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Vanishing Theorems and Almost Symmetric Spaces of Non-Compact Type*

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1. Introduction

In this paper we prove that a compact Riemannian manifold M whose curvature and torsion is almost equal to the curvature and torsion of a non-compact Riemannian symmetric space $\bar{M}$ of rank >1 and dimension >6, is diffeormorphic to a locally symmetric space $\Gamma \backslash \bar{M}$. In [9] we proved an analogous result, without the condition on rank and dimension, for compact symmetric spaces as models.

The curvature assumptions imply the existence of an approximate solution ω of the Maurer-Cartan equation $d\bar{\omega} + [\bar{\omega}, \bar{\omega}] = 0$ on an appropriate principal bundle P over M. The main step in the proof is the construction of an exact solution $\bar{\omega}$ of this equation. The obstruction to our proof is closely related to non-vanishing cohomology classes in $H^2(\Gamma, \bar{M}, \mathrm{ad})$, the second cohomology associated to discrete subgroups of semi-simple Lie groups with coefficients in the adjoint representation.

The study of these cohomology groups was initiated by Eichler and further developed by Shimura in case $\bar{M}$ is the upper half plane. The paper of Matsushima and Murakami [8], where these cohomology groups were studied on the basis of vector bundle valued harmonic forms is fundamental for our present work.

To solve the Maurer-Cartan equation $d\bar{\omega} + [\bar{\omega}, \bar{\omega}] = 0$ we follow the iteration method of Newton-Kolmogorov-Moser and solve the linearized deformation equation $d'\alpha = -\Omega$, where $\Omega = d\omega + [\omega, \omega]$ is the Cartan curvature, well enough for the iteration to converge to a Cartan connection $\bar{\omega}$ with curvature $\bar{\Omega} = d\bar{\omega} + [\bar{\omega}, \bar{\omega}] = 0$. The operator d' is closely related to the exterior derivative utilized in [8]. To obtain a suitable solution of $d'\alpha = -\Omega$ we adapt the methods of Matsushima and Murakami to our case, where the manifold M is not symmetric but nearly locally symmetric in the sense that the curvature of M is close to the curvature of a symmetric space $\bar{M}$, i.e., we introduce the adjoint operator δ' of d', the Laplace operator $\Delta' = d'\delta' + \delta' d'$, and obtain an approximate solution α of $d'\alpha = -\Omega$ by solving $\Delta'\beta = -\Omega$ and setting $\alpha = \delta'\beta$.

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To prove the existence of a unique solution β of $\Delta'\beta = -\Omega$ we rely on the positivity of Δ'. Since Δ' coincides with the Laplace operator of [8] in case $M = \Gamma \setminus \bar{M}$ and the vector bundle in question is associated to $P = \Gamma \setminus G$ by the adjoint representation, we can utilize the main result of [8] stating that Δ' is positive in case a certain quadratic form is positive. Here we are concerned with 2-forms only because the Cartan curvature is a 2-form with values in the vector bundle associated to P by the adjoint representation.

The quadratic form in question is invariant under the isotropy group and the study of its positivity can be done on the irreducible components separately. The splitting of main interest here is the splitting into curvature and torsion components. Fortunately, the main result of Simons [16] implies that the quadratic form is positive on the curvature component in case rank $\bar{M} > 1$. We prove that the same is true for the torsion component; two spaces of dimensions 5 and 6 excepted.

For rank one models the comparison theorem is a consequence of the main result of Gromov [4], where it is stated for real hyperbolic space as model only. For complex hyperbolic space as model it is also a consequence of Greene and Krantz [3]. Gromov's result is not entirely analogous to ours. His curvature condition, in case $\dim M \geqq 4$, depends only on the volume of M and not on the diameter. On the other hand, our pinching constant can be estimated by purely analytical data.

Our method is not entirely restricted to models of rank > 1. The quadratic form in question is positive in case $\bar{M} = H^3$, hyperbolic 3-space, because of the duality of first and second cohomology. For higher dimensional real and complex hyperbolic spaces the quadratic form is not strictly positive. The positivity however is not a necessary condition. What we really need is a positive lower bound for the first eigenvalue of Δ'.

Such a bound does not exist in general for real and complex hyperbolic spaces $\bar{M}$ of real dimension 4 as models, because the cohomology group $H^2(\Gamma, \bar{M}, \mathrm{ad})$ does not vanish for some discrete uniform Γ. We thank Birgit Speh for this communication. For higher dimensional hyperbolic spaces it is not known whether or not $H^2(\Gamma, \bar{M}, \mathrm{ad})$ vanishes for all discrete uniform Γ.

The main result of this paper, with the stronger restriction rank $\bar{M} > 2$, was announced in [10]. With the discovery of a counter example to the main result of [14] by Christoph Im Hof and one of us the status of the result was uncertain for some time. The correction [15] together with unpublished computations of Borel and Wallach again justifies the original approach.

2. The Result

To motivate and clarify our assumptions on the curvature we first analyse the situation in the standard case.

Let $\bar{M} = G/K$ be an irreducible Riemannian symmetric space of non-compact type with K a maximal compact subgroup $K \subset G$. The projection $G \to \bar{M}$ is a K-principal bundle representing the reduction of the bundle of frames over $\bar{M}$ to the holonomy group K. The left invariant vector fields on G define a $\mathfrak{g}$-valued 1-form, the Maurer-Cartan form $\bar{\omega}: TG \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. This 1-form is a Cartan connection form of type (G, K) for $\bar{M}$ with vanishing curvature $\bar{\Omega} = d\bar{\omega}$

$+ [\bar{\omega}, \bar{\omega}]$, where $[\,,\,]$ is the Lie bracket in $\mathfrak{g}$. The Maurer-Cartan form $\bar{\omega}$ is the prototype of a Cartan connection form. The vanishing curvature $\bar{\Omega}=0$ simply means that the Maurer-Cartan equation holds.

