L^2 Vanishing Theorems in Positive Curvature

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1. Introduction. In [EF1] and [EF2], the first two authors combined integral identities with estimates for solutions of the Riccati equation to show nonexistence of L^2 eigenforms of the Laplacian (on functions and differential forms) on certain complete noncompact manifolds of nonnegative sectional curvature. In this note we remark that the identity used to prove vanishing of L^2 harmonic forms away from the middle degree also holds for vector-bundle-valued forms and can be applied to prove nonexistence theorems for harmonic maps and Yang-Mills fields. In fact, both the identity and the applications were considered previously by H. Karcher and J. C. Wood [KW], following work of H. C. Sealey [S]. We simply want to point out that the geometric hypotheses that are needed are the same as in [EF2], and more easily stated than in [KW] or [S]. Our main result is:

Theorem A. *Let M be a complete noncompact n-manifold, E a Riemannian vector bundle on M equipped with a metric-compatible connection. Fix $0 < p < n/2$. Assume there exists a C^1 vector field X on M such that:*

(i) $|X| \leq \Phi \circ r$ *and* $|\nabla X|$ *is bounded on* M, *where* r *denotes distance to a compact set* $S \subset M$ *and* $\int_1^\infty 1/\Phi(t)\,dt = \infty$;

(ii) $2p|\nabla X| < \operatorname{div} X$ *pointwise on* M.

Then there are no L^2 harmonic forms in $\Omega_M^p \otimes E$.

The main interest of this result is that there are several geometric situations where the existence of a vector field with properties (i) and (ii) can be verified; note also that these conditions are stable with respect to C^1-small perturbations of the metric (keeping X fixed). In particular, we obtain a geometric theorem for manifolds of nonnegative curvature.

Recall that if M is a complete noncompact manifold of nonnegative curvature, a basic theorem of Cheeger and Gromoll [CG] asserts the existence of a compact totally convex (in particular, totally geodesic) submanifold S of M

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without boundary, whose normal bundle is diffeomorphic to M. The diffeomorphism is in general not given by the exponential map. S reduces to a point if M has positive curvature, or if M has a pole.

Theorem B. *Let M be a complete noncompact n-manifold with nonnegative sectional curvature; assume M has a soul S whose exponential map is a diffeomorphism, of dimension $s < n-1$.*

(i) *Fix an integer $0 < p < n/2$. If the sectional curvature in directions normal to S satisfies*

$$0 \leq K \leq \frac{c(1-c)}{r^2}, \quad r(x) = d(x, S),$$

where $c \in ((2p-1)/(n-s-1), 1)$, then there are no L^2 harmonic forms in $\Omega^p \otimes E$. In particular:

(ii) *Assume $n \geq 3$, $n-s \geq 3$. Then if $c \in (1/(n-s-1), 1)$, there are no non constant harmonic maps $v : M \to N$ with finite energy, for arbitrary targets N.*

(iii) *Assume $n \geq 5$, $n-s \geq 5$, and $c \in (3/(n-s-1), 1)$. Then for arbitrary compact Lie groups G and principal G-bundles $P \to M$, any connection on P satisfying the Yang-Mills equations whose curvature form $F \in \Omega^2_M \otimes \operatorname{ad} P$ satisfies $\int_M |F|^2(x)\, dx < \infty$, is flat.*

Part (iii) generalizes the well-known fact that there are no nontrivial Yang-Mills fields in R^n with finite action for $n \geq 5$ (in contrast with R^4, where the problem is conformally invariant and one obtains finite-action fields by pullback from S^4); see e.g., [JT].

Remarks.

1. If N has nonpositive sectional curvature, there are no harmonic maps with finite energy from M to N, where M is any complete noncompact manifold with nonnegative Ricci curvature. This was proved by R. Schoen and S. T. Yau via the Bochner identity for harmonic maps ([SY]).
2. The result (ii) is false if $n = 2$; for example, there are many holomorphic (hence harmonic) maps $C \to S^2$ with finite energy.
3. For manifolds with pole of nonnegative curvature, quadratic decay of the sectional curvatures is 'generic' in the following sense: decay *faster* than quadratic implies flatness ([GW]), and it is easy to construct rotationally symmetric examples with quadratic decay of the radial sectional curvatures. For example, the radial sectional curvatures of the metric in R^n $ds^2 = dr^2 + g^2(r)\, d\omega^2$ with

