vector $X_p \in T_pM$ in a smooth manner. If (U, x) is a coordinate system on M, then $X|_U$ can be written as $X = \Sigma_{i=1}^n X^i \frac{\partial}{\partial x^i}$ where $X^i = X(x^i)$ are smooth functions. Therefore locally with respect to a coordinate system a vector field is represented by a smooth vector valued function: $X : U \to \mathbb{R}^n$, $p \mapsto (X^1(p), \ldots, X^n(p))$

Definition 3.2. *A curve $c : [a, b] \to M$ is called an* integral curve *of a vector field X on M if $dc(\frac{\partial}{\partial t}) = \dot{c}(t) = X_{c(t)}$ for all $t \in (a, b)$.*

Let $(-\varepsilon, \varepsilon) \to M$ be an integral curve of the vector field X and let (U, x) be a coordinate system about $p = c(0)$. Then if we let $c^i = x^i \circ c$ and $F^i = X^i \circ x^{-1}$ where the X^i's are the component functions of X with respect to the base $\frac{\partial}{\partial x^i}$, i.e. $X = \Sigma_{i=1}^n X^i \frac{\partial}{\partial x^i}$ on U, we obtain the following system of ODEs for the c^i's

$$\frac{dc^i}{dt} = F^i(c^1(t), \ldots, c^n(t)),$$

for small $t \in c^{-1}(U)$. Suppose now that $x(p) = 0$ (by a translation if necessary). Then the F^i's are smooth functions defined on an open ball around 0 and hence by the fundamental existence and uniqueness theorem for systems of ODEs, there exists a unique set of solutions $c_k^i(t)$ satisfying the initial condition $c_u^i(0) = u \in \mathbb{R}^n$ (and depending smoothly on the initial condition) for $|t| < \varepsilon$ and $|u| < a$ where ε and a are small positive numbers depending only on F. Set $\phi_t(q) := X^{-1}(c_u(t))$ where $u = x(q)$, $q \in x^{-1}(B_a(0)) \subset U$ and $|t| < \varepsilon$. If $|t| < \varepsilon$, $|s| < \varepsilon$ and $|s + t| < \varepsilon$, and both $|u| = |x(q)| < a$ and $|x(\phi_s(q))| < a$, then the functions $\gamma^i(t) = c_u^i(t + s)$ are solutions of the ODE system with initial conditions $\gamma^i(0) = c_u^i(s)$. Therefore by uniqueness $\gamma^i(t) = c_{\tilde{u}}^i$ where $\tilde{u} = (c_u^1(s), \ldots, c_u^n(s))$. This proves $\phi_t(\phi_s(q)) = \phi_{t+s}(q)$. Since $\phi_0 = id$ and $\phi_t(\phi_{-t}(q)) = \phi_0(q) = q$, ϕ_t is a diffeomorphism of a small neighbourhood of p for t small enough.

Definition 3.3. *A* local 1-parameter group of local diffeomorphisms *or a* local flow *on a manifold M is a mapping $\phi : (-\varepsilon, \varepsilon) \times U \to M$ where U is an open neighbourhood in M with the following two properties:*

1. *$\forall t \in (-\varepsilon, \varepsilon)$, $\phi_t : p \mapsto \phi(t, p)$ is a diffeomorphism of U onto its image $\phi_t(U) \subset M$.*

2. *$\forall t, s, t + s \in (-\varepsilon, \varepsilon)$ with $p, \phi_s(p) \in U$, we have $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.*

A local flow defines a vector field $X_p = \frac{d}{dt}|_{t=0}\phi_t(p)$. We can therefore paraphrase our above discussion about the existence of local integral curves of a vector field as follows.

Proposition 3.4. *Let X be a smooth vector field on M. Then $\forall p \in M$, $\exists$ a neighbourhood U of p, $\varepsilon > 0$ and a local flow $\phi : (-\varepsilon, \varepsilon) \times U \to M$, of X.*

Definition 3.5. *A* global flow on M *is a map $\phi : \mathbb{R} \times M \to M$, satisfying*

1. *$\forall t \in \mathbb{R}$, $\phi_t : p \mapsto \phi(t, p)$ is a diffeomorphism;*

2. *$\forall t, s \in \mathbb{R}$, $\phi_{t+s} = \phi_t \circ \phi_s$*

Definition 3.6. *If X generates a global flow, then X is said to be complete.*

Proposition 3.7. *On a compact manifold every vector field is complete.*

Proof. By Proposition 3.4 and compactness, there exists a finite set of local flows $\phi^i : (-\varepsilon_i, \varepsilon_i) \times U_i \to M$, for $i = 1, \ldots, N$ with $\bigcup U_i = M$. Set $\varepsilon = \min_{i=1,\ldots,N}\{\varepsilon_i\}$. Then we have a flow $\phi : (-\varepsilon, \varepsilon) \times M \to M$ and hence a global flow $\phi : \mathbb{R} \times M \to M$, by iterating the flow. □

We denote by $\mathcal{F}(M)$, the algebra of all smooth real valued functions on M and by $\mathcal{X}$ the vector space of all vector fields on M.

Definition 3.8. *For $X, Y \in \mathcal{X}(M)$ define $[X, Y] \in \mathcal{X}(M)$ by setting $[X, Y]_p(\bar{f}) := X_p(Y(f)) - Y_p(X(f))$ for $f \in \mathcal{F}(M)$. $[X, Y]$ is called the* Lie Bracket of X and Y.