The flatness of $\bar{\omega}$ is reflected topologically in the fact that the G-principal bundle $G \times_K G \to \bar{M}$, obtained by extending the fibres of $G \to \bar{M}$ from K to G, is canonically trivial. The trivialization is:

$$G \times_K G \simeq G/K \times G, \quad [(a,b)] \mapsto (aK, ab),$$

where $[(a,b)]$ denotes the equivalence class $\{(ak, k^{-1}b) \mid k \in K\}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition of $\mathfrak{g}$ defined by the subgroup $K \subset G$ with Lie algebra $\mathfrak{k}$. The $\mathfrak{k}$-valued part of $\bar{\omega}$ is the Levi-Civita connection of the canonical metric on $\bar{M}$, and the $\mathfrak{m}$-valued part is the canonical soldering form related to the isomorphism $T\bar{M} \cong G \times_K \mathfrak{m}$ with K represented in $\mathfrak{m} \cong \mathbb{R}^n$ via the adjoint representation of G on $\mathfrak{g}$ restricted to K.

The following assumptions on the general n-dim riemannian manifold M serve to imitate the K-principal bundle $G \to \bar{M}$ as well as the Maurer-Cartan form $\bar{\omega}$. The first assumption on M is the existence of a reduction $\pi: P \to M$ of the bundle of orthonormal frames over M to the structure group K represented orthogonally in $\mathfrak{m} \cong \mathbb{R}^n$ as in the standard case. This is a purely topological assumption, obviously a necessary condition for M to be diffeomorphic to a quotient of $\bar{M}$. On P there is the canonical $\mathfrak{m} \cong \mathbb{R}^n$-valued 1-form θ defined by:

$$\theta: T_u P \to \mathfrak{m}, \quad X \mapsto u^{-1} \pi(X),$$

where $u \in P$ is a frame in $T_{\pi(u)}M$ and as such defines an isomorphism $u: \mathfrak{m} \to T_{\pi(u)}M$. θ is a K-equivariant 1-form vanishing on vertical vectors.

Let η denote a connection form on P. η is a metric connection for M since the structure group of the bundle P is compact; and therefore preserves some Riemannian metric. We do not assume that η is a Levi-Civita connection. For models of rank >1 the above holonomy assumption for the Levi-Civita connection would imply, by a result of Berger [1], that the manifold M is locally isometric to the model. We combine η and θ to define the $\mathfrak{g}$-valued 1-form $\omega = \eta + \theta: TP \to \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. ω is a Cartan connection of type (G, K) with curvature $\Omega = d\omega + [\omega, \omega]$, where $[\,,\,]$ is the Lie bracket of $\mathfrak{g}$. We refer to the following chapter for a definition of Cartan connections.

In the following theorem we require ω to satisfy the Maurer-Cartan equation up to a small error. To formulate this condition precisely, let $\|\Omega\|$ denote the supremum of $|\Omega|$ over the compact manifold M, where the norm $|\Omega|$ of the $\mathfrak{g}$-valued 2-form Ω is defined in terms of the Riemannian metric on M and the natural positive definite scalar product on the semi-simple Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ defined by the Cartan-Killing form of $\mathfrak{g}$ after change of sign on $\mathfrak{k}$.

Theorem. *Let $\bar{M} = G/K$ denote an irreducible non-compact Riemannian symmetric space of rank >1 and dimension >6, M a compact Riemannian manifold and Ω the curvature form of the Cartan connection ω on the K-principal bundle defined above.*

There exists a constant $A > 0$ depending on $\bar{M}$ and on an upper bound for the diameter of M such that $\|\Omega\| < A$ implies that M is diffeomorphic to a quotient $\Gamma \setminus \bar{M}$, where Γ is a discrete subgroup of the isometry group G of $\bar{M}$.

We continue with some remarks concerning this result.

No explicit assumptions on the Riemannian curvature are made in the theorem. In particular, we do not assume that M has non-positive Riemannian sectional curvature. Our assumptions imply only that the connection η has small torsion and has curvature very near to the curvature of the model $\bar{M}$.

The diffeomorphism $M \to \Gamma \setminus \bar{M}$ is a quasi-isometry with the dilatation controlled by the constant A and the diameter of M.

Our theorem, together with the result of Chen and Eberlein [2] on the finiteness of isometry types of compact Riemannian quotients with bounded diameter, shows that the number of diffeomorphism types of manifolds satisfying the assumptions of our theorem is finite.

The comparison theorem presented here bears some resemblance to the strong rigidity theorems of Mostow [12] and Siu [17]. The difference is that while rigidity theorems show that the fundamental group determines the manifold, our theorem determines properties of the fundamental group, e.g., that it is a subgroup of the isometry group G of the model $\bar{M}$. The rigidity theorems are generalizations of A. Weil's local rigidity theorem which is a consequence of the vanishing of the first cohomology group $H^1(\Gamma, \bar{M}, \mathrm{ad})$. The theorem above is related to the vanishing of $H^2(\Gamma, \bar{M}, \mathrm{ad})$. More accurately, we need a positive lower bound for the first eigenvalue of the Laplace operator Δ' defined in Chap. 4. We prove this result in Chap. 4 by showing that a certain quadratic form introduced by Matsushima and Murakami [8] in positive. As a consequence we obtain the following:

Vanishing Theorem. *Let $\bar{M}$ denote a Riemannian symmetric space of non-compact type, and rank >1, with the exception of* $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ *and* $\mathrm{SO}_0(2,3)/\mathrm{SO}(2) \times \mathrm{SO}(3)$, *$\Gamma$ a discrete uniform subgroup of the isometry group G of $\bar{M}$ and* ad *the adjoint representation of G. Then,* $H^2(\Gamma, \bar{M}, \mathrm{ad}) = (0)$.

This vanishing theorem is a special case of results conjectured by Zuckerman and proved by Vogan. Our reasons for giving a new proof for this special case are as follows:

The results of Vogan and Zuckerman have not appeared in print yet, and more importantly, their arguments are based on representation theoretic considerations, and therefore do not yield an estimate for the first eigenvalue of Δ' on an almost symmetric space. If the isometry group of $\bar{M}$ is a complex simple Lie group the vanishing theorem above is also a special case of [5].

