# A LAW OF LARGE NUMBERS FOR THE ZEROES OF HEINE-STIELTJES POLYNOMIALS

ALAIN BOURGET, DMITRY JAKOBSON, MAUNG MIN-OO AND JOHN A. TOTH

ABSTRACT. We determine the limiting density of the zeroes of Heine-Stieltjes polynomials in the thermodynamic limit and use this to prove a strong law of large numbers for the zeroes.

## 1. INTRODUCTION

In this paper, we discuss the asymptotic distribution of the zeroes of the polynomial solutions, $\phi_K$, of degree $K$ of the Heine-Stieltjes equation [Sz]:

$$\frac{d^2}{dx^2}\phi_K(x) + 2\left(\sum_{\nu=0}^{N} \frac{\beta_\nu}{x-\alpha_\nu}\right)\frac{d}{dx}\phi_K(x) = \frac{C(x)}{A(x)}\phi_K(x), \tag{1}$$

where, $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N$, $\beta_\nu > 0$ for all $\nu = 0, ..., N$, $C(x)$ is a polynomial of degree $N-1$ and $A(x) = (x-\alpha_0)\cdots(x-\alpha_N)$. The polynomial $C(x)$ is called a *Van Vleck* polynomial. Such ODE's arise naturally when separating variables in certain (quantum) completely integrable systems. More precisely, consider the generalized (real) Gaudin spin chains [HW, K, KKN, KM, Sk]. These consist of $N$ explicit pairwise-commuting, second-order elliptic differential operators $P_1^\alpha, ..., P_N^\alpha$ acting on $C^\infty(\mathbb{S}^N)$. In terms of a suitable (elliptic) coordinate system $u_1, .., u_N$ on the sphere, the joint eigenfunctions, $\psi_K(u_1, ..., u_N)$ of the $P_j^\alpha$'s are of product form $\psi_K(u_1, ..., u_N) = \prod_{j=1}^N \phi_K(u_j)$, where the $\phi_K$'s are precisely the polynomial solutions to (1) (ie. the Heine-Stieltjes polynomials). Moreover, the coefficients of the polynomial $C(x)$ are the joint eigenvalues of the $P_j^\alpha$'s with joint eigenfunction, $\psi_K$. For background, we refer to [T1, T2, Sz, WW]. It is a natural question to determine the distribution of the zeroes of $\phi_K$ in various asymptotic regimes. Recently, Martinez-Finkelstein and Saff [MS] have studied the density states of the Heine-Stieltjes zeroes as $K \to \infty$ (ie. semiclassical asymptotics).

In this paper, we study the asymptotics of the density of states of the zeroes of the Heine-Stieltjes polynomials as the dimensional parameter $N \to \infty$. More precisely, let $\Omega(K, N)$ denote the finite set of monic Heine-Stieltjes polynomials of degree $K$ depending on $N$ "spin sites" $0 < \alpha_0 < \cdots < \alpha_N < 1$. Then, it is well-known (see [Sz] and Theorem 2.1) that

$$\mathrm{card}\,(\Omega(K,N)) = \sigma(K,N) = \frac{(N+K-1)!}{K!\,(N-1)!}.$$

We consider the density of states measure

$$d\rho_{DS}(x; m, K, N, \alpha) := \frac{1}{K}\sum_{j=1}^{K} \delta\big(x - \theta_j^{(m)}(\alpha)\big), \tag{2}$$

where, $\theta_j^{(m)}$ denotes the $j$th zero of the $m$th polynomial in the set $\Omega(K, N)$. We define the averaged density of states in a similar fashion:

$$d\rho_{DS}^{AV}(x; K, N, \alpha) := \frac{1}{\sigma(K,N)}\sum_{m=1}^{\sigma(K,N)} \frac{1}{K}\sum_{j=1}^{K} \delta\big(x - \theta_j^{(m)}(\alpha)\big). \tag{3}$$

The second author was supported by NSERC, NATEQ and Alfred P. Sloan Foundation fellowship. The third author was supported by NSERC. The fourth author was supported by NSERC and Alfred P. Sloan Foundation fellowship.

Sections 3 through 7 are devoted to the study of the large $N$ asymptotics of the zeroes of $d\rho_{DS}^{AV}(x;m,K,N,\alpha)$ and $d\rho_{DS}(x;m,N,K,\alpha)$.

Our first result, Proposition 3.1, concerns the asymptotics of the averaged density of states: We show that for any $\phi \in BV([0,1])$ and any $\alpha \in \Lambda^N := \{\alpha \in [0,1]^{N+1} : 0 < \alpha_0 < \cdots < \alpha_N < 1\}$,

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{N+1}\sum_{\nu=0}^{N} \phi(\alpha_\nu) + \mathcal{O}\left(\frac{1}{N}\right), \tag{4}$$

as $N \to \infty$. In (4), we use the shorter notation $d\rho_{DS}^{AV}(\phi)(\alpha)$ to denote $d\rho_{DS}^{AV}(\phi;K,N,\alpha)$. We will use this notation throughout the rest of the text.