$$g(r) = \frac{r}{(1+r^2)^{\alpha/2}}$$

satisfying $0 \leq K \leq k/r^2$ provided $0 \leq \alpha \leq k/3$. As remarked above, small perturbations of this example provide non symmetric manifolds satisfying the hypotheses of Theorem A. On the other hand, the result is sensitive to the value of the constant in the quadratic decay condition in (i). As described in [EF2], there are examples (due to DaGang Yang) of 4-manifolds with pole of $K \geq 0$, whose radial sectional curvatures satisfy

$$0 \leq K \leq \frac{3/4}{r^2}$$

and which carry nontrivial harmonic two-forms. From this it is easy to see that Theorem B(i) does not hold if we replace $c(1-c)$ by $\frac{3}{4}$.

4. The Riemannian product of a manifold with a pole of nonnegative curvature and a compact manifold of nonnegative curvature satisfies the assumptions of Theorem B. Conversely, as shown in Corollary 2.3 of [EF2], a curvature decay condition as above implies the manifold splits locally (i.e., the universal cover is a product of a compact manifold and a manifold with pole). The conclusion of the theorem, however, is not invariant under infinite coverings.
5. The results corresponding to Theorem B in negative curvature follow from the work of H. Donnelly and F. Xavier [DX]. We state them here for completeness: let M^n be simply connected, with sectional curvatures $-1 \leq K \leq -1+\epsilon$. If $n \geq 4$ and $0 < \epsilon < 1 - 4/(n-1)^2$, there are no non constant harmonic maps with finite energy defined on M (this is also true in hyperbolic 3-space, by Sealey's results [S]). If $n \geq 6$ and $0 < \epsilon < 1 - 16/(n-1)^2$, M supports no non flat Yang-Mills connections with curvature form in L^2 (again, the case of hyperbolic 5-space is contained in [S]).

Condition (ii) in theorem A simplifies when X is a conformal vector field. In the following theorem, no curvature assumptions are required.

Theorem C. *Let $M = R^n$ endowed with a complete metric $g = e^{2v} g_0$ conformally related to the Euclidean metric g_0. Let E be a vector bundle over M with Riemannian metric and connection. Assume there exists a point $p_0 \in R^n$ such that the Euclidean spheres with center p_0 have nonnegative mean curvature with respect to the metric g. Then there are no L^2 harmonic forms in $\Omega^p \otimes E$ for $p \neq n/2$. In particular, if $n \geq 3$ there are no non constant harmonic maps $v : M \to N$ with finite energy, for arbitrary targets N; and if $n \neq 4$ there are no non flat Yang-Mills connections with curvature form in L^2, for any principal bundle over M.*

2. The Basic Identity. We begin by recalling the identity in (2.5) of [KW] and (1.2) of [EF2]; for completeness we include a short proof closer to that of [EF2].

Let M be a Riemannian manifold and $E \to M$ a vector bundle with a Riemannian metric on its fibers and a metric-compatible connection ∇. Combined with the Levi-Civita connection on M, ∇ defines a connection (still denoted ∇) and an exterior derivative on E-valued differential forms $u \in \Omega_M^p \otimes E$. For a vector field X and 1-form $\vartheta \in \Omega_M^1$, we extend the exterior product and contraction to $\Omega_M^p \otimes E$ by:

$$\vartheta \wedge (\omega \otimes e) = (\vartheta \wedge \omega) \otimes e,$$
$$i_X(\omega \otimes e) = (i_X \omega) \otimes e.$$

With these definitions, i_X is the adjoint of left exterior multiplication by ϑ, if ϑ is dual to X (as in Ω_M^p). It then follows easily from

$$du = d\omega \otimes e + (-1)^p \omega \wedge \nabla e \qquad (u = \omega \otimes e \in \Omega_M^p \otimes E)$$

that for $u \in \Omega_M^p \otimes E$, we have as for ordinary p-forms:

$$du = \sum_i (\vartheta_i \wedge \nabla_{X_i})u,$$

for an arbitrary local orthonormal frame $\{X_i\}$ with co-frame $\{\vartheta_i\}$. Denoting by $\delta : \Omega_M^p \otimes E \to \Omega_M^{p-1} \otimes E$ the adjoint of d, we also have:

$$\delta u = -\sum_i i_{X_i} \nabla_{X_i} u \qquad (\{X_i\} \text{ orthonormal}).$$