3. The Proof

The main work in the proof is to solve for a Cartan connection $\bar{\omega}$ on the principal bundle P which satisfies the Maurer-Cartan equation $d\bar{\omega} + [\bar{\omega}, \bar{\omega}] = 0$. In this chapter we set up a successive approximation for the solution $\bar{\omega}$ and show that the existence of such a flat Cartan connection implies the theorem. We defer the two main steps of the proof; the existence of a solution of the potential equation and the estimates necessary for the convergence of the iterative procedure, to the next two chapters.

The differential form $\omega: TP \to \mathfrak{g}$ defined in the previous chapter on the K-principal bundle P is a Cartan connection form of type (G, K), i.e., it has the following properties:

$\omega(A^*) = A$ for all $A \in \mathfrak{k}$, where A^* is the fundamental vertical vector field on P defined by the action of the 1-parameter group $\exp tA$ on P. (3.1)

$R_a^* \omega = \mathrm{ad}\, a^{-1} \omega$ for all $a \in K$, where R_a is the action (from the right) of a on P. (3.2)

$\omega(X) \neq 0$ for all non-zero $X \in TP$. (3.3)

The curvature of a Cartan connection ω is defined to be

$$\Omega = d\omega + [\omega, \omega], \tag{3.4}$$

where $[\,,]$ is the Lie bracket of the Lie algebra $\mathfrak{g}$.

It is well known that Ω is a horizontal ad$K$-equivariant $\mathfrak{g}$-valued 2-form on P, i.e., it satisfies $R_a^* \Omega = \mathrm{ad}\, a^{-1} \Omega$ and $\Omega(A^*, X) = 0$ for all $a \in K$, $A \in \mathfrak{k}$ and $X \in TP$. For the computations we identify horizontal ad$K$-equivariant $\mathfrak{g}$-valued forms on P with differential forms on the base space M with values in the vector bundle $E = E_{\mathrm{ad}}$.

Here, $E_{\mathrm{ad}} = P \times_K \mathfrak{g}$ denotes the vector bundle associated to the principal bundle P by the adjoint representation of G restricted to K.

Starting with an almost flat Cartan connection ω, we will set up an iteration process leading to a flat Cartan connection $\bar{\omega}$, i.e., with curvature $\bar{\Omega} = 0$. To obtain a flat Cartan connection $\bar{\omega} = \omega + \alpha$ we have to solve the equation

$$d\alpha + [\omega, \alpha] + [\alpha, \omega] + [\alpha, \alpha] = -\Omega \tag{3.5}$$

for a horizontal ad$K$-equivariant $\mathfrak{g}$-valued 1-form α on P. In addition we need an estimate $\|\alpha\| < c\|\Omega\|$, where c is a constant depending only on $\bar{M}$ and an upper bound for the diameter of M, in order to guarantee that $\bar{\omega}$ satisfies condition (3.3) for Cartan connections.

Since (3.5) is non-linear we first consider the linearized deformation equation

$$d^{\omega}\alpha = d\alpha + [\omega, \alpha] + [\alpha, \omega] = -\Omega. \tag{3.6}$$

The operator d^{ω} defined here is the exterior covariant derivative of a covariant derivative D defined on the trivial bundle $P \times \mathfrak{g}$ by the formula:

$$D_X s = Xs + [\omega(X), s], \tag{3.7}$$

where $X \in TP$, $s: P \to \mathfrak{g}$ is a section and Xs is the derivative of s in the direction X.

The exterior covariant derivative d^{ω} associated to D acting on a $\mathfrak{g}$-valued p-form α on P is given by

$$\begin{aligned} d^{\omega}\alpha(X_0, \ldots, X_p) = &\sum_{i=0}^{p} (-1)^i D_{X_i} \alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p) \\ &+ \sum_{j<k} (-1)^{j+k} \alpha([X_j, X_k], \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_p). \end{aligned} \tag{3.8}$$

The Bianchi identity for the curvature can be written simply as

$$d^{\omega}\Omega=0. \tag{3.9}$$

We now set up an iteration procedure for a sequence of Cartan connections $\{\omega_i\}$ starting with the given connection $\omega_0=\omega$ and converging to a flat Cartan connection $\bar{\omega}$.

We define inductively $\omega_{i+1}=\omega_i+\alpha_i$, $i=0, 1, \ldots$, where α_i is an approximate solution of (3.6) with Ω replaced by Ω_i. Since the curvature of ω_{i+1} is given by

$$\Omega_{i+1}=\Omega_i+d^{\omega_i}\alpha_i+[\alpha_i,\alpha_i] \tag{3.10}$$

Ω_{i+1} is of the order of magnitude $\|\Omega_i\|^2$, if we can solve (3.6) up to an error of the same order, i.e., quadratic in $\|\Omega_i\|$, for a horizontal ad$K$-equivariant 1-form α_i satisfying $\|\alpha_i\|\sim\|\Omega_i\|$.

The main estimates necessary for the iteration defined above to converge to a flat Cartan connection are formulated in the following.

Main Lemma. *Let ω be a Cartan connection of type (G,K) on a principal bundle P over a compact Riemannian manifold M with curvature Ω, where $\bar{M}=G/K$ is an irreducible Riemannian symmetric space of non-compact type, rank >1, and dimension >6.*

There exists a constant $A'>0$ depending only on $\bar{M}$ and on an upper bound for the diameter of M such that if $\|\Omega\|<A'$ then there exists an ad$K$*-equivariant horizontal,* $\mathfrak{g}$*-valued* 1*-form α on P satisfying the estimates*:

$$\begin{aligned} &\text{(i)} && \|d^{\omega}\alpha+\Omega\|\leqq c\|\Omega\|^2 \\ &\text{(ii)} && \|\alpha\|_{1,q}\leqq c\|\Omega\|, \end{aligned} \tag{3.11}$$

where c is a constant depending only on $\bar{M}$ and an upper bound for the diameter of M, $\|\ \|$ is the maximum norm with respect to the metrics in $\mathfrak{g}$ *and P and $\|\ \|_{1,q}$ is the Sobolev norm to be defined in the last chapter.*

The proof will be given in the next chapters. The estimates (3.11) imply that the sequence of curvatures $\{\Omega_i\}$ converges rapidly to zero, provided that the initial curvature $\Omega_0=\Omega$ is small enough. The series $\sum_{i=0}^{\infty}\alpha_i$, and hence the sequence of Cartan connections $\{\omega_i\}$, converges in the Sobolev space $W_{1,q}(P)$ to a 1-form satisfying properties (3.1) and (3.2) of Cartan connections. To prove (3.3) it suffices to prove that $\|\omega_0-\bar{\omega}\|<1$, in terms of the metric defined on P by ω_0. This follows from (3.11) (ii) via the Sobolev inequality. For more details we refer to [9].

