K-Area, mass and asymptotic geometry

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Preface

These notes form an outline of a series of lectures that I delivered at the Universitat Jaume I in Castellon de la Plana from July 17 to 23, 2001 for the *Advanced Course on Global Riemannian Geometry: curvature and topology*, organized by the Centre de Ricerca Mathematica, Institut d'Estudis Catalans.

The theme of these notes is centered around the use of the Dirac operator in Geometry and Physics, with the main focus on scalar curvature, Gromov's K-area and positive mass theorems in General Relativity.

The notes are written in a rather impressionistic style following Hermann Weyl's advice. I quote:

"The stringent precision attainable in Mathematics has led many authors to a mode of writing which must give the reader the impression of being shut up in a brightly illuminated cell where every detail sticks out with the same dazzling intensity, but without relief. I prefer the open landscape with a clear sky with its depth of perspective where a wealth of sharply defined nearby details gradually fade away towards the horizon".

My objective is to give a flavour of some selected aspects of the subject that I understand, rather than a comprehensive survey. I have not stated results and proofs in their most general form or in their most recent version to avoid getting bogged down in too many technical details. Instead, I have attempted to give a broader overview of the topics at an introductory level, emphasizing more the basic ideas and concepts involved, so that the reader can more easily consult the literature. Although I have included a number of important references to the original literature, my list is certainly not complete and I apologize for any serious omissions, but nowadays, it is a simple matter to access recent literature through preprint servers and the MathSciNet.

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1 Spinors and the Dirac Operator

1.1 Introduction to spinors

Spinors are objects that are more sensitive to the action of the orthogonal group than ordinary vectors. The simplest example is the Hopf bundle over S^2. The bundle is half as curved as the tangent bundle of S^2 geometrically and topologically. Parallel translation in this bundle around a great circle in S^2 rotates a spinor by an angle of π instead of 2π for a vector and the Euler characteristic of the Hopf bundle is 1, which is half that of the tangent bundle. More generally, the existence of such a double cover of the orthonormal frame bundle called a spin structure would be guaranteed by the vanishing of a suitable characteristic class.

There are two steps involved in defining a spin structure on a manifold. The first is a purely algebraic construction of Clifford algebras and spin groups associated to a vector space with a given quadratic form (scalar product). The second has to do with the way these structures can be globally defined on the tangent bundle or more generally on vector bundles. A rough definition of a spin structure is therefore an assignment of a spin group to each point on the manifold, in a manner consistent with the transition functions of the tangent bundle.

We will begin with a description of Clifford algebras, which are the devices needed to give an explicit construction of the spin groups. After that, we will turn to the problem of globalizing such objects to manifolds, i.e. to the problem of defining a spin structure on a smooth manifold. It is at this stage that global topological features of the tangent bundle become important. Before turning to the algebraic part, it is useful to mention that 'two step' constructions from an algebraic infinitesimal object to a global definition are quite canonical in differential geometry. A metric on a manifold, for example, is a smoothly varying choice of inner product on the tangent space at each point of the manifold. The construction thus involves extending inner products on each tangent space smoothly to the tangent bundle. Of course, for positive definite inner products the extension is always possible. On the other hand, to define an orientation on a manifold, we must consistently extend the orientation defined for each tangent space to the whole manifold and this is in general, not possible. The obstruction to this is called the first Stiefel-Whitney class. For the existence of a spin structure on a manifold the obstruction is called the second Stiefel-Whitney class. Now for the algebra:

A *quadratic space* is a pair (V, q), where V is a real finite-dimensional vector space and $q : V \otimes V \to \mathbb{R}$ is a non-degenerate quadratic form. We denote by $O(V, q)$ the orthogonal group of q. The real *Clifford algebra* $Cl(V, q)$ is the (unique) unital associative algebra generated by V subject to the defining relation

$$v \cdot w + w \cdot v = -2q(v, w)1$$

for every $v, w \in V$.

If c is a linear map from V to a unital algebra satisfying $c(v) = -q(v, v)$ then c extends naturally to an algebra homorphism and the Clifford algebra is the universal algebra with that characterizing property. As a vector space the Clifford algebra is isomorphic to the exterior algebra $\wedge V$ of V, but the multiplication is, of course, different.

The isometry $v \mapsto -v$ extends to an involutive automorphism χ of the algebra defining its $\mathbb{Z}/2$-grading:

$$Cl(q) = Cl^0(q) \oplus Cl^1(q)$$

. If $V = \mathbb{R}^m$ and q is the standard positive definite quadratic form, then we will simply write $Cl_m = Cl^0_m \oplus Cl^1_m$. The map $\mathbb{R}^m \to Cl^0_{m+1}$, $v \mapsto ve_{m+1}$, extends to an isomorphism of algebras: $Cl_m \cong Cl^0_{m+1}$.

If $(e_1, \ldots, e_m)$ is an orthonormal frame in V, then $\eta = e_1 \ldots e_m \in Cl(q)$is called the *volume element*. The square of η is either 1 or -1, depending on the signature of q.

Let (M, g) be a Riemannian manifold. The *Clifford bundle* of (M, g) is the total space $Cl(M, g) = \bigcup_x Cl(T_xM, g_x)$ of all the Clifford algebras of the tangent spaces.

A bundle of *Clifford modules* on (M, g) is a complex vector bundle S over M with a homomorphism of bundles of algebras $\gamma : Cl(g) \to End(S)$, i.e., for every $x \in M$, the vector space S_x is a left module over the algebra $Cl(g_x)$. Restricted to $TM \subset Cl(g)$, the map γ is a *Clifford morphism*, i.e. a homomorphism of vector bundles such that

$\gamma(v)^2 = -|v|^2 id_{S_x}$ for every $x \in M$ and $v \in T_xM$. It follows from the universal property of Clifford algebras that, conversely, given a vector bundle S over M and a Clifford morphism: $TM \to End(S)$, one can extend it to a homomorphism of bundles of algebras.