In Theorem 5.1, we strengthen this result by giving the following estimate for the variance of $d\rho_{DS}(x;m,K,N,\alpha)$ itself:

$$\frac{1}{\sigma(K,N)}\sum_{m=1}^{\sigma(K,N)} \left[ d\rho_{DS}(\phi)(\alpha) - \frac{1}{N+1}\sum_{\nu=0}^{N}\phi(\alpha_\nu)\right]^2 = \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right). \tag{5}$$

Thus, as long as the degree $K$ and the number of spin sites $N$ *both* go to infinity, the typical Heine-Stieltjes polynomial has its zeroes distributed uniformly in the interval $[0,1]$. As a corollary to Theorem 5, we arrive at the main result of the paper (Theorem 7.1), a strong law of large numbers for the zeroes of generic Heine-Stieltjes polynomials.

*Remarks:* (i) It would be interesting to determine whether the zeroes of the joint eigenfunctions, $\psi_K$, have an invariant asymptotic distribution on $\mathbb{S}^N$ itself relative to the standard round metric. We hope to address this elsewhere.

(ii) We were motivated by the previous work of two of the authors [BT] on the averaged level-spacings of the Heine-Stieltjes zeroes. There are several significant improvements here that we wish point out: (1) By a simple combinatorial monotonicity argument, we get *pointwise* results in the spin sites, $\alpha_0, ..., \alpha_N$ and completely avoid averaging over this parameter space. (2) We prove a result about the asymptotic density of states of a "typical" Heine-Stieltjes polynomial (Theorem 5.1); that is, about $d\rho_{DS}$ itself and not just $d\rho_{DS}^{AV}$. (3) We need not assume anything about the relative size of the degree, $K$, and the dimension, $N$, in any of our results.

(iii) The techniques of the present paper allow us to give an alternative proof of the results in [B]. We hope to address this in future work.

## 2. The Heine-Stieljes Theorem

The combinatorial component of our analysis is based in large part on the following classical result of Heine and Stieltjes [H, St, Sz]:

**Theorem 2.1.** *(Heine-Stieltjes) Let $A(x)$ be the polynomial of degree $N+1$ given by*

$$A(x) = (x-\alpha_0)\cdot(x-\alpha_1)\cdots(x-\alpha_N),$$

*where $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N$ and $B(x)$ a polynomial of degree $N$ satisfying the condition*

$$\frac{B(x)}{A(x)} = \frac{\beta_0}{x-\alpha_0} + \cdots + \frac{\beta_N}{x-\alpha_N},$$

*for given numbers $\beta_\nu > 0$, $\nu = 0, ..., N$. Then, there are exactly $\sigma(K,N) = \frac{(N+K-1)!}{K!\,(N-1)!}$ polynomials $C(x)$ of degree $N-1$ for which the differential equation*

$$A(x)\,\frac{d^2\phi}{dx^2} + 2B(x)\,\frac{d\phi}{dx} + C(x)\,\phi = 0 \tag{6}$$

*has a polynomial solution of degree $K > 0$. In addition, for each of the $\sigma(K,N)$ solutions, $\phi(x)$, the zeroes are simple and uniquely determined by their distribution in the intervals $(\alpha_0,\alpha_1), \ldots, (\alpha_{N-1},\alpha_N)$.*

Taking into account the Heine-Stieltjes result, we henceforth denote the zeroes of any Heine-Stieltjes polynomials $\phi_K(x) \in \Omega(K,N)$ by $\theta_1(\alpha;\underline{l}), \dots, \theta_K(\alpha;\underline{l})$, where $\alpha := (\alpha_0, \dots, \alpha_N)$ whereas $\underline{l} = (l_1, \dots, l_K)$, $1 \le l_1 \le \cdots \le l_K \le N$ denotes the configuration of the zeroes. By this we mean that $\theta_1(\alpha;\underline{l})$ is the smallest zero lying in the interval $(\alpha_{l_1-1}, \alpha_{l_1})$, the next zero $\theta_2(\alpha;\underline{l})$ is contained in the interval $(\alpha_{l_2-1}, \alpha_{l_2})$ and so on.

## 3. Mean density of states for the Heine-Stieltjes polynomials

Let $\phi \in BV([0,1])$ with total variation $V(\phi)$. As a consequence of the notation introduced in Section 2, we can rewrite Eq. (3) in the form:

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{\sigma(K,N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^{K} \phi(\theta_j(\alpha;\underline{l})). \tag{7}$$

Here, we use the notation $|\underline{l}| = K$ to denote all $K$-tuples $\underline{l} = (l_1, ..., l_K)$ of positive integers satisfying the conditions $1 \le l_1 \le \ldots \le l_K \le N$. Our first result is about the asymptotics of the averaged measure $d\rho_{AV}^{DS}$:

**Proposition 3.1.** *Let $\phi \in BV([0,1])$ and choose* any *sequence $K = K(N)$ with $N \to \infty$. Then,*

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{N+1} \sum_{j=0}^{N} \phi(\alpha_j) + E(N),$$