Proposition 3.9.
1. *$[X, Y]$ is indeed a smooth vector field;*

2. *If $f, g \in \mathcal{F}(M)$, then $[fX, gY] = fg[X, Y] - f\,X(g)\,Y - g\,Y(f)\,X$;*

3. *$[X, Y] = -[Y, X]$;* skew-symmetric

4. *$\forall\ X, Y, Z \in \mathcal{X}(M)$ we have $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$* Jacobi Identity.

Proof. A soft exercise for the reader. □

Definition 3.10. *A vector space with a skew-symmetric bilinear operator $[,]$ satisfying the Jacobi identity is called a* Lie-Algebra

We have the following local expression for $[X, Y]$. If $X = \Sigma X^i \frac{\partial}{\partial x^i}$ and $Y = \Sigma Y^j \frac{\partial}{\partial y^j}$, then

$$[X, Y] = \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Let X be a vector field of f with associated local flow ϕ_t.

Definition 3.11. *For $f \in \mathcal{F}(M)$,*

$$\mathcal{L}_X f := \lim_{t\to 0} \frac{f \circ \phi_t - f}{t} = df(X) = X(f)$$

is called the Lie derivative of f along the flow of X.

Definition 3.12. *For $Y \in \mathcal{X}(M)$, we define: or more precisely*

$$(\mathcal{L}_X Y)_p := \lim_{t\to 0} \frac{d\phi_{-t}(Y_{\phi_t(p)}) - Y_p}{t}$$

$\mathcal{L}_X Y$ *is called the* Lie derivative of Y with respect to X.

Here is a KEY FACT:

Proposition 3.13. $\mathcal{L}_X Y = [X, Y] \in \mathcal{X}(M)$

Proof. Let $f \in \mathcal{F}(M)$, $p \in M$ and define $F(t, r, s) = f(\phi_s(\psi_r(\phi_t(p))))$ for small t, r, s such that the flows ϕ and ψ associated to the vector fields X, Y respectively are defined. Let $G(t, r) = F(t, r-t)$. Then $G(t, r) = (f\circ\phi_{-t})(\psi_r(\phi_t(p))) = g_{-t}(\psi_r(p_t))$ where $f\circ\phi_{-t} = g_{-t}$, $p_t = \phi_t(p)$. For ease of notation let us write $D_1 = \frac{\partial}{\partial t}$, $D_2 = \frac{\partial}{\partial r}$ and $D_3 = \frac{\partial}{\partial s}$. For a fixed t

$$D_2G(t, 0) = \frac{d}{dr}|_{r=0} g_{-t}(\psi_r(p_t)) = Y_{p_t}(g_{-t}) = Y_{\phi_t(p)}(f \circ \phi_{-t}) = (d\phi_{-t}(Y_{\phi_t(p)})) \cdot f$$

$$D_1D_2G(0, 0) = \frac{d}{dt}|_{t=0}(d\phi_{-t}(Y_{\phi_t(p)})) \cdot f = \lim_{t\to 0} \frac{d\phi_{-t}(Y \circ \phi_t(p)) \circ \phi_t - d\phi_0 Y_p}{t} f = (\mathcal{L}_X Y)_p \cdot f$$

by definition of $\mathcal{L}_X Y$ since $\phi_0 = id$.
Now $D_2F(t, 0, 0) = \frac{d}{dr}|_{r=0}(f\circ\psi_r)(\phi_t(p)) = Y_{\phi_t(p)} f$ and $D_1D_2F(0, 0, 0) = \frac{d}{dt}|_{t=0}(Y_{\phi_t(p)} f) = \frac{d}{dt}|_{t=0}(Y f(\phi_t(p))) = X_p(Y f)$. Similarly $D_2D_3F(0, 0, 0) = Y_p(Xf)$. On the other hand, since $G(t, r) = F(t, r, -t)$ we have by the chain rule $D_1G(0, 0) = D_1F(0, 0, 0) - D_3F(0, 0, 0)$. Therefore $(\mathcal{L}_X Y)_p f = D_1D_2G(0, 0) = D_2D_1G(0, 0) = D_2D_1F(0, 0, 0) - D_2D_3F(0, 0, 0) = D_1D_2F(0, 0, 0) - D_2D_3F(0, 0, 0) = X_p(Y f) - Y_p(Xf) = [X, Y]_p f$. And so the thesis $\mathcal{L}_X Y = [X, Y]$. □

Definition 3.14. *Let ψ be a diffeomorphism of M and $X \in \mathcal{X}(M)$. Then we set $d\psi(X)$ to be the vector field $d\psi(X)_p = d\psi_q(X_q)$ where $p = \psi(q)$, in other words $d\psi(X) \circ \psi = d\psi \circ X$.*

Proposition 3.15. $d\psi([X, Y]) = [d\psi(X), d\psi(Y)]$.

Proof. We have to show that the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{d\psi} & TM \\ \Big\uparrow [X,Y] & & \Big\uparrow [d\psi(X), d\psi(Y)] \\ M & \xrightarrow[\psi]{} & M \end{array}$$

Let $q \in M$, $p = \psi(q)$ and $f \in \mathcal{F}(M)$ We have:

$$d\psi([X,Y]_q)f = [X,Y]_q(f \circ \psi) = X_q(Y(f \circ \psi)) - Y_q(X(f \circ \psi)) = X_q((d\psi \circ Y)f) - Y_q((d\psi \circ X)f)$$

$$= X_q((d\psi(Y)f) \circ \psi) - Y_q((d\psi(X)f) \circ \psi) = d\psi(X)_p((d\psi(Y)f) - d\psi(Y)_p((d\psi(X)f) = [d\psi(X), d\psi(Y)]_p f$$