Here are two examples:

(i) The bundle of *exterior algebra* on M. Put $S = \bigwedge T^*M$ and define γ by $\gamma(v)\omega = v \lrcorner \omega + v \wedge \omega$ for $v \in T_xM$ and $\omega \in S_x$ where $\lrcorner$ denotes interior multiplication with respect to the given metric. The isomorphism $Cl(M,g) \cong \bigwedge T^*(M)$ given by the map $v \mapsto \gamma(v) \cdot 1$ is called the symbol map. It is the symbol of the deRham operator $d + \delta$ on the exterior algebra of forms.

(ii) Let (M,g) be a *Kähler manifold* with complex structure J. The map: $\gamma(v+\bar{v})\omega = \sqrt{2}(\bar{v}\lrcorner\omega + v \wedge \omega)$ for $w \in W = \{v \in TM \otimes \mathbb{C} \,|\, J(v) = i\,v\}$ and put $\omega \in S = \bigwedge W$ defines a Clifford module.

If $n \geq 3$ then the fundamental group of the special orthogonal group $SO(n)$ is $\mathbb{Z}/2$ and the simply connected universal cover is a group called $Spin(n)$. We will use Clifford algebras to describe this group.

First of all, the pinor group $Pin(m)$ is the group consisting of all products of unit vectors in Cl_n. The map $\rho(v) : x \mapsto v \cdot x \cdot v^{-1}$, where v is a unit vector and x is any vector in $\mathbb{R}^m$, describes the reflection in the hyperplane $v^\perp$ and hence defines a representation from $Pin(m)$ to $O(m)$ which is a double cover. Since $O(m)$ has two connected components, we can restrict to the pre-image of the identity component $SO(m)$ to obtain the spinor group $Spin(m)$. Therefore $Spin(m) = Pin(m) \cap Cl^0_m$ and is generated by products of an even number of unit vectors in the Clifford algebra.

We can also complexify the Clifford algebra $Cl^c_m = Cl_m \otimes \mathbb{C}$ and define the complex spinor group as $Spin^c(m) = Spin(m) \otimes_{\mathbb{Z}/2} S^1$.

One basic property of the Spin group is that there exists (half-integral) representations which do not descend to $SO(n)$. The basic representation space is called the space of spinors and these are the "sensitive" objects, since the orthogonal group cannot act as a single-valued representation.

In even dimensions $m = 2n$, the algebra Cl_m is a simple matrix algebra and there is a unique faithful and irreducible (graded) Dirac representation in a complex, 2^n-dimensional vector space $\mathbb{S}$ called the spinor space such that $Cl_m \otimes \mathbb{C} = End(\mathbb{S})$. Restricted to Cl_m^0 (and hence to $Spin(n)$), this representation decomposes into the direct sum of two irreducible and inequivalent, half-spinor Weyl representations $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. The splitting is basically given by the eigenspaces of Clifford multiplication with the volume element η. In odd dimensions, we can use the isomorphism: $Cl_{2n} \cong Cl^0_{2n+1}$ to obtain the unique irreducible complex spinor representation of dimension 2^n. (The representation does not split w.r.t. $Spin(2n+1)$). There are exactly two irreducible representations of Cl_{2n+1}, of complex dimesion 2^n which become isomorphic representations when restricted to $Spin(2n+1)$, since the volume form η is now the intertwining map.

The *Stiefel-Whitney classes* $w(E)$ of a real vector bundle E of rank r are $\mathbb{Z}/2$-characteristic classes characterized by the following properties:

(i) $w(E) = 1 + w_1(E) + ... + w_r(E)$ with $w_i(E) \in H^i(M; \mathbb{Z}/2)$

(ii) If $f : M \to N$ is a map then $f^*(w(E)) = w(f^*(E))$.

(iii) $w(E \oplus F) = w(E) + w(F)$.

(iv) If L is the non orientable Möbius line bundle over S^1 then $w(L) \neq 0$.

The first Stiefel-Whitney class measures the obstruction for a vector bundle to be orientable and the non-vanishing of the second is the obstruction to the existence of a spin structure.

Let E be an oriented vector bundle with a fiber metric over a manifold M and let U_α be a simple cover of M such that E has transition functions $g_{\alpha\beta} \in SO(r)$ on $U_\alpha \cap U_\beta$ satisfying the cocycle condition $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$. We say that E admits a spin structure if E is orientable and we can define lifts $\tilde{g}_{\alpha\beta}$ of the transition functons to $Spin(r)$ such that the cocycle condition is preserved. This can be expressed in terms of the Stiefel Whitney classes simply as $w_2(E) = 0$. The set of all inequivalent spin structure is then parametrized by $H^1(M; \mathbb{Z}/2)$.

From the point of view of principal bundles, a spin structure can be realized as a principal bundle $Spin(E)$ with structure group $Spin(n)$ which is a double cover (over each fiber) of the oriented frame bundle $SO(E)$ of the vector bundle E. Given a spin manifold, the vector bundle associated to the basic spinor representation will be denoted by $\mathbb{S}$ and is called the spinor bundle.

Similarly, the necessary and sufficient topological condition to define a $Spin^c$ structure on a unitary bundle E is that $w_2(E)$ is the $\mod 2$ reduction of an integral cohomology class. This is always true for a Hermitian vector bundle E, since $w_2(E) \equiv c_1(E) \mod 2$.

Examples

1. The sphere S^n is spin for all n.

2. The complex projective space is spin if and only if m is odd. The spin structure is unique since $\mathbb{C}P^m$ is simply connected. However, as is true for all Kähler manifolds, $\mathbb{C}P^m$ has a canonical $Spin^c$ structure for all m.