Since we need to keep track of boundary terms, we verify this fact. Let Z be the vector field on M defined by $\langle Z, X\rangle = \langle u, i_X v\rangle$ for all vector fields X; here $u \in \Omega_M^p \otimes E$ and $v \in \Omega_M^{p+1} \otimes E$ are given arbitrarily. Choosing $\{X_i\}$ orthonormal so that $\nabla_{X_i} X_j(p) = 0$ at a fixed point $p \in M$, we compute:

$$\begin{aligned}
\operatorname{div} Z &= \sum_i \langle \nabla_{X_i} Z, X_i\rangle = \sum_i X_i \langle Z, X_i\rangle \\
&= \sum_i \{\langle \nabla_{X_i} u, i_{X_i} v\rangle + \langle u, \nabla_{X_i} i_{X_i} v\rangle\} \\
&= \sum_i \{\langle \vartheta_i \wedge \nabla_{X_i} u, v\rangle + \langle u, i_{X_i} \nabla_{X_i} v\rangle\} \\
&= \langle du, v\rangle - \langle u, \delta v\rangle,
\end{aligned}$$

where the fourth equality follows from

$$\nabla_X i_Y u = i_Y \nabla_X u - i_{\nabla_X Y} u,$$

for arbitrary X, Y and u. Thus for any bounded open set $D \subseteq M$ with smooth boundary ∂D, we have

$$\int_D \langle du, v\rangle = \int_D \langle u, \delta v\rangle + \int_{\partial D} \langle u, i_\nu v\rangle,$$

where ν denotes the unit outward normal vector field of ∂D. We now consider Lie derivatives with respect to a vector field X. We may extend the Lie derivative L_X in Ω^p_M and the connection ∇ on E to a first order differential operator L_X on $\Omega^*_M \otimes E$ defined by:

$$L_X u(X_1, \dots, X_p) = \nabla_X \big(u(X_1, \dots, X_p)\big) - \sum_i u\big(X_1, \dots, [X, X_i], \dots, X_p\big).$$

It follows from $[X, X_i] = \nabla_X X_i - \nabla_{X_i} X$ that on $\Omega^*_M \otimes E$:

$$L_X u = \nabla_X u + p(\nabla X) u, \tag{1}$$

where $\nabla X : \Omega^p_M \otimes E \to \Omega^p_M \otimes E$ is the linear endomorphism extending the transformation

$$(\nabla X) Z = \nabla_Z X$$

on vector fields. Explicitly,

$$\big[(\nabla X) u\big](X_1, \dots, X_p) = \frac{1}{p} \sum_i u\big(X_1, \dots, (\nabla X) X_i, \dots, X_p\big). \tag{2}$$

It is verified in p. 170 of [KW] that with this definition one has

$$L_X u = \frac{\nabla}{dt} \varphi_t^* u|_{t=0},$$

where φ_t is the flow of X. To extend the identity $L_X = d i_X + i_X d$ to $\Omega^*_M \otimes E$, we observe that since

$$L_X(\omega \otimes e) = (L_X \omega) \otimes e + \omega \otimes \nabla_X e,$$

we have for $u = \omega \otimes e \in \Omega^p_M \otimes E$:

$$d(i_X u) = d(i_X \omega \otimes e) = d i_X \omega \otimes e + (-1)^{p-1} (i_X \omega) \wedge \nabla e$$

and

$$\begin{aligned} i_X(du) &= i_X\big(d\omega \otimes e + (-1)^p \omega \wedge \nabla e\big) \\ &= (i_X d\omega)\otimes e + (-1)^p (i_X\omega)\wedge \nabla e + (-1)^{2p}\omega \otimes \nabla_X e, \end{aligned}$$

so

$$(di_X + i_X d)u = (L_X\omega)\otimes e + \omega\otimes \nabla_X e = L_X u, \tag{3}$$

which easily implies the same formula for arbitrary $u \in \Omega^p_M \otimes E$. Combining (1) and (3) we obtain the formula

$$\langle di_X u, v\rangle + \langle du, X^* \wedge v\rangle = \langle \nabla_X u, v\rangle + p\langle (\nabla X)u, v\rangle,$$

which holds pointwise on M. Integrating on $D \Subset M$, we obtain:

$$\int_D p\langle(\nabla X)u,v\rangle = \int_D \langle i_X u, \delta v\rangle + \int_D \langle du, X^*\wedge v\rangle - \int_D \langle \nabla_X u, v\rangle + \int_{\partial D} \langle i_X u, i_\nu v\rangle. \tag{4}$$