□

If ϕ_t is the local flow of X and ψ is any diffeomorphism then $d\psi X)$ generates the flow $\psi \circ \phi_t \circ \psi^{-1}$. Because if $p \in M$ then

$$\frac{d}{dt|_{t=0}}(\psi \circ (\phi_t(\psi^{-1}(p)))) = d\psi|_{t=0}(\phi_t(q)) = d\psi_q(X_q) = d\psi(X)_p.$$

If $d\psi(X) = X$, i.e. if X is invariant under ψ, then ψ commutes with the flow ϕ_t, i.e. $\psi \circ \phi_t = \phi_t \circ \psi$ for all t in a small interval $(-\varepsilon, \varepsilon)$, then $d\psi(X) = X$ (just differentiate $\psi \circ \phi_t \circ \psi^{-1}$ at $t = 0$). This implies that $d\phi_t(X) = X$ for all t.

Proposition 3.16. *Suppose $X, Y \in \mathcal{X}(M)$ generate local flows ϕ_t and ψ_s respectively. Then $\phi_t \circ \psi_s = \psi_s \circ \phi_t$ for every s, t (small enough) iff $[X, Y] = 0$.*

Proof. If $\phi_t \circ \psi_s = \psi_s \circ \phi_t$ then $d\psi(X) = X$ for all s and hence $[X, Y] = \mathcal{L}_X Y = \lim_{s \to 0} \frac{d\psi_s(X) - X}{s} = 0$.
Conversely if $[X, Y] = 0$, then

$$d\phi_t([X,Y]) = [d\phi_t(X), d\phi_t(Y)] = [X, d\phi_t(Y)] = \mathcal{L}_X(d\phi_t(Y)) = 0$$

by the previous propositions since $d\phi_t(X) = X$. Now

$$\begin{aligned} \mathcal{L}(d\phi_t(Y)) \quad &= \lim_{s \to 0} \frac{d\phi_s(d\phi_t(Y)) - d\phi_t(Y)}{s} \\ &= \lim_{s \to 0} \frac{d\phi_{t+s}(Y) - d\phi_t(Y)}{s} = \frac{d}{dt}(d\phi_t(Y)). \end{aligned}$$

Therefore $d\phi_t(Y) = d\phi_0(Y) = Y$ for all t and so ϕ_t commutes with the flow ψ_t of Y.

□

4 Tensors and differential forms

First let's do some simple (multi-)linear algebra.
For two finite dimensional real (or complex) vector spaces V and W, let $Hom(V, W)$ be the vector space of all linear maps from V to W.
The dual vector space denoted by V^* is then just $Hom(V, \mathbb{R})$.
A linear map $A \in Hom(V, W)$ induces a dual linear map $A^* \in Hom(W^*, V^*)$.
(*In general-nonsense-jargon, $*$ is a contravariant functor from the category of finite dimensional vector spaces to itself*).
V^* is isomorphic to V, but not canonically. (You have to choose a non-degenerate metric or a basis to get an isomorphism). However, V^{**} is canonically isomorphic to V via the pairing (or contraction) $(\alpha, v) \mapsto \alpha(v)$. More generally, $Hom(V, W)$ is canonically isomorphic to $Hom(W^*, V^*)$.

Definition 4.1. $V \otimes W = Hom(W^*, V)$

$v \otimes w$ denotes the map $\alpha \mapsto \alpha(w)v$ for all $v \in V, w \in W, \alpha \in W^*$.

Let us denote by V^s the direct product of s copies of V with itself. For $s = 0$ we define V^0 to be just the field $\mathbb{R}$.

Definition 4.2. *A tensor of type (r, s) is an element of $Hom(V^s, V^r)$*

So a tensor α of type $(0, k)$ is just a multilinear map: $(v_1, \ldots, v_k) \mapsto \alpha(v_1, \ldots v_k) \in \mathbb{R}$
A non-degenerate metric tensor is a tensor g of type $(0, 2)$ which is symmetric, i.e. $g(v_1, v_2) = g(v_2, v_1)$ and is an isomorphism $g : V^* \cong V$.

Definition 4.3. *An alternating (or exterior) form of degree k is a tensor of type $(0, k)$ which is totally anti-symmetric in all its arguments.*

For example, for $k = 3$ this means that $\alpha(u, v, w) = -\alpha(v, u, w) = -\alpha(w, v, u) = \alpha(u, w, v)$
For $k = n = dim(V)$, an alternating form of degree n is the determinant.

We denote the alternating forms of degree k by $\Lambda^k = \Lambda^k(V^*)$. Its dimension is $\binom{n}{k}$.

We now define the exterior product.

Definition 4.4. *For $\alpha \in \Lambda^k$ and $\beta \in \Lambda^l$, $\alpha \wedge \beta \in \Lambda^{k+l}$ is defined by*

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in \mathcal{S}'} sign(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

where $\mathcal{S}'$ denotes the subgroup of "shuffle" permutations of the set $\{1, \ldots, k+l\}$ satisfying: $\sigma(i) < \sigma(j)$ for $1 \leq i < j \leq k$ and for $k = 1 \leq i < j \leq k = l$.

This product is not commutative but it is 'graded-commutative":

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\, \beta \wedge \alpha$$

where $|\alpha| = k$ is the degree of α and $|\beta| = l$ is the degree of β
We say the exterior forms form a graded exterior algebra.