3. The product and connected sum of spin manifolds is again spin.

1.2 The Dirac Operator

The Dirac operator is the fundamental first order elliptic operator defined on a spin manifold. Its symbol is given by Clifford multiplication and its index is given by the $\hat{A}$-genus of the spin manifold.

Let M^n be a Riemannian manifold with Clifford bundle $Cl(M)$ and let S be any bundle of left modules over $Cl(M)$. Assume that S is furnished with a metric and a connection preserving the metric and compatible with the Clifford module structure, (i.e. the product rule holds). Then the Dirac operator of S is the canonical first-order differential operator defined by:

$$D\sigma = \sum_{k=1}^{n} e_k \cdot \nabla_{e_k}\sigma$$

where $\{e_k\}$ is an orthonormal base of TM and $\sigma \in \Gamma(S)$.

This defines a general Dirac operator since all we need is a Clifford module over a (not necessarily spin) manifold. However, we will be mainly interested in twisted Dirac operators that are defined on $\mathbb{S} \otimes E$ where E is a complex Hermitian bundle with a connection and $\mathbb{S}$ is the spinor bundle. These operators can sometimes be defined even if the manifold is not spin, provided the tensor product exists as a bundle. For example for Kähler manifolds we always have the $spin^c$ Dirac operator where E is a line bundle which can be thought of as the "virtual square root" of the canonical bundle. Although E and $\mathbb{S}$ do not exist globally on the manifold, $\mathbb{S} \otimes E$ and $E \otimes E$ are well defined.

For even dimensional manifolds the spinor representation has a natural splitting $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ and the Dirac operator splits as $D = D^+ + D^-$ with $D^\pm : \mathbb{S}^\pm \otimes E \longrightarrow \mathbb{S}^\mp \otimes E$ and D^- being the adjoint of D^+. Since the Dirac operator on a closed compact manifold is a self-adjoint elliptic operator it has a real discrete spectrum with finite multiplicities on a compact manifold. In particular, the index of D^+ :

$$\text{index}(D^+) = \dim(\text{Kern}(D^+)) - \dim(\text{Kern}(D^-))$$

is a topological invariant given by the famous Atiyah-Singer Index Theorem:

$$\text{index}(D^+) = \int_M \hat{A}(M) \wedge ch(E)$$

where the $\hat{A}$ genus, a certain formal power series in the Pontryagin classes of M, and $ch(E)$, the Chern character of the vector bundle E, will be defined below.

The Chern character of a complex bundle E of rank r can be defined by

$$ch(E) = \sum_{k=1}^{r} exp(x_k)$$

where the total Chern class is expressed (by the splitting principle) as:

$$C(E) = 1 + c_1(E) + ... + c_r(E) = \prod_{k=1}^{r}(1 + x_k)$$

so that c_k is given by the kth elementary symmetric function of the x_k's. The first few terms are:

$$ch(E) = \dim(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + ...$$

The Chern character satisfies:

$$ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2), \;\; ch(E_1 \otimes E_2) = ch(E_1)ch(E_2)$$

and hence defines a ring homorphism $ch : K(M) \longrightarrow H^{even}(M)$.

Similarly, the total $\hat{A}$ genus is given by:

$$\hat{A}(M) = \prod_{k=1}^{r} \frac{x_k/2}{\sinh(x_k/2)}$$

where now the total Pontryagin class of TM is formally expressed as:

$$p(M) = 1 + p_1(M) + ... + p_r(M) = \prod_{k=1}^{r}(1 + x_k^2)$$

so that p_k is given by the kth elementary symmetric function of the x_k^2's.

The first few terms are:

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{2^7 3^2 5}(-4p_2 + 7p_1^2) + ...$$

We can represent the Chern character of E by the differential form

$$ch(E) = Tr\Big(exp(\frac{F^\nabla}{2\pi i})\Big)$$

where F^∇ is the curvature of a connection ∇ for E, regarded as an $End(E)$-valued two form.

Similarly, $\hat{A}(M)$ is represented by the closed differential form:

$$\hat{A}(M) = \sqrt{\det}\Big(\frac{R/2}{sinh(R/2)}\Big)$$

where R is the Riemannian curvature of the metric g, regarded as and $End(TM)$-valued two form and $\sqrt{\det}$ is the Pfaffian.

1.3 The Lichnerowicz Formula

The Bochner technique of proving vanishing theorems for harmonic forms rely on expressing the relevant Laplacian as a sum of a non-negative operator (the rough Laplacian) and a purely algebraic terms depending only on the curvature. For the square of the Dirac operator, the corresponding result is the famous formula of Lichnerowicz:

$$D^2 = \nabla^*\nabla + \frac{R}{4}$$

where ∇ is the Levi-Civita connection, ∇^* its adjoint and R is the scalar curvature.

The surprising element here is the simplicity of the curvature term. Only the simplest invariant, namely the scalar curvature appears. As we will see in the proof below, this is partly a consequence of the fact that the spin representation is very "democratic" in the sense that all weights are equal.

The Lichnerowicz formula implies that a compact spin manifold with positive scalar curvature has no non-zero harmonic spinors. As a consequence, by the index theorem, compact spin manifolds with a non-zero $\hat{A}$-genus do not carry metrics of positive scalar curvature. Also it is interesting to note that the curvature expressions for the characteristic classes appearing in the index theorem are quite elaborate and it is not easy to see why the simple condition $R > 0$ should imply directly that the top form in the index density is exact (on a compact manifold). The index density is the (super-)trace of the heat kernel $\exp(-tD^2)$ as $t \to 0^+$ whereas the harmonic projectors describe the behaviour of the heat kernel as $t \to \infty$, so there should be a natural way that the index density and the Lichnerowicz formula are related by varying t in the heat kernel.