*where,*

$$|E(N)| \le \frac{2V(\phi)}{N}.$$

*Proof of Proposition 3.1:* The first step is to write $\phi = \phi_+ - \phi_-$ where $\phi_\pm \in BV([0,1])$ are monotone, non-decreasing functions. Split the expression

$$d\rho_{DS}^{AV}(\phi)(\alpha) = d\rho_{DS}^{AV}(\phi_+)(\alpha) - d\rho_{DS}^{AV}(\phi_-)(\alpha). \tag{8}$$

We do the computations for $d\rho_{AV}^{DS}(\phi_+;K,N)$ and $d\rho_{AV}^{DS}(\phi_-;K,N)$ separately. We call $\alpha_j$ *interior* if $j \ne 0, N$. The points $\alpha_0$ and $\alpha_N$ are boundary points. Then,

$$d\rho_{AV}^{DS}(\phi_\pm)(\alpha) = \frac{1}{\sigma(K,N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^{K} \phi_\pm(\alpha_{l_j-1}) + E(N), \tag{9}$$

where,

$$\begin{aligned} E(N) &= \frac{1}{\sigma(K,N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^{K} [\phi_\pm(\theta_j(\alpha;\underline{l})) - \phi_\pm(\alpha_{l_j-1})] \\ &\le \frac{1}{\sigma(K,N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^{K} [\phi_\pm(\alpha_{l_j}) - \phi_\pm(\alpha_{l_j-1})]. \end{aligned} \tag{10}$$

Here, the line (10) follows from the monotonicity of $\phi_\pm$ and the inequalities,

$$\alpha_{l_j-1} < \theta_j(\alpha;\underline{l}) < \alpha_{l_j},$$

that are consequences of the Heine-Stieltjes Theorem.

Next, we claim that

$$\frac{1}{\sigma(K,N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^{K} [\phi_\pm(\alpha_{l_j}) - \phi_\pm(\alpha_{l_j-1})] = \frac{\text{card}(\{\underline{l}; l_1 = 1\})}{\sigma(K,N)} \phi_\pm(\alpha_0) - \frac{\text{card}(\{\underline{l}; l_K = N\})}{\sigma(K,N)} \phi_\pm(\alpha_N). \tag{11}$$

The identity in (11) is proved as follows: Let $m > K$ and fix any $m - K$ zeroes $x_1, ..., x_{m-K}$ with $m$ zeroes $x_{m-K+1}, ..., x_K \in (\alpha_j, \alpha_{j+1})$. Then by Heine-Stieltjes, for any such $m$ we can choose a corresponding configuration with zeroes $y_1, ..., y_{m-K}$ in exactly the same relative position versus the $\alpha$'s as $x_1, ..., x_{m-K}$

and put $y_{m-K+1}, ..., y_K \in (\alpha_{j-1}, \alpha_j)$. It follows that $\phi_\pm(\alpha_j)$ appears with the same number of plus and minus signs in (11) as long as $\alpha_j$ is an interior point. So, we are left with computing the total number of occurrences of $-\phi_\pm(\alpha_0)$ and $\phi_\pm(\alpha_N)$ in (11). This number is

$$\frac{\operatorname{card}(\{l; l_1 = 1\})}{\sigma(K,N)} = \frac{\operatorname{card}(\{l; l_K = N\})}{\sigma(K,N)} = \frac{\sum_{m=0}^{K} m \cdot \sigma(N-1, K-m)}{K \cdot \sigma(N,K)}. \tag{12}$$

The computation of the sum in the right-hand side of (12) follows from the following elementary counting lemma:

**Lemma 3.2.** *We have that*

$$\sum_{m=0}^{K} m \cdot \sigma(N-1, K-m) = \frac{K \cdot \sigma(N,K)}{N}.$$

*Proof of Lemma 3.2:* Switching two intervals $[\alpha_{i-1}, \alpha_i]$ and $[\alpha_{j-1}, \alpha_j]$ (i.e. a permutation of i and j) leaves a given configuration invariant iff the two intervals contain the same number of zeroes of the configuration. Otherwise it switches the two numbers. Therefore either a configuration has the same number of zeroes in the two intervals or it is mapped under the permutation to a different (unique) configuration having exactly the two numbers switched. Therefore if we take the union of all the zeroes of all the configurations each $\alpha$ interval contains exactly the same number of zeroes and since the total number of zeroes is $N\sigma(K,N)$ each of the $N$ intervals has to contain exactly $M = \frac{K\sigma(K,N)}{N}$ zeroes. □

Applying Lemma 3.2 together with Eqs. (8), (9), (10), (11) and (12) gives

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{N+1} \sum_{\nu=0}^{N} \phi(\alpha_\nu) + E(N),$$

where,

$$|E(N)| \leq \frac{\phi_+(\alpha_N) - \phi_+(\alpha_0)}{N} + \frac{\phi_-(\alpha_N) - \phi_-(\alpha_0)}{N} \leq \frac{2V(\phi)}{N}.$$

This completes the proof of Proposition 3.1. □

## 4. Approximation

We proved in Proposition 3.1 that as the number of $\alpha$'s becomes arbitrary large as $(N \to \infty)$,

$$\frac{1}{\sigma(K,N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^{K} \delta(\theta_j(\alpha; \underline{l})) \to \frac{1}{N+1} \sum_{i=0}^{N} \delta(\alpha_i),$$

with the error being $O(1/N)$. For brevity, we denote the measure $(1/K)\sum_{j=1}^{K} \delta(\theta_j(\alpha; \underline{l}))$ by $d\rho_{DS}(\underline{l})$ and the limiting measure $1/(N+1)\sum_{i=0}^{N} \delta(\alpha_i)$ by $d\mu_N$.