As in [9] we finish the proof by utilizing the flat Cartan connection $\bar{\omega}$ to define an integrable distribution on $\tilde{P}\times\tilde{G}$ where $\tilde{P}$ and $\tilde{G}$ are the universal covers of P and G respectively. The maximal integral submanifold through $(\tilde{p},e)$, where $\tilde{p}\in\tilde{P}$ is an arbitrary point and e is the identity in $\tilde{G}$, defines the graph of a diffeomorphism $\tilde{F}:\tilde{P}\to\tilde{G}$. We then pass to quotients to obtain the required diffeomorphism $f:M\to\Gamma\setminus\bar{M}$. Because we are dealing only with Cartan connections, the arguments in [9] involving the center of mass construction are not needed here.

4. A Vanishing Theorem

To solve for an α satisfying the linearized deformation equation (3.6) approximately in the sense of the Main Lemma we will use Hodge theory for the associated Laplacian. In the standard case $M=\Gamma\setminus\bar{M}$, $P=\Gamma\setminus G$, the bundle E_{ad} is flat and the exterior covariant derivative $d^{\bar{\omega}}$ defines cohomology groups $H^p(\Gamma,\bar{M},\text{ad})$ of Γ with coefficients in the adjoint representation. These cohomology groups were first studied on the basis of Hodge theory and vector bundle valued harmonic forms by Matsushima and Murakami [8]. We will closely follow their computations and show in this chapter that a sufficient condition for the vanishing of $H^2(\Gamma,\bar{M},\text{ad})$, namely that a certain algebraic quadratic form associated to the adjoint representation is positive definite, is satisfied if rank $\bar{M}>1$, and $\dim\bar{M}>6$.

Since we are interested in solving $d^{\omega}\alpha=-\Omega$ only approximately, i.e., up to an error whose order of magnitude is quadratic in the curvature, we are allowed to replace d^{ω} by a slightly modified operator d', as we did in [9], in order to exploit the similarity to the model space and also to profit from the computations of Matsushima and Murakami [8]. We obtain d' from d^{ω} by replacing the vector field bracket on P in the last term of the formula (3.8) by the Lie algebra bracket of $\mathfrak{g}$. For the rest of the paper the symbols X_i will denote parallel vector fields on P, i.e., $\omega(X_i)=\text{const.}\in\mathfrak{g}$. We define:

$$d'\alpha(X_0,\dots,X_p)=\sum_{i=0}^{p}(-1)^i D_{X_i}\alpha(X_0,\dots,\hat{X}_i,\dots,X_p)$$
$$+\sum_{j<k}(-1)^{j+k}\alpha(\{X_j,X_k\},\dots,\hat{X}_j,\dots,\hat{X}_k,\dots,X_p), \qquad (4.1)$$

where $\{X_j,X_k\}=\omega^{-1}([\omega(X_j),\omega(X_k)])$.
The difference between d^{ω} and d' is then given by

$$(d'-d^{\omega})\alpha(X_0,\dots,X_p)=\sum_{j<k}(-1)^{j+k}\alpha(\omega^{-1}(\Omega(X_j,X_k)),\dots,\hat{X}_j,\dots,\hat{X}_k,\dots,X_p). \qquad (4.2)$$

As is easily verified, d' maps the subspace of horizontal ad$K$-equivariant forms into itself and from now on we will restrict our attention to such forms. d' restricted to these forms is then given by the following formula of Matsushima and Murakami [8, (4.12)] with $\varrho=\text{ad}$.

$$d'\alpha(X_0,\dots,X_p)=\sum_{i=0}^{p}(-1)^i X_i\alpha(X_0,\dots,\hat{X}_i,\dots,X_p)$$
$$+\sum_{i=0}^{p}(-1)^i[\omega(X_i),\alpha(X_0,\dots,\hat{X}_i,\dots,X_p)], \qquad (4.3)$$

where now $\omega(X_i)=\text{const}\in\mathfrak{m}$.

The adjoint of d' with respect to the scalar product we are using on $\mathfrak{g}$ and the corresponding metric induced by ω on P is given by the formula [8, (6.3)]

$$\delta'\alpha(X_2,\dots,X_p)=-\sum_{k=1}^{n}e_k\alpha(e_k,X_2,\dots,X_p)$$
$$+\sum_{k=1}^{n}[\omega(e_k),\alpha(e_k,X_2,\dots,X_p)], \qquad (4.4)$$

where $\{e_k\}_{k=1,\dots,n}$ is a parallel orthonormal basis of horizontal vectors.

For further reference we note here the following three properties of d' and δ'. The first is a consequence of the Bianchi identity (3.9) and of (4.2). For the proof of the other two properties we refer to [9, (3.10), (3.11)].

$$\|d'\Omega\| < c\|\Omega\|^2, \text{ where } c \text{ depends only on } \mathfrak{g}. \tag{4.5}$$

$$d'd'\alpha(X_0, \ldots, X_{p+1}) = \sum_{i<j}(-1)^{i+j}\omega^{-1}(\Omega(X_i, X_j))\alpha(X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \tag{4.6}$$

$$\delta'\delta'\alpha(X_3, \ldots, X_p) = \sum_{k,l=1}^{n} \omega^{-1}(\Omega(e_k, e_l))\alpha(e_k, e_l, X_3, \ldots, X_p), \tag{4.7}$$

where $\{e_k\}_{k=1\ \ldots, n}$ is a parallel orthonormal basis of horizontal vectors.

Let $\Delta' = d'\delta' + \delta' d'$ be the Laplacian and we define

$$\alpha = \delta'\beta, \tag{4.8}$$

where β is the unique solution of the potential equation

$$\Delta'\beta = -\Omega. \tag{4.9}$$

The rest of this chapter is devoted to proving that Δ' is positive definite on E-valued 2-forms, which proves the vanishing theorem and shows that (4.9) has a unique solution β.