For the twisted Dirac operator with values in a vector bundle E, the Lichnerowicz formula for D^2 is computed to be:

$$D^2(\sigma \otimes \phi) = \nabla^*\nabla(\sigma \otimes \phi) + \frac{R}{4}\sigma \otimes \phi + \mathcal{R}(\sigma \otimes \phi)$$

for $\sigma \otimes \phi \in \Gamma(\mathbb{S} \otimes F)$, where $\nabla^*\nabla$ is the rough Laplacian, R is the scalar curvature and the last term is explicitly given by:

$$\mathcal{R}(\sigma \otimes \phi) = \frac{1}{2} \sum_{j,k=1}^{m} \gamma(e_a)\sigma \otimes R^{\nabla}(e_a)\phi$$

where $\{e_a\}, a = 1, ...m = \frac{n(n-1)}{2}$ is now an orthonormal base with respect to the metric g for the two-vectors $\bigwedge^2(T_pM)$ at the point in question, R^{∇} is the curvature tensor of the connection in the bundle E and γ is Clifford multiplication for g.

Proof

We define the second order covariant derivative: $\nabla^2_{u,v} = \nabla_u \nabla_v - \nabla_{\nabla_u v}$ where for simplicity of notation, ∇ is used for all the covariant derivatives $\nabla^2_{u,v}$ is tensorial in u,v and its antisymmetric part is the curvature: $R(u,v) = \nabla^2_{u,v} - \nabla^2_{v,u}$ (Here we use the fact that the Levi-Civita connection is torsion free). Now using a frame satisfying $\nabla_{e_k} e_l = 0$ at a given point we compute:

$$\begin{aligned} D^2 &= \sum_{k,l} e_k \cdot \nabla_{e_k}\big(e_l \cdot \nabla_{e_l}\big) \\ &= \sum_{k,l} e_k \cdot e_l \cdot \nabla_{e_k} \nabla_{e_l} \\ &= \sum_{k=l} e_k \cdot e_l \nabla_{e_k} \nabla_{e_l} + 2 \sum_{k<l} e_k \cdot e_l \nabla_{e_k} \nabla_{e_l} \\ &= -\sum_{k} \nabla_{e_k} \nabla_{e_k} + \sum_{k<l} e_k \cdot e_l \cdot R(e_k, e_l) \end{aligned}$$

The first term is the rough Laplacian $\nabla^*\nabla$ and the second term acting on $\sigma \otimes \phi$ can be simplified as:

$$\sum_{k<l} e_k \cdot e_l \cdot R(e_k, e_l)(\sigma \otimes \phi) = -\sum_{a=1}^{m} e_a \cdot \hat{R}(e_a)\sigma \otimes \phi + \sum_{a=1}^{m} e_a \cdot \sigma \otimes R^{\nabla}(e_a)\phi$$

where $\{e_a\}, a = 1, ..., m = \frac{n(n-1)}{2}$ is now an orthonormal base for for $\bigwedge^2(TM)$ and $\hat{R}$ is the curvature operator of the Riemannian manifold. (Note the sign change!).

The second term is the same as $\mathcal{R}$ and the first term can be further simplified by choosing an orthonormal base $\{e_a\}$ for $\bigwedge^2(TM)$ that diagonalizes the curvature operator, so that $\hat{R}(e_a) = \lambda_a \, e_a$ and $\hat{R}(e_a)\sigma = \frac{1}{2}\lambda_a \, e_a \cdot \sigma$:

$$-\sum_{a=1}^{m} e_a \cdot \hat{R}(e_a)\sigma = -\frac{1}{2}\sum_{a=1}^{m} \lambda_a \, e_a \cdot e_a \cdot \sigma = \frac{R}{4}\sigma$$

This proves the Lichnerowicz formula.

2 Gromov's K-Area

2.1 Definition of K-Area

The K-area is, roughly speaking, the inverse of the norm of the smallest curvature obtainable among all topologically essential unitary bundles equipped with connections on a given Riemannian manifold. K stands here both for K-theory and also for curvature (Krümmung). To measure the norm of the curvature, a metric g on the manifold is used. However the definition does not involve the Riemannian curvature of the metric and hence the K-area is a pure C^0-invariant of g. It measures the K-theoretic 2-dimensional size of the manifold. One can modify the definition by taking the supremum with respect to a suitable class of metrics, e.g., adapted metrics for a symplectic manifold. One can also restrict ourselves to a special class of bundles to get more refined invariants. We refer to [G] for more details.

Since we would like to speak about symplectic connections, we will describe a rather general set-up to define the K-area. Let G be a connected Lie group, not necessarily finite-dimensional, whose tangent bundle is equipped with a bi-invariant norm defining a left invariant metric on G. We identify the Lie algebra $\mathfrak{g}$ of G with the space of right-invariant vector fields on G. Suppose that G acts on a connected manifold. The standard situation is when F is a vector space E and we have a representation of G. Another case which is also important is when F is a symplectic manifold and G is a subgroup of the infinite dimensional group of all Hamiltonian symplectomorphisms: $Ham\,(F,\omega)$.

Consider a fiber bundle $\pi : P \to M$, over a Riemannian manifold (M,g) with fiber F associated to a G-principal bundle. and G -connections on these bundles, i.e. connections whose parallel transports belong to the structural group G. Let R^∇ denote the curvature of a connection ∇ on the bundle P.