We now want to estimate the *variance*, over the space of all configurations of size $K$, of the difference between the two measures $d\rho_{DS}(\underline{l})$ and $d\mu_N$. We choose a test function $\phi \in \mathrm{BV}([0,1])$, and wish to estimate

$$\operatorname{Var}(\phi) := \frac{1}{\sigma(K,N)} \sum_{|\underline{l}|=K} \left( \int \phi \, d\rho_{DS}(\underline{l}) - \int \phi d\mu_N \right)^2. \tag{13}$$

First, we rewrite the right-hand side of line (13) as

$$\sum_{|\underline{l}|=K} \left[ \frac{1}{\sigma(K,N)K^2} \sum_{i,j=1}^{K} \phi(\theta_i(\alpha; \underline{l}))\phi(\theta_j(\alpha; \underline{l})) + \frac{2}{\sigma(K,N)K} \int \phi \, d\rho_{DS}(\underline{l}) \cdot \int \phi d\mu_N + \left( \int \phi d\mu_N \right)^2 \right].$$

We then use Proposition 3.1 for the sum of the second term in Eq. (14) to conclude that

$$\mathrm{Var}(\phi) = \frac{1}{\sigma(K,N)K^2} \sum_{|\underline{l}|=K} \sum_{i,j=1}^{K} \phi(\theta_i(\alpha;\underline{l}))\phi(\theta_j(\alpha;\underline{l})) - \left(\frac{1}{N+1}\sum_{i=0}^{N}\phi(\alpha_i)\right)^2 + \mathcal{O}\left(\frac{1}{N}\right). \tag{14}$$

As for the proof of Proposition 3.1, we write $\phi = \phi_+ - \phi_-$ where $\phi_\pm$ are monotone increasing, and use the estimates

$$\phi_\pm(\alpha_{l_j-1}) \leq \phi_\pm(\theta_j(\alpha;\underline{l})) \leq \phi_\pm(\alpha_{l_j}), \tag{15}$$

that follows from the Heine-Stieltjes Theorem (see Section 2). We apply (15) to every term in the sum $\sum_{i,j=1}^{K} \phi_\pm(\theta_i(\alpha;\underline{l}))\phi_\pm(\theta_j(\alpha;\underline{l}))$ and add the resulting estimates for all $|\underline{l}| = K$. We get a lower bound for the sum $\sum_{|\underline{l}|=K}\sum_{i,j=1}^{K} \phi_\pm(\theta_i(\alpha;\underline{l}))\phi_\pm(\theta_j(\alpha;\underline{l}))$ of the form

$$L(\phi_\pm) := \sum_{i=0}^{N} L(i)(\phi_\pm(\alpha_i))^2 + \sum_{i<j} L(i,j)\phi_\pm(\alpha_i)\phi_\pm(\alpha_j), \tag{16}$$

where $L(i)$ denotes the number of times $\phi_\pm^2(\alpha_i)$ appears on the left-hand side of (15), and $L(i,j)$ denotes the number of times $\phi_\pm(\alpha_i)\phi_\pm(\alpha_j)$ appears on the left-hand side of (15), when added over all configurations.

Similarly, we get an upper bound for the sum $\sum_{|\underline{l}|=K}\sum_{i,j=1}^{K} \phi_\pm(\theta_i(\alpha;\underline{l}))\phi_\pm(\theta_j(\alpha;\underline{l}))$ of the form

$$U(\phi) := \sum_{i=0}^{N} U(i)(\phi_\pm(\alpha_i))^2 + \sum_{i<j} U(i,j)\phi_\pm(\alpha_i)\phi_\pm(\alpha_j), \tag{17}$$

where $U(i)$ denotes the number of times $\phi_\pm^2(\alpha_i)$ appears on the right-hand side of (15), and $L(i,j)$ denotes the number of times $\phi_\pm(\alpha_i)\phi_\pm(\alpha_j)$ appears on the right-hand side of (15), when added over all configurations.

We now proceed to calculate $L(i), U(i), L(i,j), U(i,j)$.

**Proposition 4.1.** *We have that*

(a) $0 = U(0) = L(N)$.
(b) $U(i) = L(j) = \sigma(N+2, K-1) + \sigma(N+2, K-2)$ *for* $i \neq 0, j \neq N$.
(c) $0 = U(i,0) = L(j,N)$, *for all* $i > 0$ *and* $j < N$.
(d) $U(i,j) = L(i,j) = 2\sigma(N+2, K-2)$, *for all* $i < j$ *not described in (c).*

*Proof of Proposition 4.1:* The proof of (a) and (c) is clear: there are no $\theta$-s to the right of $\alpha_N$ or to the left of $\alpha_0$.