In case $P = \Gamma \setminus G$, Δ' coincides of course with the Laplacian Δ of [8]. Moreover, since the formal expression of Δ' with respect to a parallel orthonormal basis of horizontal vectors is exactly the same as that of Δ on the model space, the main computation of [8] proves that Δ' splits as a sum of two operators

$$(\Delta'\beta, \beta) = (\Delta'_V\beta, \beta) + \int_M \langle \Delta_{\mathrm{ad}}\beta, \beta\rangle, \tag{4.10}$$

where $(\,,)$ denotes the global L_2-scalar product, Δ'_V is the Laplacian associated to some covariant differentiation V so that $(\Delta'_V\beta, \beta) \geqq 0$ and $\langle \Delta_{\mathrm{ad}}\beta, \beta\rangle$ is a quadratic form defined pointwise on $\Lambda^p(TM) \otimes E_{\mathrm{ad}}$.

The main result of Raghunathan [15], which is the corrected version of an older paper [14], gives a criterion for $\langle \Delta_\varrho\beta, \beta\rangle$ to be positive definite, ϱ being an irreducible representation of G. According to unpublished computations of Borel and Wallach, this criterion is satisfied if $p < \operatorname{rank}\bar{M}$. Here we need the case $p = 2$ and $\varrho = \mathrm{ad}$. For this case we now give a direct proof that Δ_{ad} is positive definite on 2-forms if $\operatorname{rank}\bar{M} > 1$ and $\dim \bar{M} > 6$. Our proof relies on the main result of Simons [16], where he gives an intrinsic algebraic proof of some classification theorems on holonomy groups due to Berger [1], and on certain computations of Matsushima [7] and Kaneyuki and Nagano [6] concerning the cohomology of $\Gamma \setminus \bar{M}$ with trivial coefficients.

$\Delta_\varrho = d_\varrho\delta_\varrho + \delta_\varrho d_\varrho$ where d_ϱ and δ_ϱ for $\varrho = \mathrm{ad}$ are given by the following formulas:

$$d_\varrho\alpha(X_0, \ldots, X_p) = \sum_{i=0}^{p}(-1)^i[X_i, \alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p)], \tag{4.11}$$

$$\delta_\varrho\alpha(X_2, \ldots, X_p) = \sum_{k=1}^{n}[e_k, \alpha(e_k, X_2, \ldots, X_p)], \tag{4.12}$$

where $\alpha, \beta \in \Lambda^p(\mathfrak{m}^*) \otimes \mathfrak{g}$, $X_i \in \mathfrak{m}$ and $\{e_k\}_{k=1, \ldots, n}$ is an orthonormal basis for $\mathfrak{m}$.

As is well known Δ_ϱ is invariant under the action of K on $\Lambda^p(\mathfrak{m}^*)\otimes\mathfrak{g}$, and therefore we consider Δ_{ad} acting on the ad$K$-invariant subspaces $\Lambda^2(\mathfrak{m}^*)\otimes\mathfrak{k}$ and on $\Lambda^2(\mathfrak{m}^*)\otimes\mathfrak{m}$ separately.

A 2-form $r\in\Lambda^2(\mathfrak{m}^*)\otimes\mathfrak{k}$ can be thought of as a map $r:\Lambda^2(\mathfrak{m})\to\mathfrak{k}\subset\Lambda^2(\mathfrak{m})$, where we identify $\mathfrak{m}^*$ with $\mathfrak{m}$ via the scalar product and $\mathfrak{k}$ is imbedded in $\Lambda^2(\mathfrak{m})$ via the adjoint action. $d_{\mathrm{ad}}r=0$ is then equivalent to the Bianchi identity and hence a closed 2-form defines a tensor with the symmetries of a curvature tensor $r:\Lambda^2(\mathfrak{m})\to\mathfrak{k}\subset\Lambda^2(\mathfrak{m})$.

The main results of Simons [16, Theorems 2.4, and 6] now state that if rank $G/K>1$, then every curvature tensor $r:\Lambda^2(\mathfrak{m})\to\mathfrak{k}\subset\Lambda^2(\mathfrak{m})$ is a scalar multiple of the curvature tensor of the symmetric space $\bar{M}=G/K$. If r is harmonic then r has to vanish, since $\delta_{\mathrm{ad}}r=0$ is equivalent to the vanishing of the Ricci tensor associated to r, and the Ricci curvature of the symmetric space $\bar{M}$ is well known to be non-zero.

This proves that there are no non-trivial algebraic harmonic curvature tensors r if rank $\bar{M}>1$, and hence Δ_{ad} is positive definite on $\Lambda^2(\mathfrak{m})\otimes\mathfrak{k}$.

We now investigate 2 forms $t\in\Lambda^2(\mathfrak{m})\otimes\mathfrak{m}$, which we call torsion forms. For the computations we introduce orthonormal bases $\{e_k\}_{k=1,\dots,n}$ foı $\mathfrak{m}$ and $\{e_a\}_{a=1,\dots,N-n}$ for $\mathfrak{k}$. Using the formulas (4.11) and (4.12), the Laplacian $\Delta_\varrho=d_\varrho\delta_\varrho+\delta_\varrho d_\varrho$, with $\varrho=\mathrm{ad}$, acting on a 2 form t is then computed to be

$$\begin{aligned}(\Delta_\varrho t)(e_i,e_j)&=\sum_{k=1}^{n}[e_k,[e_k,t(e_i,e_j)]]\\&\quad+\sum_{k=1}^{n}[[e_i,e_k],t(e_k,e_j)]\\&\quad-\sum_{k=1}^{n}[[e_j,e_k],t(e_k,e_i)]\end{aligned}\tag{4.13}$$

We write $c_{ij}^a=c_{ia}^j=\langle[e_i,e_j],e_a\rangle$ for the structure constants of $\mathfrak{g}$ and set $R_{ijk}^l=c_{ij}^a c_{ka}^l$ where we use the Einstein summation convention. R is the curvature tensor of $\bar{M}$ and hence satisfies the usual identities. With this index notation, (4.13) can be written as

$$\begin{aligned}(\Delta_\varrho t)_{ij}^m&=c_{ka}^m c_{kl}^a t_{ij}^l+c_{al}^m c_{ik}^a t_{kj}^l-c_{al}^m c_{jk}^a t_{ki}^l\\&=R_{klk}^m t_{ij}^l-R_{ikl}^m t_{kj}^l+R_{jkl}^m t_{ki}^l.\end{aligned}\tag{4.14}$$

We compute the first term of the right hand side in the following

Lemma.
$$\sum_{k=1}^{n}[e_k,[e_k,x]]=-\sum_{a=1}^{N-n}[e_a,[e_a,x]]=\frac{1}{2}x\tag{4.15}$$

for all $x\in\mathfrak{m}$.