Dual to this we also have an "interior" product (not to be confused with an "inner product")

Definition 4.5. *For $\alpha \in \Lambda^k$ $(k > 0)$ and $v \in V$, $\iota_v \alpha \in \Lambda^{k-1}$ is defined by*

$$\iota_v \alpha(v_1, \ldots, v_{k-1}) = \alpha(v, v_1, \ldots, v_{k-1})$$

Now let $V = T_pM$ be the tangent space of a differential manifold at a point p in a a neighbourhood U where we have local coordinates $(x^1, \ldots, x^n)$. Then we have a well defined basis $e_i = \frac{\partial}{\partial x^i}$ for V and a corresponding dual basis $e^i = dx^i$ for $V^\star$ at each point (*algebra is geometry at a point*).
We can then use the basis $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ for Λ^k where I is a multi-index $I = (i_1, \ldots i_k)$. *(very multicultural but anti-symmetric!)*
To deal with the whole manifold, we bundle them up and think of an exterior form as a section of a vector bundle (actually of a graded-algebra bundle):

$$\Lambda^k(TM^*) = \bigcup_{p \in M} \Lambda^k(T_pM^*)$$

and think of exterior differential forms as sections of that bundle written in local coordinates as:

$$\alpha = \alpha_I dx^I$$

where the α^I are smooth (i.e. C^∞) functions on M and we will be using the Einstein summation convention (sum over repeated "dual" indices) from now on.
We denote all smooth exterior differential k-forms by $\Omega^k(M)$ and the whole algebra by $\Omega^\star(M)$.
Although it is "legalistically" incorrect I will sometimes just write Λ with Ω (*in order to confuse you? lol*)

Definition 4.6. *A nowhere vanishing n-form* ν *on an* n*-dimensional manifold* M^n *is called an* **orientation** *for* M*.* M *is then oriented.* M *is said to be orientable if we can find an orientation for* M*. We usually assume that* M *is connected.*

Remark 4.7. M *is orientable if and only if we can find an atlas such that all the coordinate changes preserves orientation, i.e., all the Jacobian matrices for coordinate changes have positive determinants.*

5 Exterior derivative and Cartan's Formula

Definition 5.1. *A derivation (anti-derivation respectively) of degree $l \in \mathbb{Z}$ is a linear operator:*

$$D : \Omega^k(M) \to \Omega^{k+l}(M) \qquad k = 0, 1, \ldots$$

satisfying the product rule

$$D(\alpha \wedge \beta) = D\alpha \wedge \beta + \alpha \wedge D\beta \qquad \textit{for a derivation}$$

and

$$D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^{|\alpha|}\, \alpha \wedge D\beta \qquad \textit{for an anti- derivation}$$

Definition 5.2. *The commutator (respectively the anti-commutator) of two derivations (anti-derivations respectively) is defined by:*

$$[D_1, D_2] = D_1 D_2 - D_2 D_1 \qquad \textit{commutator}$$

$$\{D_1, D_2\} = D_1 D_2 + D_2 D_1 \qquad \textit{anti-commutator}$$

Given the product rule it is obvious to the naked eye that **a derivation or an anti-derivation is uniquely determined by its action on functions and on the one forms** (which locally are always of the form df for some function $f \in \Omega^0(M) = \mathcal{F}$). Ergo we have the following simple but very important fact:

Lemma 5.3. *There exists a unique anti-derivation d of degree one with the following two properties:*

(i) df is just the "usual" differential of f for a function f

(ii) $d^2 = 0$

d is called the exterior derivative and is the most fundamental operator for doing calculus on manifolds it (the name of this course!). Of course $d^2 = d \circ d = 0$ is here the key and the mother of a lot of theorems and computations as we will see.

Definition 5.4. *For a smooth map $\phi : M \to N$ and for $\alpha \in \Omega^k(N)$, we define $\phi^* \alpha \in \Omega^k(M)$ (the pull-back, since forms are contravariant) by:*

$$\phi^* \alpha(v_1, \ldots, v_k) := \alpha(d\phi(v_1), \ldots, d\phi(v_k)) \qquad k > 0$$

and by $\phi^* f = f \circ \phi$ for functions ($k = 0$)

ϕ^* is a linear operator order 0. It is not a derivation (or anti-derivation) but it is a homomorphism:

$$\phi^*(\alpha \wedge \beta) = \phi^*\alpha \wedge \phi^*\beta$$

.

More importantly, it commutes with d:

$$[\phi^*, d] = \phi^* \, d - d \, \phi^* = 0$$

$[\phi^*, d]$ is an anti-derivation of order 1 and it commutes with d because $d^2 = 0$. Now all we have to do is to check that it vanishes on functions but that is just the chain rule!
$\phi^* df = df \circ d\phi = d(f \circ \phi) = d\phi^\star(f)$.

We can now define the Lie derivative of a vector field acting on differential forms:

Definition 5.5. *For a vector field X and a k-form $\alpha \in \Omega^k$ we define:*

$$\mathcal{L}_X \, \alpha = \lim_{t \to 0} \frac{\phi_t^* \alpha - \alpha}{t}$$

where ϕ_s is the flow of the vector field X.

Since ϕ^* is a homomorphism, the product rule says that the Lie derivative is a derivation of order 0:

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta$$

Since ϕ^* commutes with d so does the Lie derivative:

$$[\mathcal{L}_X, d] = 0$$

Now comes one of my favourite formulas in all of Mathematics:

$$\boxed{\mathcal{L}_X = \{\iota_X, d\} = \iota_X d + d \, \iota_X} \tag{5.1}$$

The proof is trivial. Both sides are derivations that commute with d, so we just have to check it on functions and by definition $\mathcal{L}_X \, f = df(X) = \iota_X d \, f$, since $\iota_X \, f = 0$.