To a pair of vectors $v, w \in T_xM$, the curvature tensor associates an element $R^\nabla(v,w) \in \mathfrak{g}$, defining a G-vector field on the fiber $\pi^{-1}(x)$. The fiber $\pi^{-1}(x)$ can be identified with F and since $\|\cdot\|$ is bi-invariant norm on $\mathfrak{g}$, $\|R^\nabla(v,w)\|$ is well-defined independent of the identification. For

$G = U(N)$ acting on a vector space E through a representation ρ, we will use the following supremum norm:

$$\|A\| = \max_{|u|=1} |\rho(A)(u)|,$$

where the maximum is taken over all unit vectors in E.

For a given bundle with a connection ∇ we define:

$$\|R^{\nabla}\ (F)\| = \sup_{|v \wedge w|=1} \|R^{\nabla}(v, w)\|,$$

where the maximum is taken over all unit bi-vectors $v \wedge w \in \Lambda^2(TM)$ with respect to the metric g.

Gromov's K-area of a compact even dimensional Riemannian manifold (M^{2m}, g) is now defined by taking the supremum of $\|R^{\nabla}(E)\|^{-1}$ over all unitary bundles E with structure groups $U(N)$ (all N), that have a non-vanishing Chern number. That E has a non-vanishing Chern number is equivalent to the fact that the classifying map for E: $\chi_E : M \longrightarrow BU(N)$ is not homologous to zero. By an algebraic calculation involving Chern classe, it can also be shown (see [GR3]) to be equivalent to the non-vanishing of the index of the Dirac operator twisted with the bundle E. We will call such bundles *homologically essential*.

Definition.

$$K\text{-area}\,(M^{2m}, g) = \sup_{E,\nabla} \|R^{\nabla}(E)\|^{-1},$$

where the maximum is taken over all homologically esssential unitary bundles E of all dimensions and over all connections ∇.

Since the classifying space carries a universal connection on its universal bundle and every unitary connection is induced by a map into BU, one can think of minimizing the"surface area" among all homologically essential classifying maps.

In order to extend the definition to odd dimensional manifolds we first define K-area for non compact even dimensional manifolds exactly as above, except that we use bundles E which are trivial outside a compact set and also characteristic classes with compact support. Now for an odd dimensional manifold (M^{2m+1}, g), we stabilize by taking products with $\mathbb{R}^{2k+1}$ for all k and define the stable K-area as:

$$K\text{-area}_{st}\,(M^{2m+1}, g) = \sup_k K\text{-area}\,(M \times \mathbb{R}^{2k+1}, g \times \bar{g})$$

The K-area has some fundamental properties (see [G] for more details):

(i) K-area scales like a two dimensional area and if $g_1 \geq g_2$ on 2-vectors, then $K\text{-area}\,(M, g_1) \geq K\text{-area}\,(M, g_2)$.

(ii) The K-area of a simply connected manifold is finite, since on a simply connected manifold an almost flat connection can be deformed to a flat connection.

(iii) A finite covering (which is trivial outside a compact set) has the same K-area. This implies in particular that the K-area of a torus is ∞, since it can cover a multiple of itseslf by homotheties.

(iii) More generally, the K-area of a closed manifold of non-positive sectional curvature whose fundamental group is residually finite is ∞, since such manifolds admit finite coverings can cover a multiple of itseslf by homotheties.

That the torus has infinite K-area can also be seen from the fact that there are homologically essential bundles on a covering torus with arbitrarily small curvature. A large even dimensional (covering) torus can always be mapped to a standard round sphere: $f : T^{2m} \to S^{2m}$ such that $|df|$ is very small. Now we can pull back the spinor bundle $\mathbb{S}^+$ on the sphere (this bundle has non-zero top Chern class and is the fundamental generator of the K-theory of even-dimensional spheres) to the torus via the map f. $f^*(\mathbb{S})$ is then homologically essential but will have very small curvature on a large torus, showing that the K-area of a torus is arbitrarily large.

2.2 The fundamental estimate in terms of scalar curvature

Although the definition of K-area does not involve the curvature tensor of the Riemannian metric, there is a deep and perhaps surprising connection to the scalar curvature. In fact, part of the motivation that led Gromov to this new invariant is to give a new interpretation of the proof given by Gromov and Lawson of the following fundamental global theorem on scalar curvature rigidity of the torus.

Theorem 1. *Let g be a Riemannian metric on T^n with scalar curvature $R(g) \geq 0$ everywhere. Then g is flat.*

This was proved first by Schoen-Yau [SY1] for low dimensions (≤ 7) and then by Gromov-Lawson [GL1,2,3] in all dimensions. The proof by Schoen and Yau is a somewhat simpler version of their argument to establish the positive mass theorem and uses the second variation formula for stable minimal surfaces. The proof by Gromov-Lawson on the other hand, uses spinors and is closer in spirit to Witten's subsequent proof of the positive mass theorem. However, since the result is global, the argument is more elaborate than Witten's and is based on the Index Theorem. Gromov's definition of K-area gives an elegant re-interpretion of the basic idea of their proof, expressing it as a fundamental inequality relating the scalar curvature to the K-area.

The main technique is to obtain a bound for the K-area from above, in terms of the inverse of the infimum of the scalar curvature, provided the manifold carries a metric with positive scalar curvature. This can be achieved by analyzing the Lichnerowicz formula and using the index theorem. Since the torus has infinite K-area, we see that a torus cannot carry a metric of positive scalar curvature. Some extra geometric work is then needed to get the full scalar curvature rigidity result for the torus.

The fundamental K-area inequality can be stated as follows:

Theorem 2. *Every complete Riemannian spin manifold (M^n, g) with scalar curvature $R(g) \geq \kappa^2$ everywhere satisfies:*

$$K\text{-}area_{st}\,(M,g) \leq \frac{c(n)}{\kappa^2}$$

for some universal constant $c(n)$ depending only on the dimension.

The proof is an immediate cosequence of the Lichnerowicz formula, the index theorem and the definitions. We refer again to [G] for details.