Proof.
$$\begin{aligned}\delta_{ij}=\langle e_i,e_j\rangle&=+\,tr\,ade_i\cdot ade_j\\&=c_{ia}^k c_{jk}^a+c_{ik}^a c_{ja}^k\end{aligned}$$

and hence

$$\begin{aligned}\langle[e_k,[e_k,e_i]],e_j\rangle=c_{ka}^j c_{ki}^a\quad&=-c_{ak}^j c_{ai}^k\\=-\langle[e_a,[e_a,e_i]],e_j\rangle\quad&=\tfrac{1}{2}\delta_{ij}.\end{aligned}$$

To compute the last two terms of (4.14) we first set $t_{ij}^k = \alpha_{ij}^k + \beta_{ij}^k$, where $\alpha_{ij}^k = \frac{1}{3}(t_{ij}^k + t_{jk}^i + t_{ki}^j)$ is antisymmetric in all its indices and $\beta_{ij}^k = t_{ij}^k - \alpha_{ij}^k$ satisfies the Bianchi identity: $\beta_{ij}^k + \beta_{jk}^i + \beta_{ki}^j = 0$.

This defines an ad$K$-invariant orthogonal splitting of $\Lambda^2(\mathfrak{m}) \otimes \mathfrak{m}$ and we consider Δ_ϱ acting on these two spaces separately.

If t_{ij}^k is antisymmetric in all its indices, then

$$-R_{ikl}^m t_{kj}^l = (R_{kli}^m + R_{lik}^m) t_{kj}^l = R_{kli}^m t_{kj}^l + R_{kil}^m t_{lj}^k = R_{kli}^m t_{kj}^l + R_{ikl}^m t_{kj}^l,$$

and therefore

$$-R_{ikl}^m t_{kj}^l = \tfrac{1}{2} R_{kli}^m t_{kj}^l. \tag{4.16}$$

Substituting (4.15) and (4.16) in (4.14), we obtain after taking the scalar product with t

$$\langle \Delta_\varrho t, t\rangle = \tfrac{1}{2}|t|^2 + \tfrac{1}{2} R_{kli}^m t_{kj}^l t_{ij}^m.$$

The last term is non-negative, because the curvature operator $R: \Lambda^2(\mathfrak{m}) \to \Lambda^2(\mathfrak{m})$ of $\bar{M}$ is non-positive. This shows that Δ_ϱ is positive definite on antisymmetric t.

We are now left with t satisfying the Bianchi identity. For such t, we consider the splitting $t_{kj}^l = s_{kl}^j + \frac{1}{2} t_{kl}^j$ where $s_{kl}^j = \frac{1}{2} t_{kj}^l + \frac{1}{2} t_{lj}^k$ is symmetric in k and l. Since the splitting is orthogonal, we have

$$|s|^2 = \tfrac{3}{4}|t|^2. \tag{4.17}$$

To simplify the last two terms in (4.14) we use the following

Lemma. $R_{ikl}^m s_{kl}^j$ *is symmetric in m and i*

$R_{ikl}^m t_{kl}^j$ *is antisymmetric in m and i.*

The proof of this lemma is an easy computation using the Bianchi identity of the curvature tensor R.

In view of these lemmas, scalar multiplication of (4.14) by t now gives

$$\begin{aligned}\langle \Delta_\varrho t, t\rangle &= \tfrac{1}{2}|t|^2 - R_{ikl}^m s_{kl}^j s_{im}^j - \tfrac{1}{4} R_{ikl}^m t_{kl}^j t_{im}^j \\ &= \tfrac{2}{3}|s|^2 - R_{ikl}^m s_{kl}^j s_{im}^j + \tfrac{1}{8} R_{lki}^m t_{lk}^j t_{im}^j \\ &\geqq \tfrac{2}{3}|s|^2 - R_{ikl}^m s_{kl}^j s_{im}^j,\end{aligned} \tag{4.18}$$

where we used (4.17) and the Bianchi equation on R for the second equation and the fact that the curvature operator of $\bar{M}$ is non-positive for the last inequality.

We will now estimate the largest eigenvalue of the operator $Q(s)_{im} = R_{ikl}^m s_{kl}$ acting on symmetric 2-tensors $s \in \mathcal{S}^2(\mathfrak{m})$. Q can also be defined by the formulas

$$\langle Q(u \odot v), x \odot y\rangle = \tfrac{1}{2}\langle R(xu)v + R(xv)u, y\rangle \tag{4.19}$$

and

$$Q(u \odot v) = -\sum_{a=1}^{N-n} [e_a, u] \odot [e_a, v], \tag{4.20}$$

where we use the notation $u \odot v = \frac{1}{2}(u \otimes v + v \otimes u) \in \mathcal{S}^2(\mathfrak{m})$. This operator has been considered previously, in particular by Matsushima [7] and Kaneyuki and Nagano [6] in connection with the computation of the Betti numbers of $\Gamma \setminus \bar{M}$. Following [6] we simplify the computation of the maximal eigenvalue of Q by considering the corresponding operator $\hat{Q}$ of the compact simple Lie group U, where U/K is the compact symmetric space dual to $\bar{M}$. The curvature tensor of U/K is just $-R$. Moreover, if $\hat{R}$ is the curvature tensor of U defined by $\langle \hat{R}(x,y)z,w\rangle = -\langle [[x,y],z],w\rangle$, where the metric $\langle\, ,\rangle$ is minus the Killing form B of $\mathfrak{u}$, the Lie algebra of U, then $\hat{R}(x,u)v\in\mathfrak{m}$ if $x,u,v\in\mathfrak{m}$. Therefore, the largest eigenvalue of Q is not larger than the largest eigenvalue of the operator

$$\begin{aligned} -\hat{Q}: \mathscr{S}^2(\mathfrak{u}) &\to \mathscr{S}^2(\mathfrak{u}) \\ x\odot y &\to \sum_{k=1}^{N} [u_k,x]\odot[u_k,y], \end{aligned} \tag{4.21}$$

where $\{u_k\}$ is an orthonormal base for $\mathfrak{u}$.