Let $m_j(\underline{l})$ denote the number of zeroes occuring in the interval $(\alpha_{j-1}, \alpha_j); 1 \leq j \leq N$ for the fixed configuration $\underline{l}$. The contribution to $U(i), 1 \leq i \leq N$ from this configuration is equal to $m_i(\underline{l})^2$. Thus,

$$U(i) = \sum_{|\underline{l}|=K} m_i(\underline{l})^2.$$

Similarly, the contribution from $\underline{l}$ to $L(i), 0 \leq i \leq N-1$, is equal to $m_{i+1}(\underline{l})^2$, Thus,

$$L(i) = \sum_{|\underline{l}|=K} m_{i+1}(\underline{l})^2.$$

The statement of (b) now follows from Corollary 6.4.

We finally turn to the proof of (d). The contribution from a configuration $\underline{l}$ to $U(i,j)$ is equal to $2m_i m_j, 1 \leq i < j \leq N$. Accordingly,

$$U(i,j) = \sum_{|\underline{l}|=K} 2m_i(\underline{l})m_j(\underline{l}).$$

Similarly, the contribution from $\underline{l}$ to $L(i,j), 0 \leq i < j \leq N-1$, is equal to $2m_{i+1}(\underline{l})m_{j+1}(\underline{l})$. Thus,

$$L(i,j) = 2 \sum_{|\underline{l}|=K} m_{i+1}(\underline{l})m_{j+1}(\underline{l}).$$

The statement of (d) now follows from Corollary 6.4. □

We next show that $U(\phi_\pm)$ and $L(\phi_\pm)$ provide an asymptotically good approximation for the sum

$$\sum_{|\underline{l}|=K} \sum_{i,j=1}^{K} \phi_\pm(\theta_i(\alpha;\underline{l}))\phi_\pm(\theta_j(\alpha;\underline{l})).$$

Indeed,

**Lemma 4.2.** *Let* $\Delta(\phi_\pm) := (U(\phi_\pm) - L(\phi_\pm))/(K^2\sigma(K,N))$. *Then* $\Delta(\phi_\pm) \to 0$ *as* $N \to \infty$.

*Proof of Lemma 4.2:* From Eqs. (17), (16) and Proposition 4.1, we conclude that

$$\begin{aligned}\Delta(\phi_\pm) &= \frac{\sigma(N+2,K-1)+\sigma(N+2,K-2)}{K^2\sigma(K,N)}(\phi^2(\alpha_N) - \phi_\pm^2(\alpha_0)) \\ &\quad + \frac{2\sigma(N+2,K-2)(\phi_\pm(\alpha_N)-\phi_\pm(\alpha_0))}{K^2\sigma(K,N)} \cdot \sum_{i=1}^{N-1} \phi_\pm(\alpha_j).\end{aligned}$$

Since by definition $\sigma(K,N) = \binom{N+K-1}{N-1}$, we see that

$$\frac{\sigma(N+2,K-2)}{K^2\sigma(K,N)} = \frac{K-1}{KN(N+1)}, \quad \frac{\sigma(N+2,K-1)}{K^2\sigma(K,N)} = \frac{N+K}{KN(N+1)}. \tag{18}$$

It follows from line (18) and the fact that the functions $\phi_\pm$ are bounded, i.e. $|\phi_\pm(\alpha_\nu)| \leq M$ for some fixed $M > 0$ and all $\nu$,

$$\Delta(\phi_\pm) \leq 2M^2 \left[\frac{N+2K-1}{KN(N+1)} + \frac{(K-1)N}{KN(N+1)}\right] = \mathcal{O}\left(\frac{2}{N^2} + \frac{1}{KN} + \frac{1}{N}\right) = \mathcal{O}\left(\frac{1}{N}\right).$$

This shows that $\Delta(\phi_\pm) \to 0$ as $N \to \infty$ and completes the proof of Lemma 4.2. □

## 5. Variance

As a consequence of Eq. (14) and Lemma 4.2, we see that

$$\operatorname{Var}(\phi_\pm) = \frac{U(\phi_\pm)}{K^2\sigma(K,N)} - \left(\frac{1}{N+1}\sum_{i=0}^{N}\phi_\pm(\alpha_i)\right)^2 + \mathcal{O}\left(\frac{1}{N}\right).$$

The last expression can be rewritten as

$$\operatorname{Var}(\phi_\pm) = S_1 + S_2 + S_3 + \mathcal{O}(1/N),$$

where

$$S_1 := \sum_{i=1}^{N}\left(\frac{\sigma(N+2,K-2)+\sigma(N+2,K-1)}{K^2\sigma(K,N)} - \frac{1}{(N+1)^2}\right)\phi_\pm^2(\alpha_i),$$

where

$$S_2 := 2\sum_{1\leq i<j\leq N}\left(\frac{\sigma(N+2,K-2)}{K^2\sigma(K,N)} - \frac{1}{(N+1)^2}\right)\phi_\pm(\alpha_i)\phi_\pm(\alpha_j),$$

and where

$$-S_3 := \frac{\phi_\pm(\alpha_0)}{(N+1)^2}\sum_{j=0}^{N}\phi_\pm(\alpha_j).$$

It is clear that $S_3 = \mathcal{O}(1/N)$. We proceed to estimate $S_1$ and $S_2$. Using (18), we see that

$$S_2 \leq \frac{2M^2N(N-1)}{2}\left(\frac{K-1}{KN(N+1)} - \frac{1}{(N+1)^2}\right) = \mathcal{O}\left(\frac{1}{K} + \frac{1}{N}\right).$$