It is a natural question now to find Riemanian manifolds which are "extremal" with respect to this K-area inequality. The first sharp result of this nature was obtained by Llarull [L1] who made a careful analysis of the proof by Gromov and Lawson in the case of the sphere to obtain the following theorem on the scalar curvature rigidity of spheres.

Theorem 3. *Let g be a Riemannian metric on S^n satisfying $g \geq \bar{g}$ on all 2-vectors and with scalar curvature $R(g) \geq R(\bar{g}) \equiv n(n-1)$ everywhere, where $\bar{g}$ is the standard metric of constant sectional curvature $K \equiv 1$. Then $g \equiv \bar{g}$ everywhere.*

This is the main result in Llarull's work [L1][L2], where one can also find various extensions and generalizations. The theorem basically says that round spheres are extremal for the K-area inequality among spin manifolds. We will sketch a simplified proof of Llarull's theorem in even dimensions. An appropriate modification of the proof yields the following results for compact symmetric spaces.

Theorem 4. *Let $(M^{2n}, \bar{g})$ be a compact Hermitian-symmetric space of constant scalar curvature $R(\bar{g})$ with Kähler form ω. If g is any Riemannian metric on M satisfying $|\omega|_g < |\omega|_{\bar{g}}$ then there is a point on M where the scalar curvatures satisfy $R(g) < R(\bar{g})$.*

Theorem 5. *Let $(M^{2n}, \bar{g})$ be an even-dimensional compact symmetric space of constant scalar curvature $R(\bar{g})$. Assume that either the Euler class or the signature of M is non-zero. If g is any Riemannian metric on M satisfying $g > \bar{g}$ on all 2-vectors then there is a point on M where the scalar curvatures satisfy $R(g) < R(\bar{g})$.*

Before proving these results, I would like to mention a recent preprint by S.Goette and U. Semmelmann [GS], where theorem 4 has been generalized to Kähler manifolds with positive Ricci curvature. They show that these manifolds are extremal for the K-area inequality in the category of *spin*^{c}-manifolds.

The assumption on the metrics in Llarull's theorem can be stated more geometrically by saying that all surfaces in M have larger area with respect to g than the standard metric $\bar{g}$. For Hermitian-symmetric spaces, we relax the assumption on the metrics and compare them only on the Kähler form, i.e., only the areas of holomorphic curves need to be compared.

The key step in obtaining sharp estimates for the K-area inequality is to find the optimal homologically essential twisting bundle for the Dirac operator. We then have to make make careful estimates of the curvature terms that appear in the Lichnerowicz formula and then appeal to the index theorem. In order to illustrate this we now present a rather detailed proof of Llarull's theorem for even dimensional spheres. This is a somewhat simplified version of his proof and generalizes easily to prove Theorem 4.

For the purpose of local calculations, we may always assume, that the manifold is spin. Let $\mathbb{S}(g) = \mathbb{S}^+(g) \oplus \mathbb{S}^-(g)$ denote the bundle of spinors of an even dimensional spin manifold (M^{2n}, g), so we have two spinor bundles with respect to the two metrics g and $\bar{g}$, where $\bar{g}$ is the standard metric. We consider the twisted Dirac operator D on the bundle $\mathbb{S}(g) \otimes E$, where we choose the coefficient bundle to be $E = \mathbb{S}^+(\bar{g})$ (or $\mathbb{S}^-(\bar{g})$), the spinor bundle with respect to the spherical, or more generally, a symmetric background metric $\bar{g}$. Here we use the metric g and its Levi-Civita connection to define the Dirac operator on the spinors in $\mathbb{S}(g)$, but for the twisting bundles $\mathbb{S}^{\pm}(\bar{g})$, the Levi-Civita connection of the metric $\bar{g}$ is used.

We will regard the two spinor bundles $\mathbb{S}(g)$ and $\mathbb{S}(\bar{g})$ as isomorphic complex vector bundles over M, with two different metrics but more importantly, admitting two different Clifford multiplications by vectors and exterior forms on M. To distinguish the two distinct Clifford multiplications, we will denote them by: $\sigma \mapsto \bar{\gamma}(v)\sigma$ for the metric $\bar{g}$ and $\sigma \mapsto \gamma(v)\sigma$ for the metric g, where v is a tangent vector (or more generally for $v \in \Lambda^*(TM)$).

After diagonalizing the metric g with respect to $\bar{g}$, so that we have two orthonormal bases: $\{\bar{e}_i\}$ for $\bar{g}$ and $\{e_i = \frac{1}{\lambda_i}\bar{e}_i\}$ for g, we can define $\gamma(e_i) = \bar{\gamma}(\bar{e}_i)$ and extend it canonically to the whole Clifford algebra with respect to g to give us a new representation on the same Hermitian vector space $\mathbb{S}$. γ then satisfies $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)$ and $\gamma(u)$ is skew adjoint.

The twisted Dirac operator is then given by:

$$D(\sigma_1 \otimes \sigma_2) = \sum_{k=1}^{2n} \left\{ \gamma(e_k)\nabla_{e_k}\sigma_1 \otimes \sigma_2 + \gamma(e_k)\sigma_1 \otimes \overline{\nabla}_{e_k}\sigma_2 \right\}$$

where $\{e_k\}$ is an orthonormal base for the tangent vectors with respect to the metric g, ∇ is the Levi-Civita connection of g , $\overline{\nabla}$ is the Levi-Civita connection of the metric $\bar{g}$ and $\sigma_1 \otimes \sigma_2 \in \Gamma(\mathbb{S} \otimes \mathbb{S}^+)$.