The operator $\hat{Q}$ is of course closely related to the Casimir operator $\varrho(c)$ of the representation ϱ of $\mathfrak{u}$ on the symmetric 2-tensors induced by the adjoint representation. In fact, if we define $\varrho(c)$ with respect to the Killing form B on $\mathfrak{u}$, we have

$$\begin{aligned} \varrho(c)(x\odot y) &= -\sum_{k=1}^{N} [u_k,[u_k,x]]\odot y \\ &\quad -2\sum_{k=1}^{N} [u_k,x]\odot[u_k,y] \\ &\quad -\sum_{k=1}^{N} x\odot[u_k,[u_k,y]] \\ &= 2x\odot y - 2\hat{Q}(x\odot y) \end{aligned}$$

and hence

$$-\hat{Q} = \tfrac{1}{2}\varrho(c) - \mathrm{id}. \tag{4.22}$$

As is well known, $\varrho(c)$ acts as a scalar on each irreducible component of the representation. The value of this scalar on an irreducible component V_λ of highest weight λ, with respect to an ordering of the roots, is computed by the following formula, compare Raghunathan [13].

$$\varrho(c)|_{V_\lambda} = B(\lambda,\lambda+2\delta)\cdot\mathrm{id}, \tag{4.23}$$

where 2δ is the sum of the positive roots of the given ordering.

The highest weight which occurs in the representation ϱ of $\mathfrak{u}$ on $\mathscr{S}^2(\mathfrak{u})$ is 2μ, where μ is the highest weight of the adjoint representation. Moreover, since the Casimir operator of the adjoint representation with respect to the Killing form is the identity we have the normalization

$$B(\mu,\mu+2\delta)=1, \tag{4.24}$$

hence the largest eigenvalue of $-\hat{Q}$ is given by

$$\tfrac{1}{2}B(2\mu,2\mu+2\delta)-1 = B(\mu,2\mu+2\delta)-B(\mu,\mu+2\delta)=B(\mu,\mu).$$

To show that Δ_ϱ is positive definite, it suffices therefore, by (4.18), to prove that $B(\mu,\mu)<\frac{1}{3}$.

Because of the normalization (4.24) this is equivalent to

$$B(\mu,\delta)>B(\mu,\mu). \tag{4.25}$$

A simple check of the root systems for the simple Lie algebras shows that (4.25) holds for all $\mathfrak{u}$ except for the cases A_1, A_2, and B_2. The symmetric spaces of type III corresponding to these cases which are of rank >1 are just the two spaces $SL(3;\mathbb{R})/SO(3)$ and $SO_0(2,3)/SO(2)\times SO(3)$. The symmetric spaces of type IV are covered in [5] and also by the results of Simons [16].

This proves our vanishing theorem, and shows that the first eigenvalue of Δ' is bounded from below by a positive constant depending only on the model space $\bar{M}$, provided rank $\bar{M}>1$ and $\dim\bar{M}>6$.

5. The Estimates

The basic strategy for deriving the estimates of the Main Lemma is the same as in [9]. Many of the estimates obtained there can be used here also. The difference is that the operator Δ' that occurs in the comparison theorems for compact symmetric spaces is coercive, and the solutions of the important equations there satisfy a maximum principle. This is not the case here and we have only the L_2-positivity of Δ', proved in the preceding chapter at our disposal. To balance this deficiency we require an upper bound on the diameter of M.

To prove the main lemma we use L_q-interior regularity estimates for solutions of elliptic equations. These estimates are usually stated for balls in $\mathbb{R}^n$. As a first step therefore, we have to convert the global L_q-norm of β on M into a norm defined by integration over a suitable ball. The usual procedure of covering M with normal neighborhoods and summing up the local estimates by means of a partition of unity is not good enough for our purposes since we need estimates which are independent of the injectivity radius of M. Instead, we use the following argument which arose in a discussion with Christoph Im Hof. We replace the differential forms on M by their pullbacks via a suitable exponential map. The exponential map defined in [9, p. 348] restricted to horizontal vectors is suitable because of the estimate [9, (5.5)]. It proves that the Jacobi fields differ from those of the model $\bar{M}$ by an error term that can be controlled by $\|\Omega\|$. In particular, since $\bar{M}$ has non-positive curvature, this exponential map $\exp_x: T_xM\to M$ has maximal rank on a ball of arbitrarily large radius, provided the constant A' of the main lemma is sufficiently small. We now consider balls B_r of a fixed radius r in each tangent space T_xM, where r is chosen large compared to the diameter $d(M)$ of M, say $r>10d(M)$, but still small enough compared to the distance to the first conjugate locus, e.g., such that the estimate $0.9<\|d\exp_0(\overline{d\exp})^{-1}\|<1.1$ holds, where $\overline{d\exp}$ is the differential of the exponential map of $\bar{M}$. The tangent spaces of M and $\bar{M}$ are of course matched using the frames in P.

Let β_x denote the pullback via $\exp_x$ of β to the ball $B_r\subset T_xM$. We define the Sobolev norm $\|\ \|_{m,q}$ by setting

$$\|\beta\|_{m,q}=\sup_{x\in M}\|\beta_x\|_{m,q}, \quad \|\beta_x\|_{m,q}=\sum_{|\mu|\leq m}\left(\int_{B_r}|D^\mu\beta_x|^q\right)^{1/q}. \tag{5.1}$$

First, we wish to show that $\| \;\|_{o,q}$ and the L_q-norm $\| \;\|_q$ obtained by integration over M are equivalent norms, i.e., that the inequalities

$$c_1 \| \;\|_{o,q}^q \leqq \| \;\|_q^q \leqq c_2 \| \;\|_{o,q}^q \tag{5.2}$$

hold for some constants c_1, c_2 such that the ratio c_2/c_1 is bounded from above by some constant depending only on $\bar{M}$ and on an upper bound for $d(M)$, provided the constant A' of the main lemma is sufficiently small. We take c_1 to be the infimum over $x \in M$ of the inverse of the number of fundamental domains for M in $T_x M$ having non-empty intersection with B_r and c_2 equal to the supremum over $x \in M$ of the inverse of the number of fundamental domains contained in B_r.