Finally, using (18) once more, we see that

$$S_1 \leq NM^2\left(\frac{N+2K-1}{KN(N+1)} - \frac{1}{(N+1)^2}\right) = \mathcal{O}\left(\frac{1}{K} + \frac{1}{N}\right).$$

Summing up, we have

$$\mathrm{Var}(\phi_\pm) = \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right). \tag{19}$$

We now come to the main theorem of this section.

**Theorem 5.1.** *Let $\phi \in BV([0,1])$. Then, $Var(\phi) \to 0$ as $K \to \infty$ and $N \to \infty$. More precisely, we have*

$$Var(\phi) = \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right).$$

*Proof of Theorem 5.1:* Since $\phi = \phi_+ - \phi_-$, it follows from Eq. (19) that

$$\begin{aligned}\mathrm{Var}(\phi) &= \mathrm{Var}(\phi_+) + \mathrm{Var}(\phi_-) - 2\mathrm{Cov}(\phi_+, \phi_-)\\ &= \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right) - 2\mathrm{Cov}(\phi_+, \phi_-),\end{aligned} \tag{20}$$

where $\mathrm{Cov}(\phi_+, \phi_-)$ denotes the covariance of $\phi_+$ and $\phi_-$. Moreover, we can apply Proposition 3.1 to conclude that

$$\begin{aligned}\mathrm{Cov}(\phi_+, \phi_-) &= \frac{1}{\sigma(K,N)} \sum_{|\underline{l}|=K} \left[\int \phi_+ d\rho_{DS}(\underline{l}) - \int \phi_+ d\mu_N\right]\left[\int \phi_- d\rho_{DS}(\underline{l}) - \int \phi_- d\mu_N\right]\\ &\leq \frac{2\|\phi\|_0}{\sigma(K,N)} \left|\sum_{|\underline{l}|=K} \left[\int \phi_+ d\rho_{DS}(\underline{l}) - \int \phi_+ d\mu_N\right]\right|\\ &\leq \frac{4\|\phi\|_0 V(\phi)}{N}\\ &= \mathcal{O}\left(\frac{1}{N}\right).\end{aligned} \tag{21}$$

The rest of the proof is an immediate consequence of Eqs. (20) and (21). □

## 6. Calculating the coefficients

Recall that a configuration $\underline{l}$ has $m_j(\underline{l})$ zeroes in $(\alpha_{j-1}, \alpha_j)$, $1 \leq j \leq N$. Let $0 < p < N$ and let $\{i_1, i_2, \dots, i_p\}$ be a sequence of distinct indices, $1 \leq i_j \leq N$. Finally, let $a_1, a_2, \dots, a_p \in \mathbb{N}$. We want to calculate

$$G(i_1, \dots, i_p; a_1, \dots, a_p) := \sum_{|\underline{l}|=K} \prod_{j=1}^{p} (m_{i_j}(\underline{l}))^{a_j}. \tag{22}$$

We shall give an answer in terms of the coefficients of the polynomial defined as follows: let $a_1 + a_2 + \dots + a_p = A$. Define a polynomial $P(x) := b_0 + b_1 x + b_2 x^2 + \dots$ by

$$\prod_{j=1}^{p}\left[\left(x\frac{d}{dx}\right)^{a_j}\left(\frac{1}{1-x}\right)\right] = \frac{P(x)}{(1-x)^{p+A}}. \tag{23}$$

The polynomial $P(x)$ in (23) is a product of "elementary" polynomials $P_q(x)$ defined by the formula

$$\left(x\frac{d}{dx}\right)^{q}\left(\frac{1}{1-x}\right) := \frac{P_q(x)}{(1-x)^{q+1}}.$$

We let $P_q(x) = \sum_{m=1}^{q} b_{q,m} x^m$. In proposition 6.1, we compute the numbers $b_{q,m}$:

**Proposition 6.1.** *Let $q \geq 1$, and let $b_{q,m}$ be as above. Then*

$$b_{q,m} = \sum_{k=1}^{m} k^q (-1)^{m-k} \binom{q+1}{m-k}.$$

*Proof of Proposition 6.1:* By definition,

$$P_q(x) = (1-x)^{q+1} \cdot \left( x \frac{d}{dx} \right)^q \left( \frac{1}{1-x} \right) = (1-x)^{q+1} \left( \sum_{k=1}^{\infty} k^q x^k \right).$$

The proposition now follows by expanding $(1-x)^{q+1}$ and multiplying the two series. □

We are now in a position to compute the expression $G(i_1, \dots, i_p; a_1, \dots, a_p)$ defined in (22).