Setting $u = v$, integration by parts yields:

$$\int_D \left[p\langle(\nabla X)u,u\rangle - \frac{1}{2}(\operatorname{div} X)\,|u|^2\right] = \int_D \langle i_X u, \delta u\rangle + \int_D \langle du, X^*\wedge u\rangle + \int_{\partial D}\left[\langle i_X u, i_\nu u\rangle - \frac{1}{2}\langle X,\nu\rangle\,|u|^2\right]. \tag{5}$$

This immediately implies the following lemma (cf. Prop 2.5 of [KW]):

Lemma 1. *Let $u \in \Omega^p_M \otimes E$ be harmonic (that is, $du = 0 = \delta u$). Then for any smooth bounded domain $D \Subset M$ and smooth vector field X we have:*

$$\int_D \left\{2p\langle(\nabla X)u,u\rangle - (\operatorname{div} X)\,|u|^2\right\} = \int_{\partial D}\left\{2\langle i_X u, i_\nu u\rangle - \langle X,\nu\rangle\,|u|^2\right\}.$$

Remark. For any $T \in \operatorname{End}(TM)$ and its extension to Ω^p_M, the function $\langle T\omega,\omega\rangle$ depends only on the symmetric part of T, defined by

$$\langle T^s X, Y\rangle = \frac{1}{2}\{\langle TX,Y\rangle + \langle X, TY\rangle\}.$$

This follows from the fact that

$$T\omega = \frac{1}{p}\sum_{i,j}\langle TX_i, X_j\rangle \vartheta_j \wedge i_{X_i}\omega \tag{6}$$

($\{X_i\}$ orthonormal, with co-frame $\{\vartheta_i\}$; our definition differs from that of [EF2] by a factor of $1/p$, included so that the identity on TM extends to the identity on Ω^p_M). Thus

$$\begin{aligned}\langle T\omega,\omega\rangle &= \frac{1}{p}\sum_{i,j}\langle TX_i,X_j\rangle\langle i_{X_i}\omega, i_{X_j}\omega\rangle \\ &= \frac{1}{p}\sum_{i,j}\langle T^sX_i,X_j\rangle\langle i_{X_i}\omega, i_{X_j}\omega\rangle = \langle T^s\omega,\omega\rangle,\end{aligned}$$

since $\langle i_{X_i}\omega, i_{X_j}\omega\rangle$ is symmetric in i and j. This clearly extends to the action on $\Omega^p_M\otimes E$.

Proof of Theorem A. Let $S(R) = \{x \in M \mid r(x) = R\}$. For some sequence $R_i \to \infty$ we must have:

$$\int_{S(R_i)} |X|\,|u|^2 \to 0;$$

otherwise we would have $\Phi(R)\int_{S(R)}|u|^2$ bounded below as $R\to\infty$, which contradicts $u\in L^2$ given the assumption on Φ. Since $|\nabla X|$ is bounded and $u\in L^2$, Lemma 1 implies:

$$\int_M \left[2p\big\langle(\nabla X)^s u,u\big\rangle - (\operatorname{div} X)|u|^2\right] = 0.$$

But hypothesis (ii) of Theorem A implies the integrand is nonpositive on M, hence vanishes identically. This implies $u\equiv 0$ (again by (ii)). □

3. Applications. In this section we prove Theorems B and C.

Proof of Theorem B.

(i) We apply Theorem A to the vector field $X = \nabla(r^2/2)$. Denote the eigenvalues of $(\nabla X) = \text{Hessian}(r^2/2)$ by:

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_{n-1} \le \lambda_n = 1.$$

Recall (see [EF1, Lemma 1.1]) that under the assumption

$$0 \le K \le \frac{c(1-c)}{r^2} \quad \text{on } M - S,$$

we have on $M - S$:

(7) $\lambda_1 = \lambda_2 = \cdots \lambda_s = 0$ if $s = \dim S > 0$, $c \leq \lambda_{s+1} \leq \cdots \lambda_{n-1} \leq \lambda_n = 1$.

