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Smoothing Riemannian Metrics

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1. Introduction

In riemannian geometry the local assumptions imposed on a manifold are usually formulated in terms of the riemannian curvature tensor. Traditionally, only C^0-bounds, but not bounds on the derivatives of the curvature tensor are accepted as reasonable assumptions. This poses the following dilemma: On one hand one feels that an additional bound on the first few derivatives of the curvature would not essentially change the value of a theorem. On the other hand, such an assumption would reduce the technical difficulties considerably, especially for a proof in the framework of global analysis. In this paper we resolve this dilemma by proving that C^0-bounds on the curvature of a riemannian metric on a compact manifold yield C^k-bounds, for any $k \in \mathbb{N}$, for the curvature of a suitable metric nearby.

The smoothing of the metric is achieved by using the evolution equation introduced by Hamilton in [3], where existence of a solution for a short time is proved by means of a Nash-Moser type inverse function theorem. To show long time existence Hamilton also derived a priori estimates [3, § 14] for higher derivatives of the curvature of the evolved metric. However, his estimates depend on the higher derivatives of the curvature of the initial metric. Here, we only need to follow the evolution for a short time, but we prove that the derivatives of the evolved curvature can be estimated in terms of a C^0-bound on the initial curvature. This is done by applying De Giorgi-Nash-Moser type of interior regularity theorems for parabolic equations. We refer to Ladyženskaja-Solonnikov-Ural'ceva [5]. In order to use the results of [5] we need local coordinates on a riemannian manifold in which the local expression of the metric and its Christoffel symbols are controlled in terms of the curvature. As shown in DeTurck-Kazdan [1], harmonic coordinates yield optimal estimates. For our purposes we will use the results obtained by Jost-Karcher [4] on harmonic coordinates defined by special (almost linear) boundary conditions.

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2. The Result

Let $\mathfrak{M}=\{(M,g);\ |\mathrm{Rm}(g)|\leqq 1\}$ denote the set of all compact riemannian manifolds of dimension n whose riemannian curvature tensor $\mathrm{Rm}(g)$ is bounded by one in the C^0-norm. We will construct smoothing operators $S_\varepsilon\colon \mathfrak{M}\to\mathfrak{M}$ assigning to a metric g on M a new metric $S_\varepsilon(g)$ on the same manifold whose curvature tensor $\mathrm{Rm}(S_\varepsilon(g))$ has improved regularity properties. Let $\nabla^k \mathrm{Rm}(S_\varepsilon(g))$ denote the k-th covariant derivative of the riemannian curvature tensor with respect to the Levi-Civita connection of the metric $S_\varepsilon(g)$.

Theorem. *Given any* $\varepsilon>0$ *and* $k\in\mathbb{N}$, *there exists a constant* $C=C(n,k,\varepsilon)$ *and a smoothing operator* $S_\varepsilon\colon\mathfrak{M}\to\mathfrak{M}$ *such that:*

(i) $\|S_\varepsilon(g)-g\|_{C^0}<\varepsilon$,

(ii) $\|\nabla^k \mathrm{Rm}(S_\varepsilon(g))\|_{C^0}<C(n,k,\varepsilon)\,\|\mathrm{Rm}(g)\|_{C^0}$.

In this theorem the C^0-norms can be taken in terms of g or $S_\varepsilon(g)$ since, by (i), they are nearly the same.

A typical application of this theorem is the following: If (M,g) is an ε'-flat manifold, i.e. a manifold with diameter $\leqq 1$ and sectional curvature $|K|\leqq\varepsilon'$, then the theorem yields a nearby metric on M whose curvature tensor is bounded in the C^k-norm by $C(n,k,\varepsilon)\cdot\varepsilon'$. With this additional regularity the technical difficulties in constructing a locally homogeneous structure for M, compare [6], are substantially reduced. Another instance where the theorem is useful comes up when we study a riemannian metric by investigating its bundle of orthonormal frames or even higher order frame bundles. In this process derivatives are lost and the theorem provides us with a method of gaining the derivatives needed. A third application is to the study of compactness properties of the set of n-dimensional compact riemannian manifolds with bounded curvature, diameter and injectivity radius, see [2, Chap. 8c]. Any sequence in this set has a convergent subsequence, but only in the set of $C^{1,1}$-riemannian manifolds, compare [2, p. 129]. To obtain smooth riemannian manifolds as limits we can smooth the sequence before taking the limit.