**Theorem 6.2.** *Let $b_j$-s be the coefficients of a polynomial $P$ defined by (23), and let $K$ be the total number of zeroes in the configuration. Then*

$$G(i_1, \dots, i_p; a_1, \dots, a_p) = \sum_{j=0}^{K} b_j \sigma(N+A, K-j).$$

*Proof of Theorem 6.2:* Define a bijection $\rho$ between the set $\{\underline{l} : |\underline{l}| = K\}$ and monomials of degree $K$ in $N$ variables by

$$\rho(\underline{l}) := x_1^{m_1} x_2^{m_2} \dots x_N^{m_N}.$$

Here $K = m_1 + m_2 + \dots + m_N$. We need the following lemma.

**Lemma 6.3.** *The sum*

$$\sum_{|\underline{l}|=K} \rho(\underline{l})$$

*consists of all monomials of total degree $K$ in the infinite series*

$$F(x_1, \dots, x_N) := \prod_{i=1}^{N} \frac{1}{(1-x_i)} = \prod_{i=1}^{N} (1 + x_i + x_i^2 + \dots).$$

*We remark that $F(x_1, \dots, x_N)$ thus defines the generating series for the set of* all *configurations with $N$ intervals.*

*Proof of Lemma 6.3:* Choose $x_1^{m_1}$ from the first bracket $(1 + x_1 + x_1^2 + \dots)$, $x_2^{m_2}$ from the second bracket etc. □

If we want to compute the *number* of such monomials, we just let $x_j = x$ and compute the coefficient of $x^K$ in the series expansion of $1/(1-x)^N$. The answer is $\sigma(K, N) = \binom{N+K-1}{N-1}$.

To compute $G(i_1, \dots, i_p; a_1, \dots, a_p)$, we remark that

$$\prod_{j=1}^{p} \left( x_j \frac{\partial}{\partial x_j} \right)^{a_j} (\rho(\underline{l})) = \prod_{j=1}^{p} (m_j(\underline{l}))^{a_j} \rho(\underline{l}).$$

Summing over all $|\underline{l}| = K$, we see that we need the sum of coefficients of monomials of degree $K$ in the series

$$\prod_{j=1}^{p} \left( x_j \frac{\partial}{\partial x_j} \right)^{a_j} F(x_1, \dots, x_N).$$

To compute the sum, we let $x_j = x$; the generating series then becomes equal to

$$\frac{P(x)}{(1-x)^{N+A}},$$

by the definition (23) of $P(x)$. Since the degree of $(1-x)$ in the denominator is equal to $N+A$, we see that the answer will correspond to configurations with $N+A$ intervals. The answer then follows from the multiplication of $P(x)$ by the series $1/(1-x)^{N+A}$. $\square$

The following computations are used in Section 5.

**Corollary 6.4.** *For any* $1 \leq i \leq N$, $\sum_{|\underline{l}|=K} m_i(\underline{l}) = \sigma(N+1, K-1)$ *and* $\sum_{|\underline{l}|=K} (m_i(\underline{l}))^2 = \sigma(N+2, K-1) + \sigma(N+2, K-2)$. *For any* $1 \leq i < j \leq N$, $\sum_{|\underline{l}|=K} m_i(\underline{l}) m_j(\underline{l}) = \sigma(N+2, K-2)$.

*Proof of Corollary 6.4:* The corollary follows from the Theorem 6.2 and the following two identities:

$$\left(y\frac{\partial}{\partial y}\right)\left(\frac{1}{1-y}\right) = \frac{y}{(1-y)^2} \quad \textit{and} \quad \left(y\frac{\partial}{\partial y}\right)^2\left(\frac{1}{1-y}\right) = \frac{y+y^2}{(1-y)^3}. \quad \square$$

## 7. Asymptotic distribution of zeroes

There is a rather simple corollary of Theorem 5.1 that we would like to address in this section. Fix a configuration $\underline{l}$ with $|\underline{l}| = K$ and consider the average

$$\overline{\theta}(\alpha; \underline{l}) := \frac{1}{K}\sum_{j=1}^{K} \theta_j(\alpha; \underline{l}). \tag{24}$$

Recall, a sequence of measurable subsets $U_N; N = 1, 2, ...$ of a probability space $(X, \mu)$ is said to be of full density if $\lim_{N\to\infty} \mu(U_N) = 1$.

Then, by the Chebyshev inequality it follows from Theorem 5.1 that for any $\alpha \in \Lambda^N$, there exists a full density subset of configurations $D_N(\alpha) \subset \Omega(K, N)$, such that for any $\underline{l} \in D_N(\alpha)$ and $\delta > 0$,

$$\overline{\theta}(\alpha; \underline{l}) = \frac{1}{N+1}\sum_{\nu=0}^{N} \alpha_\nu + \mathcal{O}_\delta\left(\max\{N^{-1/2+\delta}, K^{-1/2+\delta}\}\right). \tag{25}$$

Let $[0,1]^\infty$ denote the set of all infinite sequences $\{\alpha_j\}_{j=0}^\infty$ with $\alpha_j \in [0,1]$. We embed $[0,1]^{N+1}$ in $[0,1]^\infty$ via the map $(\alpha_0, ..., \alpha_N) \mapsto (\alpha_0, ..., \alpha_N, 0, 0, ....)$. Similarly, we define $\Lambda^\infty \subset [0,1]^\infty$ to consist of all strictly increasing sequences with $\alpha_0 < \alpha_1 < \cdots$ and $\Lambda^N \subset \Lambda^\infty$ in the same way.