To simplify notation, we denote the product connection by ∇, i.e.:

$$\nabla_v(\sigma_1 \otimes \sigma_2) = \nabla_v\sigma_1 \otimes \sigma_2 + \sigma_1 \otimes \overline{\nabla}_v\sigma_2.$$

The last term in the Lichnerowicz formula:

$$D^2(\sigma_1 \otimes \sigma_2) = \nabla^*\nabla(\sigma_1 \otimes \sigma_2) + \frac{R}{4}\sigma_1 \otimes \sigma_2 + \mathcal{R}(\sigma_1 \otimes \sigma_2)$$

can be expressed as:

$$\mathcal{R}(\sigma_1 \otimes \sigma_2) = -\frac{1}{2}\sum_{a=1}^{m} \gamma(e_a)\sigma_1 \otimes \bar{\gamma}\left(\bar{R}(e_a)\right)\sigma_2$$

where $\{\bar{e}_a\},\ a = 1, \ldots, m = n(2n-1)$ is an orthonormal base with respect to g for $\Lambda^2(T_pM)$ and $\bar{R}$ is the curvature operator of the symmetric metric. We note the right hand side is independent of the orthonormal base chosen and $\mathcal{R}$ is a well-defined self-adjoint algebraic operator on $\mathbb{S} \otimes \mathbb{S}$.

Let g now satisfy the condition: $g \geq \bar{g}$ on all 2-forms. This simply means that $g(v, v) \geq \bar{g}(v, v)$ for all $v \in \Lambda^2(TM)$. This implies that operator $\mathcal{R}$ dominates the corresponding operator $\overline{\mathcal{R}}$ defined by:

$$\overline{\mathcal{R}}(\sigma_1 \otimes \sigma_2) = -\frac{1}{2}\sum_{a=1}^{m} \bar{\gamma}(\bar{e}_a)\sigma_1 \otimes \bar{\gamma}\left(\bar{R}(\bar{e}_a)\right)\sigma_2$$

where $\{\bar{e}_a\},\ a = 1, \ldots, m$ is now an orthonormal base for $\bar{g}$, in the sense that $\mathcal{R} - \overline{\mathcal{R}}$ is positive semi-definite on $\mathbb{S} \otimes \mathbb{S}$ with respect to the metric $g \otimes \bar{g}$ (and hence also w.r.t. $\bar{g} \otimes \bar{g}$). This is easily seen by choosing $\{e_a\}$to be an orthonormal base of eigenforms that diagonalizes the metric g with respect to the background metric $\bar{g}$ on $\Lambda^2(T_pM)$, so that $e_a = \frac{1}{\lambda_a}\,\bar{e}_a$ with $\lambda_a \geq 1$ and with $\gamma(e_a) = \bar{\gamma}(\bar{e}_a)$. $\overline{\mathcal{R}}$ is then given by:

$$\overline{\mathcal{R}}(\sigma_1 \otimes \sigma_2) = -\frac{1}{2}\sum_{a=1}^{m} \frac{1}{\lambda_a}\left(\gamma(e_a)\sigma_1 \otimes \bar{\gamma}\left(\bar{R}(\bar{e}_a)\right)\sigma_2\right)$$

If $\mathcal{K}_p \subset \mathcal{O}(T_pM) = \Lambda^2(T_pM)$ denotes the holonomy (= isotropy) subalgebra of the symmetric space at the point in question, then the curvature operator of $\bar{g}$ is just the orthogonal projection onto $\mathcal{K}_p$ followed by an invertible symmetric map $S : \mathcal{K} \to \mathcal{K}$. Therefore:

$$\overline{\mathcal{R}}(\sigma_1 \otimes \sigma_2) = -\frac{1}{2}\sum_{a=1}^{m} \bar{\gamma}(\bar{e}_a)\sigma_1 \otimes \rho_a \bar{\gamma}(e_a)\sigma_2$$

where the ρ_a's are the non-zero eigenvalues of the curvature operator and $m = \dim(\mathcal{K})$.

On each simple factor $\mathcal{K}_s$ of $\mathcal{K}$, the eigenvalues of the operator $\bar{R}$ are all equal $(= \rho_s \neq 0)$, so

$$\overline{\mathcal{R}}_s(\sigma_1 \otimes \sigma_2) = -\frac{\rho_s}{2}\sum_{a=1}^{m_s} \bar{\gamma}(\bar{e}_a)\sigma_1 \otimes \bar{\gamma}(\bar{e}_a)\sigma_2$$

where $m_s = \dim(\mathcal{K}_s)$, and $\{\bar{e}_a\}$ is an orthonormal base for $\mathcal{K}_s \subset \Lambda^2(T_pM)$. $\overline{\mathcal{R}}_s$ is independent of the choice of the orthonormal base.

We now estimate the minimum eigenvalue of the operator $\overline{\mathcal{R}}_s$. The Casimir operator $\mathcal{C}_s$ of the representation on $\mathbb{S}\otimes\mathbb{S}$ induced by the isotropy representation restricted to each simple component $\mathcal{K}_s$ is given by:

$$\begin{aligned}
\mathcal{C}_s(\sigma_1 \otimes \sigma_2) &= -\sum_{a=1}^{m_s} \left(\bar{\gamma}(\bar{e}_a)\right)^2 (\sigma_1 \otimes \sigma_2) \\
&= -\sum_{a=1}^{m_s} \left\{\left(\bar{\gamma}(\bar{e}_a)\right)^2 \sigma_1 \otimes \sigma_2 + 2\bar{\gamma}(\bar{e}_a)\sigma_1 \otimes \bar{\gamma}(\bar{e}_a)\sigma_2 + \sigma_1 \otimes \left(\bar{\gamma}(\bar{e}_a)\right)^2 \sigma_2\right\} \\
&= 2m_s(\sigma_1 \otimes \sigma_2) + \frac{4}{\rho_s}\overline{\mathcal{R}}_s(\sigma_1 \otimes \sigma_2)
\end{aligned}$$