Secondly, in order to be able to use the estimates of [9] we view the E_{ad}-valued forms on M as horizontal ad$K$-equivariant forms on P with values in $\mathfrak{g}$. Since the fibres have constant volume, the L_q-norm of such a form on P is just some constant multiple of the L_q-norm on M.

Thirdly, before we can use the estimates of [9] we have to note that the adjoint δ' we use here is not the δ' used in [9] but rather its projection onto horizontal forms. However, restricted to ad$K$-equivariant horizontal forms, the difference of these two operators is an algebraic operator of order zero. Similarly, the Δ' we use here differs from that of [9] only by lower order terms. In fact we still have an equation in the form of [9, (5.6)]

$$\Delta \beta_{ij} + (L'(\beta))_{ij} = -\Omega_{ij}, \tag{5.3}$$

where the leading term is the Laplacian acting on the component functions and L' is a first-order differential operator, because the derivatives of β_{ij} in the vertical directions are just algebraic expressions as β is an ad$K$-equivariant horizontal form.

Using the interior regularity estimate of [11, Theorem 5.5.5′] we obtain

$$\|\beta\|_{1,q} \leqq c(\|\Omega\|_{0,q} + \|\beta\|_{0,2}). \tag{5.4}$$

where c depends only on $\bar{M}, q, r$ and an upper bound for $\|\Omega\|$.

From now on we fix some $q > N = \dim P$ and denote by c any constant depending on $\bar{M}$ and on upper bounds for $d(M)$, $\|\Omega\|$ and on $\frac{1}{\lambda}$ where λ is the smallest eigenvalue of Δ' on 2-forms.

The right hand side of (5.4) is now estimated as follows

$$\|\Omega\|_{0,q} \leqq \|\Omega\| (\operatorname{vol} B_r)^{1/q} \leqq c\|\Omega\|, \tag{5.5}$$

$$\|\beta\|_{0,2}^2 \leqq \frac{1}{c_1} \|\beta\|_2^2 \leqq \frac{1}{\lambda^2} \frac{1}{c_1} \|\Omega\|_2^2 \leqq \frac{1}{\lambda^2} \|\Omega\|^2 \operatorname{vol}(B_{r+d(M)}). \tag{5.6}$$

Since the union of the $\frac{1}{c_1}$ fundamental domains having a non-empty intersection with B_r lies in a ball of radius $r + d(M)$. (5.4), (5.5) and (5.6) now give

$$\|\beta\|_{1,q} \leqq c\|\Omega\|. \tag{5.7}$$

The estimates [9, (5.11), (5.12)] are still valid since they do not involve the maximum principle and because we do not have to switch from the L_q-norms on M to the L_q-

norms in balls. We therefore have

$$\|\alpha\|_{1,q} + \|d'\beta\|_{1,q} \leqq c(\|\Omega\| + \|\Omega\|^2 + \|\gamma\|_{0,q}), \tag{5.8}$$

where $-\gamma = d'\alpha + \Omega = -\delta' d'\beta$.

γ satisfies the elliptic system:

$$\begin{aligned} d'\gamma &= -d'd'\alpha - d'\Omega \\ \delta'\gamma &= \delta'\delta'(d'\beta). \end{aligned} \tag{5.9}$$

Using (4.5), (4.6), (4.7) and the L_2-version of (5.8) we obtain from (5.9) the estimate

$$\|d'\gamma\|_{0,2} + \|\delta'\gamma\|_{0,2} \leqq c\|\Omega\|(\|\Omega\| + \|\gamma\|_{0,2}), \tag{5.10}$$

where we have omitted the term $\|\Omega\|^2$ because it is negligible compared to $\|\Omega\|$. We now use (5.2) together with the fact that $\frac{c_2}{c_1}$ is bounded from above to obtain

$$\begin{aligned} \|\gamma\|_{0,2} &\leqq \frac{1}{\sqrt{c_1}}\|\gamma\|_2 \leqq \sqrt{\frac{c_2}{c_1}}\frac{1}{\sqrt{\lambda}}(\|d'\gamma\|_{0,2} + \|\delta'\gamma\|_{0,2}) \\ &\leqq c\|\Omega\|(\|\Omega\| + \|\gamma\|_{0,2}). \end{aligned}$$

This yields, if $\|\Omega\|$ is small enough, the estimate

$$\|\gamma\|_{0,2} \leqq c\|\Omega\|^2. \tag{5.11}$$

As in [9] we rewrite (5.9) in the form

$$\Delta'\gamma = \delta'\phi + d'\psi, \tag{5.12}$$

where ϕ and ψ are the right hand sides of (5.9). The L_q-version of (5.10) is then

$$\|\phi\|_{0,q} + \|\psi\|_{0,q} \leqq c\|\Omega\|(\|\Omega\| + \|\gamma\|_{0,q}). \tag{5.13}$$

Applying Stampacchia [18, Theorem 4.2] we obtain

$$\|\gamma\| \leqq c(\|\gamma\|_{0,2} + \|\Omega\|(\|\Omega\| + \|\gamma\|_{0,q})),$$

which in view of (5.11) yields

$$\|\gamma\| \leqq c\|\Omega\|(\|\Omega\| + \|\gamma\|_{0,q}),$$

and the final result for small enough $\|\Omega\|$ is

$$\|\gamma\| \leqq c\|\Omega\|^2. \tag{5.14}$$

From this and (5.8) we obtain

$$\|\alpha\|_{1,q} \leqq c\|\Omega\|,$$

which is the second estimate of the main lemma. The first estimate is a consequence of (5.14), (4.2), the second estimate and the Sobolev inequality.

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