Given any $\sigma \in S_N$, the symmetric group on $N$-elements, it is clear that the estimate in (25) extends pointwise to $\sigma(\Lambda^N)$ since

$$\frac{1}{N+1}\sum_{\nu=0}^{N} \alpha_{\sigma(\nu)} = \frac{1}{N+1}\sum_{\nu=0}^{N} \alpha_\nu.$$

On $\sigma(\Lambda^N)$ the functions $\theta_j(\alpha; \underline{l})$ are defined in the same way as on $\Lambda^N$ except that $\alpha_{\sigma(j)}$ replaces $\alpha_j$ for $j = 0, ..., N$. So, the estimate (25) holds Lebesgue a.e. on $[0,1]^{N+1}$ and consequently, Lebesgue a.e. on $[0,1]^\infty$ since the $\overline{\theta}(\alpha; \underline{l})$ only depend on the first $N+1$ coordinates $(\alpha_0, ..., \alpha_N)$. We delete here the set $D$ consisting of points $\alpha = (\alpha_0, ..., \alpha_N) \in [0,1]^{N+1}$ where at least two of the $\alpha_j$'s coincide.

Since the monomials $\alpha_\nu; \nu = 0, ..., N$ are independent, identically distributed random variables on $[0,1]^\infty$, by the strong law of large numbers,

$$\frac{1}{N+1}\sum_{\nu=0}^{N} \alpha_\nu = \frac{1}{2} + \mathcal{O}_\delta(N^{-1/2+\delta}) \tag{26}$$

with probability one on $[0,1]^\infty$. Combining (25) and (26) gives:

**Theorem 7.1.** *Let $N, K \to \infty$. Then, for Lebesgue almost every $\alpha \in [0,1]^\infty$ there exist a full density of configurations, $D_N(\alpha)$, such that for any $\underline{l} \in D_N(\alpha)$, $|\underline{l}| = K$, and any $\delta > 0$,*

$$\overline{\theta}(\alpha; \underline{l}) = \frac{1}{2} + \mathcal{O}_\delta \left( \max\{N^{-1/2+\delta}, K^{-1/2+\delta}\} \right).$$

*Remark:* It would be interesting to see whether one can prove a central limit theorem for the $\theta_j(\alpha)$'s. For this, we would need an estimate for $\theta_j(\alpha)$ in terms of the monomials $\alpha_j; j = 0, ..., N$ that is better than the bound that one gets from the Heine-Stieltjes theorem. We hope to address this in future work.

## References

[B] A. Bourget, Nodal Statistics for the Van Vleck Polynomials, Comm. Math. Phys. 230 (2002), 503-516.

[BT] A. Bourget and J.A. Toth. Asymptotic statistics of zeroes for the Lamé ensemble. Comm. Math. Phys. 222 (2001), no. 3, 475–493.

[H] M. Heine, Handbuch der Kugelfunctionen, Bd. I (2nd ed.), G. Reimer, Berlin 1878.

[HW] J. Harnad and P. Winternitz, Harmonics on hyperspheres, separation of variables and the Bethe ansatz, Lett. Math. Phys. (33) (1995), 61-74.

[K] V.B. Kuznetsov, Quadrics on real Riemannian spaces of constant curvature: Separation of variables and connection with Gaudin magnet, J. Math. Phys. 33(9) (1992), 3240-3254.

[KKN] E.G. Kalnins, V.B. Kuznetsov and W. Miller Jr. Quadrics on complex Riemannian spaces of constant curvature: Separation of variables and the Gaudin magnet, J. Math. Phys. 35(4) (1994), 1710-1731.

[KM] E.G. Kalnins and W. Miller Jr. Separable coordinates, integrability and the Niven equations, J. Phys. A 25(4) (1992), 5663-5675.

[MS] A.Martinez-Finkelstein and E. B. Saff, Asymptotic properties of Heine-Stieltjes and Van Vleck polynomials, J. Approx. Theory 118 (2002), no. 1, 131-151.

[Sk] E.K. Sklyanin, Separation of variables in the Gaudin model, J. Sov. math. 47 (1989) (1987), 2473-2488.

[St] T.J. Stieltjes, Sur certains polynômes que vérifient une équation différentielle linéaire du second ordre et sur la théorie des fonctions de Lamé, Acta Math. (6) (1885), 321-326.

[Sz] G. Szegö, Orthogonal Polynomials, Vol. 23 (third edition), Amer. Math. Soc., Providence, RI, 1967.

[T1] J.A. Toth, The quantum C. Neumann problem, Internat. Math. Res. Notices (5) (1993), 137-139.

[T2] J.A. Toth, Various quantum mechanical aspects of quadratic forms, J. Funct. Anal. (130) (1995), 1-42.

[WW] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, (fourth edition), Camb. Univ. Press, Cambridge U.K., 1963.

First and third authors: Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1, Second and fourth authors: Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montreal, Quebec, Canada H3A 2K6