This implies

$$2p\lambda_n - \sum_{j=1}^{n} \lambda_j = (2p-1) - \sum_{j=1}^{n-1} \lambda_j < 0 \quad \text{if } (n-s-1)c > 2p-1.$$

Thus X satisfies the hypotheses of Theorem A, and the conclusion of part (i) follows.

(ii) Assume $1/(n-s-1) < c < 1$. Then any $u \in L^2(\Omega^1_M \otimes E)$ which is harmonic vanishes identically on M. In particular, let $v : M \to N$ be a harmonic map. Then $u = dv \in \Omega^1_M \otimes E$ (where $E = v^*TN$), and v has finite energy if and only if $u \in L^2$. We have $du = 0$ (since u is the differential of a map) and $\delta u = \delta dv = -\sum_i (\nabla_{X_i} dv) X_i = 0$ by the harmonic map equation. Hence if v has finite energy, u vanishes identically (and v is constant).

(iii) Assume $3/(n-s-1) \leq c_2 < 1$. Then any $u \in L^2(\Omega^2_M \otimes E)$ which is harmonic vanishes identically on M. Let $P \to M$ be a principal G-bundle (G a compact Lie group). The curvature of a principal connection on P is a 2-form F with values in the associated Lie algebra bundle $E = P \times \mathcal{G}$, defined by the adjoint representation $\mathrm{Ad} : G \to \mathrm{End}\,\mathcal{G}$. The critical points of the functional $\int_M |F|^2(x)\,dx$ satisfying the Yang-Mills equations $dF = 0$, $\delta F = 0$. The fact that $F = 0$ if it is in L^2 follows immediately from part (i). □

Proof of Theorem C. Let (M,g) be as in Theorem C. For a conformal vector field X on M, define $DX = (\nabla X)^s$. We have:

$$\frac{2}{n}(\operatorname{div} X) g = L_X g = 2DX. \tag{8}$$

Using (6), we see that pointwise on M:

$$\langle (DX)u, u\rangle = \frac{\operatorname{div} X}{np} \sum_i |i_{X_i} u|^2 = \frac{\operatorname{div} X}{n} |u|^2.$$

It is straightforward to check using (8) that for any vector field Z on M we have

$$DZ = e^{2v}\big(D_0 Z + Z(v) g_0\big).$$

Taking the trace with respect to the metric g, we get that

$$\mathrm{div}_g Z = \mathrm{div}_{g_0} Z + nZ(v). \tag{9}$$

The vector field $X = \nabla_{g_0}(r^2/2)$ where $r(x)$ is the Euclidean distance from x to p_0 is a conformal vector field in (R^n, g_0). Hence X is a conformal vector field on (M,g). (9) implies $\mathrm{div}_g X = n((1+r)\partial v/\partial r)$, hence we obtain from Lemma 1:

$$\int_{B_R} \left(\frac{2p}{n} - 1\right) n \left(1 + r\frac{\partial v}{\partial r}\right) \left|u\right|_g^2 d\mathrm{vol}_g = \int_{\partial B_R} \left[2re^{-v}\left|i_{\partial/\partial r}u\right|_g^2 - e^v r\left|u\right|_g^2\right] d\sigma_g,$$

where B_R denotes the Euclidean ball of radius R centered at p_0. The boundary integral is bounded above by

$$(2p+1)R\int_{\partial B_R} e^{(1-2p)v}\left|u\right|_0^2 e^{(n-1)v}\, d\sigma_{g_0}, \tag{10}$$

where we have expressed everything in terms of the Euclidean metric g_0. Since

$$\int_{R^n} e^{-2p}\left|u\right|_0^2 e^{nv}\, d\mathrm{vol}_{g_0} < \infty,$$

the boundary term (10) must converge to zero for some sequence $R_i \to \infty$.

The mean curvature h_{r_0} of $r = r_0$ with respect to the metric g is $e^{-v}(1/r_0 + \partial v/\partial r)$; thus under the hypotheses of Theorem C, $(1 + (r)\partial v/\partial r) \geq 0$ on M, and we conclude:

$$\int_M \left(\frac{2p}{n} - 1\right) n \left(1 + r\frac{\partial v}{\partial r}\right) \left|u\right|_g^2 d\mathrm{vol}_g = 0.$$

Thus the integrand must vanish identically. Since $1 + (r)\partial v/\partial r$ is positive for small r, by unique continuation for harmonic forms, u must vanish identically. □

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