6 DeRham Cohomology and the Poincare Lemma

Definition 6.1. *A differential form is said to be* **closed** *if it is in the kernel of d.*
A differential form is said to be **exact** *if it is in the image of d.*
We will denote by $Z^k(M)$ all the vector space of closed differential forms on M and by $B^k(M)$ the vector space of all exact differential forms on M.

So α is closed iff $d\alpha = 0$ and α is exact iff there exists a form β such that $\alpha = d\beta$. Since $d^2 = 0$, $B^k(M)$ is a subspace of $Z^k(M)$ i.e., all exact forms are necessarily closed, but the converse is in general not true (remember not all vector fields with zero curl are gradient of functions!). It depends on the topology of M.

Definition 6.2. *The deRham cohomology groups of a differential manifold M are defined to be*

$$H^k_{dR}(M) = Z^k(M)/B^k(M)$$

for a non-negative integer k.

It is a NON-TRIVIAL FACT that these vector spaces $H^k(M)$ (in fact they are rings under the wedge product) are all finite-dimensional for compact manifolds. For ease of notation, we will skip the sub-index dR for the rest of the notes.

The dimension of the vector space $H^k(M)$ is called the k-th Betti number of M and is denoted by $b_k(M)$.

H^* is a contravariant functor from manifolds to vector spaces (rings, in fact), since a differentiable map $f :\to N$ induces a homomorphism $f^* : H^*(N) \to H^*(M)$ such that $(f \circ g)^* = g^* \circ f^*$

I should point out that cohomology groups are not always that easy to compute and one needs a lot of tools (and tricks). *defining what a bike is easier than riding one?*

For non-compact manifolds (such as $\mathbb{R}^n$!), it is more meaningful to use deRham cohomology with compact support which is defined as follows:

Definition 6.3. *The deRham cohomology with compact support of a differential manifold M is defined to be*

$$H^k_c(M) = Z^k_c(M)/B^k_c(M)$$

where Z^k_c denotes closed forms with support inside a compact set (i.e. they vanish outside a compact set) and B^k_c denotes closed forms α which can be written as $\alpha = d\beta$ where β is compactly supported.

In order to define homotopy and explain the homotopy invariance of cohomology, let us now look at the product $M \times \mathbb{R}$ where M is some smooth manifold. We will denote by t the coordinate on $\mathbb{R}$ and by x a (local) coordinate on M, so (x,t) is a generic point in $M \times \mathbb{R}$ and let j_t be the inclusion map: $M \to M \times \mathbb{R} : x \mapsto j_t(x) = (x,t)$.
Let X be the vector field $\frac{\partial}{\partial t}$. Then the corresponding flow ϕ_t of X on $M \times I$ is just the "time" translation: $\phi_t(x,s) = (x, s+t)$ and $\phi_s \circ j_t = j_{s+t}$
Let us define a linear operator of degree -1 (*a chain homotopy for a high-brow topologist*):

$$I\beta = \int_0^1 {j_t}^*(\iota_{\frac{\partial}{\partial t}}\beta)\, dt$$

acting on a differential form β.

Now here is the **FUNDAMENTAL LEMMA** (mother of all lemmas!) which is a consequence (or better an integrated version) of the magical Cartan formula but which implies both Poincare's lemma and Stokes' theorem after some easy manipulations.

Lemma 6.4.

$${j_1}^* - {j_0}^* = \{d, I\}$$

i.e.

$${j_1}^*\beta - {j_0}^*\beta = d(I\beta) + I(d\beta)$$

and hence, if $d\beta = 0$ *(* β *is a closed form), then*

$${j_1}^*\beta - {j_0}^*\beta = d\alpha \qquad \textit{where} \;\; \alpha = \int_0^1 {j_t}^*(\iota_{\frac{\partial}{\partial t}}\beta)\, dt$$

Of course, 0 and 1 can be replaced by any $a < b$

Proof. $\mathcal{L}_{\frac{\partial}{\partial t}} = \iota_{\frac{\partial}{\partial t}} d + d\,\iota_{\frac{\partial}{\partial t}}$ and so ${j_t}^*\mathcal{L}_{\frac{\partial}{\partial t}} = {j_t}^*\iota_{\frac{\partial}{\partial t}} d + {j_t}^* d\,\iota_{\frac{\partial}{\partial t}} = {j_t}^*\iota_{\frac{\partial}{\partial t}} d + d{j_t}^*\iota_{\frac{\partial}{\partial t}}$

Now $\phi_s \circ j_t = j_{s+t}$ so $j^*_{s+t} = j^*_t\, \phi^*_s$ and hence $\frac{d}{dt}{j_t}^* = {j_t}^*\mathcal{L}_{\frac{\partial}{\partial t}}$ so we have:

$$\frac{d}{dt}({j_t}^*\beta) = d\,{j_t}^*\iota_{\frac{\partial}{\partial t}}\beta + {j_t}^*\iota_{\frac{\partial}{\partial t}} d\beta$$

Now integrate from 0 to 1 □

Definition 6.5. *Two smooth maps* $f_0 : M \to N$ *and* $f_1 : M \to N$ *are said to be (differentiably) homotopic if there exists a smooth map* $H : M \times [0.1] \to N$ *such that* $j_0 \circ H = f_0$ *and* $j_1 \circ H = f_1$

Lemma 6.6. *If α is a closed form on N (i.e. $d\alpha = 0$) and if $f_0, f_1 : M \to N$ are homotopic, then ${f_1}^*\alpha - {f_0}^*\alpha$ is exact (i.e. this difference is in the image of d).*

This is a special case of the Fundamental Lemma. Saying the same thing in fancy words we get:

Lemma 6.7. *Homotopic maps induce the same homomorphism between the deRham cohomology groups*

Definition 6.8. *M is said to be contractible if the identity map $id : M \to M$ is homotopic to a constant map $M \to \{pt\}$, where pt is a point in M.*

So

Lemma 6.9. *If M is contractible, then M has the same deRham cohmology groups as that of a point: $H^k_{dR}(M) \cong H^k_{dR}\{pt\})$ for all k.*

Note that $H^0_{dR}\{pt\} = \mathbb{R}$ and $H^k_{dR}\{pt\} = 0$ for all $k > 0$.