3. The Proof

Let g_{ij}, R_{ij} and R_{ijkl} denote, in the index notation, a riemannian metric g, its Ricci tensor Rc and its full riemannian curvature tensor Rm on a compact manifold M. We define the evolution of the metric $g=g(0)$ in time t by the equation

$$\frac{\partial}{\partial t} g_{ij}=-2R_{ij}. \tag{1}$$

According to Hamilton [3, Thm. 14.1], this equation has a unique solution for a maximal time interval $0\leqq t<T^*\leqq\infty$. If $T^*<\infty$, then $\sup|R_{ijkl}|\to\infty$ as $t\to T^*$. The evolution of the riemannian curvature tensor is then governed by the following heat equation: [3, Thm. 7.1]. (We will freely use the notation of [3]).

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl}$$
$$+2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})$$
$$-(C_{ijkl} - C_{jikl} + C_{klij} - C_{lkij}), \tag{2}$$

where $\Delta = \partial_p g^{pq} \partial_q$ is the "rough" Laplacian, ∂_p denotes the covariant derivative in the direction $\frac{\partial}{\partial x^p}$, and where

$$\begin{aligned} B_{ijkl} &= g^{pr} g^{qs} R_{piqj} R_{rksl} \\ C_{ijkl} &= g^{pq} R_{qi} R_{pjkl}. \end{aligned} \tag{3}$$

Our aim is now to estimate, by using this parabolic equation, the curvature together with its derivatives at a slightly later time $T > 0$ in terms of the C^0-norm of the initial curvature at time $t = 0$. We begin by estimating the growth of the C^0-norm of R_{ijkl}. The evolution inequality for $|\mathrm{Rm}|^2$, which follows from Eq. (2) for R_{ijkl}, is given by [3, Cor. 13.3]:

$$\frac{\partial}{\partial t} |\mathrm{Rm}|^2 - \Delta |\mathrm{Rm}|^2 \leqq C_1 |\mathrm{Rm}|^3, \tag{4}$$

where C_1 is a constant depending only on the dimension n.

Applying the maximum principle to this inequality we obtain:

$$\sup_{x \in M,\, 0 \leqq \tau \leqq t} |\mathrm{Rm}(x, \tau)| \leqq \|\mathrm{Rm}_0\|_{C^0} (1 - C_2 t)^{-1}, \tag{5}$$

where Rm_0 is the initial curvature.

We now choose $t = T$ small enough so that

$$\|\mathrm{Rm}(x, t)\|_{C^0(M \times [0, T])} \leqq C_3 \|\mathrm{Rm}_0\|_{C^0}, \tag{6}$$

where $C_3 \to 1$ as $T \to 0$.

Now, since the metric evolves by Eq. (1) and because the C^0-norm of R_{ij} is dominated by that of Rm, we can choose for any given $\varepsilon > 0$ a time $T > 0$ such that $S_\varepsilon(g) = g(T)$ satisfies the estimate (i) of our theorem. In order to guarantee that $S_\varepsilon(g) \in \mathfrak{M}$ we may have to rescale the metric slightly.

To prove the smoothing property (ii) of our theorem, we shall apply interior regularity estimates to the heat equation (2) satisfied by the curvature. Such estimates are usually formulated, as in [5], for domains in euclidean space and since we want our estimates to be independent of the injectivity radius of M we use the standard trick of pulling back the metric and the equations we are considering, via the exponential map, from M to a ball in the tangent space at any point $p \in M$ which we are interested in. Moreover, because of our bounds on the curvature of M, we have a universal lower bound for the size r of the ball $B_r(0)$ in the tangent space where we can pull back the metric. The heat equation (2) for the curvature is now considered as a parabolic system with initial value $\mathrm{Rm}_0 = \mathrm{Rm}(g_0)$ defined in this ball $B_r(0)$, which is now regarded as a riemannian manifold with a time-dependent metric $g(t)$. In the course of applying the regularity results of [5] it will be necessary to define and control the Hölder norms of the quantities involved, in particular the coefficients, of Eq. (2) with respect to suitable coordinates. For our purposes it is most convenient to use *harmonic coordinates* as defined by Jost-Karcher in [4]. Such coordinates exist by [4, Satz 5.1] in a neighborhood $B_\rho(p)$ of any point p of a riemannian manifold M, where the radius ρ can be estimated from below in terms of curvature bounds and the injectivity radius. In our case, the injectivity radius does not enter into the estimate since we consider the metric lifted to a ball in the tangent space by means of the exponential map. These coordinates have the property that the Hölder norm of the metric and the Christoffel symbols Γ^i_{jk} can be estimated in terms of the curvature. More precisely, [4, Satz 3.4] implies the following estimates:

$$\begin{aligned} \|g_{ij}-\delta_{ij}\|_{C^{k+1+\alpha}(B_\rho)} &< C_4 \|\mathrm{Rm}\|_{C^k(B_\rho)} \\ \|g_{ij}-\delta_{ij}\|_{C^{k+2+\alpha}(B_\rho)} &< C_5 \|\mathrm{Rm}\|_{C^{k+\alpha}(B_\rho)}, \end{aligned} \tag{7}$$

where the constants C_4, C_5 are independent of Rm.

To study the parabolic equation (2) we define, as in [5], the following Hölder norms:

$$\|u\|_{C^{k+\alpha,0+\frac{\alpha}{2}}(Q_T)} = \sum_{|i|\leq k} \sup_{(x,t)\in Q_T} |D^i_x u(x,t)| + \sup_{|i|=k} [D^i_x u]_{C^{0+\alpha,0+\frac{\alpha}{2}}(Q_T)}$$

with

$$[v]_{C^{0+\alpha,0+\frac{\alpha}{2}}(Q_T)} = \sup_{\substack{(x,t),(y,s)\in Q_T \\ (x,t)\neq(y,s)}} \frac{|v(x,t)-v(y,s)|}{|x-y|^\alpha + |s-t|^{\alpha/2}}, \quad Q_T = B_\rho \times [0,T].$$

In local coordinates Eq. (2) is of the form

$$\begin{aligned} &\frac{\partial}{\partial t} u^\lambda - D_i(g^{ij} D_j u^\lambda) + b^i D_i u^\lambda + D_i(A^{i\lambda}_\mu u^\mu) + c^\lambda_\mu u^\mu = F^\lambda \\ &u^\lambda(x,0) = u^\lambda_0(x), \end{aligned} \tag{8}$$

where $u^\lambda = R_{ijkl}$, the coefficients b^i, $A^{i\lambda}_\mu$, c^λ_μ are polynomials in g^{ij} and Γ^i_{jk} and F is a linear combination of the tensors B_{ijkl} and C_{ijkl} defined in (3).

This is a parabolic system with the leading term in diagonal form. We also note that the non-linearity occurs only in the terms of order zero. As the first step of a bootstrap argument we wish to deduce, using estimate (6), a local $C^{0+\alpha,0+\frac{\alpha}{2}}$-bound on R_{ijkl} by means of a De Giorgi-Nash-Moser type theorem for linear parabolic equations with discontinuous coefficients. Here, the coefficients are actually C^∞ but the higher derivatives are not yet under control. Using harmonic coordinates, we have, by the first estimate of (7) for the case $k=0$, $g^{ij}\in C^{1+\alpha}$ and $\Gamma^i_{jk}\in C^{0+\alpha}$. This means that we have no estimate on the space derivatives of the coefficient $A^{i\lambda}_\mu$ in (8) and therefore the solution u of (8) is to be understood in the weak sense. We define u^λ to be a weak solution of the parabolic system (8) if the following holds:

$$\begin{aligned}
&\text{(i)}\quad \|u\|_{E(Q_T)} \equiv \sup_{0 \leqq t \leqq T} \|u\|_{L_2(B_\rho)} + \int_0^T \int_{B_\rho} (|Du|^2 + |u|^3)\, dx\, dt < \infty \\
&\text{(ii)}\quad \int_{B_\rho} u^\lambda \varphi^\lambda dx + \int_0^t \int_{B_\rho} g^{ij} D_j u^\lambda D_i \varphi^\lambda dx\, dt \\
&\qquad + \int_0^t \int_{B_\rho} b^i D_i u^\lambda \varphi^\lambda dx\, dt - \int_0^t \int_{B_\rho} A^{i\lambda}_\mu u^\mu D_i \varphi^\lambda dx\, dt \\
&\qquad + \int_0^t \int_{B_\rho} c^\lambda_\mu u^\mu \varphi^\lambda dx\, dt - \int_{B_\rho} u^\lambda_0(x) \varphi^\lambda(x,0)\, dx = \int_0^t \int_{B_\rho} F^\lambda \varphi^\lambda dx\, dt
\end{aligned} \tag{9}$$

for all φ with finite $E(Q_T)$-norm that vanish on $\partial B_\rho \times [0, T]$. In this form the coefficient $A^{i\lambda}_\mu$ occurs without derivatives.

For a solution of (9) the following local Hölder estimate holds, compare [5, p. 579–582]:

$$\|u\|_{C^{0+\alpha, 0+\frac{\alpha}{2}}(Q'_T)} \leqq C_6(\|g\|_{C^1(B_\rho)}, \|u\|_{C^0(Q_T)}), \tag{10}$$

where Q'_T is the smaller cylinder $B_{\rho_0} \times [t_0, T]$ with $t_0 > 0$ and $\rho_0 < \rho$.

For this estimate, which is our first step in a bootstrap argument, we use the harmonic coordinates defined by the metric at time $t=0$. By the first estimate of (7) for $k=0$, we then have a bound for $\|g\|_{C^1}$. For the remaining steps we fix equidistant partitions $0 < t_0 < \ldots < t_k < T$, $\frac{\rho}{2} = \rho_k < \ldots < \rho_0 < \rho$ and set $Q'_l = B_{\rho_l} \times [t_l, T]$, for $l = 0, \ldots, k$, where $2k$ is the number of derivatives we want to estimate. At each step l, we wish to have the following interior Schauder estimate:

$$\|\mathrm{Rm}\|_{C^{2l+2+\alpha, 0+\frac{\alpha}{2}}(Q'_{l+1})} \leqq C_7(\|\mathrm{Rm}\|_{C^{2l+\alpha, 0+\frac{\alpha}{2}}(Q'_l)}). \tag{11}$$

Such an estimate holds, see [5, p. 351–355], provided we have bounds on the $C^{2l+\alpha, 0+\frac{\alpha}{2}}$-Hölder norm of the coefficients of (8). This is now achieved by writing the equation with respect to canonical harmonic coordinates of [4] determined by the metric frozen at the end of the previous step, i.e. at time $t = t_l$. By induction we know that the curvature is $C^{2l+\alpha, 0+\frac{\alpha}{2}}$ in Q'_l with respect to these coordinates. A slight extension of the proof of [4, Satz 5.1] to the case where one has a metric g depending on a parameter t then establishes the following estimate:

$$\|g_{ij} - \delta_{ij}\|_{C^{2l+2+\alpha, 0+\frac{\alpha}{2}}(Q'_l)} \leqq C_8 \|\mathrm{Rm}\|_{C^{2l+\alpha, 0+\frac{\alpha}{2}}(Q'_l)}. \tag{12}$$

In fact, harmonic coordinates are defined in [4] by solving an elliptic boundary value problem and hence they depend Hölder continuously on the data. (12) now implies the required regularity of the coefficients for the estimate (11) to hold. This completes the induction and we obtain

$$\|\mathrm{Rm}\|_{C^{2k+\alpha}(B_{\rho/2})} \leqq C_9(T, \rho, \|\mathrm{Rm}_0\|_{C^0}). \tag{13}$$

Since the non-linearity in Eq. (8) occurs only on the right hand side F which is quadratic in Rm, we may replace the bound C_9 by a linear function in $\|\mathrm{Rm}_0\|_{C^0}$ for T small enough.

Finally, we remark that all the constants C_i do not depend on the metric g explicitly but only on bounds for the norm of g in a function space. Therefore, our estimates hold uniformly in the space $\mathfrak{M}$ of riemannian manifolds with bounded curvature.

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