This is the content of Poincare's Lemma, but let me introduce a notion that is used to state Poincare's lemma in textbooks:

Definition 6.10. *A open set $U \subset \mathbb{R}^n$ is said to be "star-shaped" with respect to a point $p_0 \in U$ if for every point $p \in U$, the straight line segment from p_0 to p lies in U.*

A star-shaped U is obviously contractible since $H(p,t) = tp + (1-t)p_0$ is a homotopy from the constant map to the identity map. Hence we have what is called Poincare's Lemma in the textbooks:

Lemma 6.11. *If $U \subset \mathbb{R}^n$ is star-shaped, then all closed k-forms are exact ($k > 0$).*

Example 6.12. $H^k(\mathbb{R}^n) \simeq \{0\}$ *if* $k \neq 0$ *and* $H^0(\mathbb{R}^n) \simeq \mathbb{R}$.

7 Stokes' Theorem

Up to now we've been only talking about derivatives. In elementary calculus on the real line, we all learned that integration and differentiation are related by the Fundamental Theorem of Calculus:

$$\int_{[a,b]} df = \int_a^b df = \int_a^b f'(x)dx = f(b) - f(a) = \int_{\partial[a,b]} f$$

Stokes theorem, which is a powerful generalisation of this to manifolds, replaces the function f by a differential form ω, the derivative d by the exterior derivative, still denoted by d and we now integrate on "chains", which are nothing but (finite) linear combinations of differentiable maps from "standard objects" of the appropriate dimension in $\mathbb{R}^k$ to the manifold. For these lecture notes, I will use the k-dimensional cube $I^k = [0,1]^k \subset \mathbb{R}^k$ as the standard object where I want to integrate (topologists prefer to use the k-simplex). So the computations are actually performed on these standard objects and the rest is done "functorially" (I didn't say perfunctorially!) by pulling back to a space where you know how to integrate.We will integrate a k-form α on a k chain c, and this will be bilinear pairing producing a real number:

$$< c\,,\, \alpha >= \int_c \omega \in \mathbb{R}$$

Definition 7.1. *A k-dimensional chain on a manifold M is a (finite) linear combination*

$$c = \sum_{i=1}^{N} \lambda_i c_i$$

where each c_i is a smooth map $c_i : I^k \to M$ and the λ_i are real numbers. We set

$$\int_c \alpha = \sum_{i=1}^{N} \lambda_i \int_{I^k} {c_i}^* \alpha$$

where α is a k-form on M.

So now all we have to do is to define the integral of a k-form α on I^k. This is easy since α is a "top form" we can write it as $\alpha = a(x_1, \ldots, x_k)\, dx^1 \wedge \ldots \wedge dx^k$ in Euclidean coordinates, where a is a smooth function and we simply set

$$\int_{I^k} \alpha = \int_0^1 \cdots \int_0^1 a(x^1, \cdots, x^k) dx^1 \ldots dx^k$$

(you can use Lebesgue integrals if you know that, but our functions are differentiable so Riemann integrals will do the job)

Stokes' theorem is the Fundamental Theorem of Calculus applied to integrating exterior forms on chains. It says that the geometric boundary operator ∂ on chains is the "transpose" or the "adjoint" operator of the exterior derivative. So what is ∂. It is a linear operator which maps a k-chain to a $k-1$-chain and by "functoriality" we only need to define it for the standard object I^k:

Definition 7.2.

$$\partial I^k = \sum_{j=1}^{k} (-1)^{j+\epsilon}\, \imath_{(j,\epsilon)}(I^k)$$

where $\epsilon = 0, 1$ *and* $\imath_{(j,\epsilon)}$ *is the inclusion of the* (j,ϵ)*-th face:*

$$\imath_{(j,\epsilon)} : (x^1, \ldots, x^{k-1}) \mapsto (x^1, \ldots, x^{j-1}, \epsilon, x^j, \ldots, x^{k-1})$$

For $c : I^k \to M$*, we set*

$$\partial c = \sum_{j=1}^{k} (-1)^{j+\epsilon}\, c_{(j,\epsilon)}$$

where $c_{(j,\epsilon)} = \imath_{(j,\epsilon)} \circ c$

It is now an easy exercise to check (it's in Spivak's book on p.99 for example) the crucial fact that $\partial^2 = \partial \circ \partial = 0$

We are now ready to state Stokes' Theorem:

Theorem 7.3. *Let* ω *be a* k*-form and* c *be a* $(k+1)$*-dimensional chain on a differential manifold* M*. Then*

$$\int_c d\omega = \int_{\partial c} \omega$$

Proof:
By the above "functorial" way of defining the objects, we only need to consider the case where $c = I^{k+1}$ itself.
We prove by induction on k. For $k = 0$,

$$\int_{[0,1]} df = f(1) - f(0) = \int_{\partial[0,1]} f$$

is just the Fundamental Theorem of Calculus, so we start from there and proceed by using the Fundamental lemma, which is the integrated version of Cartan's magic formula.
Let $\omega_\epsilon = \imath_{1,\epsilon}^* \, \omega$ (the restriction of ω to the top and bottom faces). Then

$$\begin{aligned} \int_{I^k} \omega_1 - \int_{I^k} \omega_0 &= \int_{I^k} dI\omega + \int_{I^k} Id\omega \\ &= \int_{\partial I^k} I\omega + \int_{I^{k+1}} d\omega \end{aligned}$$

by induction and the definition of I and hence:

$$\int_{I^{k+1}} d\omega = \int_{I^k} \omega_1 - \int_{I^k} \omega_0 - \int_{\partial I^k} I\omega = \int_{\partial I^{k+1}} \omega$$

Stokes' theorem can be succinctly stated as:

$$\boxed{< c\,, d\omega > = < \partial c\,, \omega >} \tag{7.2}$$

A more common version of Stokes' Theorem which can be found in the literature is the following:

Theorem 7.4. *If M^n is a smooth, compact,* **oriented** *manifold with a smooth boundary ∂M, then for any $(n-1)$-form ω:*

$$\int_M d\,\omega = \int_{\partial M} \omega$$

I will explain the proof (using something technical called a "partition of unity") in class.

8 Cohomology with compact support and Degree theory

To discuss Degree Theory and Poincare Duality, we need to define deRham cohomology using compactly supported differential forms. Of course, on a compact manifold every thing is compactly supported. Let M be a differential manifold.

Definition 8.1.

$$H_c^k(M) = \frac{Z_c^k(M)}{B_c^k(M)}$$

where

$$Z_c^k(M) = \{\alpha \in \Omega^k(M) \mid d\alpha = 0, \text{ and } supp(\alpha) \text{ is compact}\}$$

$$B_c^k(M) = \{\alpha \in \Omega^k(M) \mid \alpha = d\beta, \text{ and } supp(\beta) \text{ is compact}\}$$

Theorem 8.2.

$$H_c^k(\mathbb{R}^n) \cong \{0\} \;\; \text{for } k \neq n \;\; \text{and } H_c^n(\mathbb{R}^n) \cong \mathbb{R}$$

More generally we have:

Theorem 8.3.

$$H_c^n(M^n) \cong \mathbb{R}$$

for any connected, orientable n-dimensional manifold.

The cohomology class of an orientation ν is a basis for this 1−dimensional vector space and defines a specific isomorphism: $H_c^n(M^n) \cong \mathbb{R}$, since $\int_M \nu \neq 0$

Definition 8.4. *Given a* **proper** *map $f : M^n \to N^n$ between two connected* **oriented** *manifolds, the degree of f is defined to be the real number that is the 1×1 matrix describing the linear map $f^\star : H_c^k(M) = \mathbb{R} \to \mathbb{R} = H_c^k(M)$*

So

$$f^\star[\nu_N] = deg(f)[\nu_M]$$

where [ ] denotes the cohomology class of a closed form.

A map is called proper if pre-images of compact sets are compact.

Theorem 8.5. *Given a proper map $f : M^n \to N^n$ between two connected* **oriented** *manifolds, the degree of f can be computed by the formula:*

$$deg(f) = \sum_{p \in f^{-1}(q)} sign(f;p) \in \mathbb{Z} \qquad for\, any\, \textbf{regular}\, value\, q \in N$$

where $sign(f;p) = \begin{cases} +1 & \text{if } df_p \text{ is orientation preserving} \\ -1 & \text{if } df_p \text{ is orientation reversing} \end{cases}$

Note that $f^{-1}(q)$ is a finite discrete set of points in M.

The main thing to remember here is that (i) the degree is an integer and that (ii) it is a sum of local contributions. This is prototypical of a lot of"index-type" statements in differential geometry and algebraic topology

The proofs of these theorems are not difficult but involves a bit of technical work and will be presented in class. Come to the lectures, as I always say!

9 Poincare Duality and Hodge Duality

Theorem 9.1. *If M^n is an* **oriented** *n-dimensional manifold, then for $0 \leq k \leq n$, we have an isomorphism:*

$$H_c^k(M) \cong H^{n-k}(M)^\star$$

The key fact is to prove that

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta \in \mathbb{R}$$

where α is a closed k-form with compact support and β is any closed $(n-k)$-form is a well-defined **non-degenerate** bilinear pairing.

There are different proofs of Poincare duality. A straightforward but rather technical method uses a covering of M by a good neighbourhoods that intersect nicely and a gluing argument (Mayer-Vietoris) which I will briefly explain in class.

A more elegant way to understand Poincare duality on a compact manifold is through Hodge duality using a metric on M and a volume form ν. An inner product $<\,,\,>$ on any vector space induces an inner product on the exterior algebra (using Gram determinants):

$$< v_1 \wedge \ldots \wedge v_k \,, w_1 \wedge \ldots \wedge w_k >= \det\left(< v_i, w_j >\right)$$

The Hodge star operator is a linear isomorphism between k-forms and $(n-k)$-forms.

Definition 9.2. *For a k-form α on a manifold M with an orientation ν and a metric $<\,,\,>$, $\star\alpha$ is the $(n-k)$-form defined by*

$$\alpha \wedge \beta \; = < \star\alpha \,,\, \beta > \nu \qquad \textit{valid for any } \; (n-k)\textit{-form}\, \beta$$

It is now tempting to think that the Hodge star of a closed form is exactly the Poincare dual. The problem is that the Hodge star of a closed form is not necessarily closed, so we want to look at closed forms whose Hodge duals are also closed. This leads us to the next section!

10 The Laplacian and Hodge Theory

For simplicity, I will assume in this section that M^n is an n-dimensional, orientable, closed (i.e. compact without boundary) Riemannian (i.e. equipped with a positive-definite metric $< >$) manifold oriented with the volume form ν coming from the metric. This defines a positive-definite inner product on forms (L_2-inner product) as follows:

$$< \alpha, \beta > = \int_M < \alpha, \beta > \nu$$

The Orwellian double-notation here is deliberate! You can always figure out in a given context what I mean like the word kids nowadays use the word "like" lol

The adjoint operator δ of the exterior derivative d is defined by:

$$\delta = \pm \star\, d \star$$

where the sign is $(-1)^{k(n+1)+1}$ (*I think*) on k-forms. The sign is chosen so that δ satisfies

$$< \alpha, d\beta > = < \delta\alpha, \beta >$$

after integration (**global**)

Do the **local** computation

$$< d\alpha, \beta > - < \alpha, \delta\beta > = \pm d(\alpha \wedge \star\beta)$$

to figure out the correct sign using the fact that $\star\star = (-1)^{k(n-k)}$ on k-forms.

Now apply Stokes' theorem to get the global formula. Doing this is known as integration by parts as you learned it in elementary calculus on $\mathbb{R}$. In case M has a boundary, then you will retain a boundary integral

$$\int_M < \alpha, d\beta > \nu_M - \int_M < \delta\alpha, \beta > \nu_M \;=\; \pm \int_{\partial M} \alpha \wedge \star\beta$$

The Laplace operator acting on forms is defined to be

$$\Delta = d\,\delta + \delta\,d$$

.

Δ is a self-adjoint elliptic operator (the symbol is the metric).

$$< \Delta\alpha, \beta >=< \alpha, \Delta\beta >$$

and is non-negative

$$< \Delta\alpha, \alpha >= |d\alpha|^2 + |\delta\alpha|^2 \geq 0$$

Furthermore, Δ commutes with $\star$.

The Kernel of the linear operator Δ is the vector space of **harmonic forms**, i.e. forms that satisfy:

$$\Delta\alpha = 0$$

which is on a closed manifold equivalent to being closed and co-closed, so both α and $\star\alpha$ are closed. *Again we are being Orwellian about the double meaning of the word "closed" !*

$$\Delta\alpha = 0 \qquad \Leftrightarrow \qquad d\alpha = 0 \text{ and } \delta\alpha = 0$$

The first basic Theorem in the subject (Hodge Theory) is that there is a unique harmonic form representing any given deRham cohomology class. The second basic theorem is that the set of harmonic forms (on a closed manifold) is finite-dimensional. The third basic Theorem is that the L_2 Hilbert space of forms has an orthogonal decomposition:

$$\Omega \cong \text{Harmonic} \oplus \text{Image of } d \oplus \text{Image of } \delta$$

It's trivial to see now that Poincare Duality is simply given by the Hodge star operator on harmonic forms.

11 Mayer-Vietoris and Künneth

There are two fundamental formulas that can be used to compute cohomology groups.

Theorem 11.1. Mayer-Vietoris exact sequence
If U and V are two open sets (with non-empty intersection) in a manifold, then we have a **long exact sequence** *of cohomology groups:*

$$\cdots \to H^{k-1}(U \cap V) \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to \cdots$$

Here the word exact just means that the kernel of any arrow (linear map) is the image of of the proceeding arrow. More generally a sequence of vector spaces and arrows connecting them like that is called a **co-chain complex** if the composition of two subsequent arrows is the zero map, that is to say the kernel of an arrow contains the image of the proceeding arrow. If the arrows are in the opposite direction it's called a **chain complex**.
One can think of a co-chain complex as a graded vector space with an operator of degree $+1$ (-1 for a chain complex) whose square is zero. Exterior differential forms $\omega \in \Omega^*(M)$ on a manifold with the exterior derivative operator d is a co-chain complex and the set of chains $c \in C_*(M)$ on a manifold with the boundary operator ∂ is a chain complex. Integration and Stokes' theorem describes the duality (pairing) between these two complexes.
A (co-) chain map (homomorphism) from one (co-)chain complex to another is a degree-preserving map (so a sequence of linear maps between the corresponding vector spaces of the same degree) that commutes with the operator d. The pull back ϕ^* of a map $\phi : M \to N$ is an example.

The long exact Mayer-Vietoris sequence is an abstract algebraic consequence of the simple fact that the following sequence of co-chain maps between the co-chain complexes is **exact**:

$$0 \to \Omega^*(U \cup V) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$$

The second arrow is given by restricting the forms to each piece and the third arrow is taking the difference after restricting the forms to the intersection. The exactness in the middle simply means that forms come from the union iff they agree on the intersection. For the surjectivity at the end you can use two functions that form a simple "partition of unity". Mayer-Vietoris now follows from the mantra "*short exact sequence of chain complexes gives rise to a long exact sequence of their cohomology groups*"

Theorem 11.2. Künneth formula
If M and N are two compact oriented manifolds, then:

$$H^k(M \times N) \cong \bigoplus_{i+j=k} H^i(M) \otimes H^j(N)$$

In terms of Betti numbers: $b_k(M \times N) = \sum_{i+j=k} b_i(M) b_j(N)$

To prove this for manifolds, you can use Hodge theory. Pull back harmonic forms from each factor to the product manifold and then take the wedge product. Use the product metric on $M \times N$ to show that what you get is still harmonic.