where we used the Clifford identity $\left(\bar{\gamma}(\bar{e}_a)\right)^2 = -Id$. Now $\mathcal{C}_s$ is positive semi-definite on $\mathbb{S} \otimes \mathbb{S} \approx \Lambda^*(TM)$. In fact, it is positive definite on all non-trivial irreducible components of the representation and is equal to zero only on the trivial representations that occur, in particular for Λ^0 and Λ^n. In any case, we have the following basic algebraic estimate:

$$\overline{\mathcal{R}}_s \geq -m_s \frac{\rho_s}{2} Id$$

This implies:

$$\mathcal{R} \geq \overline{\mathcal{R}} = \sum_s \overline{\mathcal{R}}_s \geq -\sum_s m_s \frac{\rho_s}{2} Id = -\frac{\overline{R}}{4} Id \geq -\frac{R}{4} Id$$

provided $R = R(g) \geq R(\bar{g}) = \overline{R}$. Moreover, $\mathcal{R}$ is strictly $> -\frac{R}{4} Id$ unless all the inequalities above are strict equalities everywhere on M. This yields, by the Lichnerowicz formula, a vanishing theorem for both $\mathbb{S}^+(\bar{g})$ and $\mathbb{S}^-(\bar{g})$ valued harmonic spinors on M, provided that $g \geq \bar{g}$ on $\mathcal{K}_p \subset \Lambda^2(T_pM)$ for each p and $R(g) \geq R(\bar{g})$ everywhere, with strict inequality holding at least at one point.

By the Index Theorem the topological indices of the two twisted Dirac operators are given by:

$$\int \hat{A}(M) \cdot ch(\mathbb{S}^\pm) = \frac{1}{2}(e(M) \pm \tau(M)) \tag{1}$$

where $e(M)$ is the Euler characteristic and $\tau(M)$ is the signature of M.

This proves Theorem 5, since the twisted bundles are globally defined for any Riemannian manifold and we need the factors (the spinor bundles) only for local calculations.

In the case of the sphere,the curvature operator is just the identity map on 2-vectors, so rigidity follows from the fact that if strict equality holds in the inequalities above we must have $g \equiv \bar{g}$ on ${\Lambda^2}_p$ at all points. Since $g \geq \bar{g}$ on Λ^2 this now implies that the two metrics are identical on each tangent space. The odd dimensional spherical case can be proved by applying the even-dimensional case to $M \times S^1$, which is given an appropriate metric with a long S^1. We refer to [L1].

To prove Theorem 4, we consider the twisted Dirac operator D on the bundle $\mathbb{S}(g) \otimes E$, where we choose the coefficient bundle E to be the line bundle $L(\bar{g})$, whose square is the canonical bundle of the Hermitian symmetric space. Here we use the metric g and its Levi-Civita connection to define the Dirac operator on the spinors in $\mathbb{S}(g)$, but for the twisting bundle $L(\bar{g})$, the connection induced by the Levi-Civita connection of the symmetric metric $\bar{g}$ is used. Under the assumption that $|\omega|_{\bar{g}} > |\omega|_g$ the operator $\mathcal{R}$ dominates the corresponding operator $\overline{\mathcal{R}}$:

$$\overline{\mathcal{R}}(\sigma \otimes l) = -\frac{\overline{R}}{4|\omega|_g^2}\bar{\gamma}(\omega)\sigma \otimes \bar{\gamma}(\omega)l$$

which in turn dominates $-\overline{R}/4$ so that $\mathcal{R} + \overline{R}/4$ is positive definite on $\mathbb{S} \otimes L$. The assumption $R \geq \bar{R}$ would then imply that there are no harmonic spinors by Lichnerowicz' formula. The tensor product of the spinor bundle with the line bundle, whose square is the inverse of the canonical bundle exists globally on any Hermitian manifold, and is the spinor bundle of the canonically associated *spin*^{c} structure. The index for the corresponding Dirac operator is given by the Todd genus which is non-zero for compact Hermitian symmetric spaces and Theorem 4 follows.

2.3 Connections with symplectic invariants

In symplectic geometry, there is a notion of fibrations $\pi : P \to M$ with a symplectic manifold F as fiber, where the structure group is the group of (exact) Hamiltonian symplectomorphisms of the fiber. These are called symplectic fibrations. If the base manifold (M, ω_M) is also symplectic, there is a weak coupling construction, originally due to Thurston, of defining a symplectic structure on the total space P. An efficient way to describe this procedure is through the use of the curvature of a symplectic connection and results in what is known as minimal coupling form. We rerfer to [GLS] for details. Parallel translation w.r.t. a symplectic connection is a symplectomorphism of the fiber and hence the symplectic curvature can be described as a two-form on the base manifold M with values in the Hamiltonian vector fields on the fiber at the given point and hence can be identified with a function (the Hamiltonian) on the fiber. We normalize Hamiltonians on compact symplectic manifolds to have mean value zero. If the fiber F is compact and simply-connected, each symplectic connection Γ gives rise to a unique *closed* 2-form ω^Γ on the total space P characterized by the following properties:

(i) ω^Γrestricts to the symplectic form on the fibers.

(ii) The horizontal space and the vertical space of the connection Γ are perpendicular w.r.t. ω^Γ.

(iii) On the horizontal space, ω^Γ coincides with the symplectic curvature of the connection. Γ

The cohomology class of ω^Γ is independent of the connection Γ and hence is a symplectic invariant of the fibration. In general, the form ω^Γ is not symplectic (it could be degenerate in horizontal flat directions). However if we define the weak coupling form:

$$\omega_\epsilon = \epsilon\omega^\Gamma + \pi^*(\omega_B)$$

then for sufficiently small ϵ, this would define a closed non-degenerate symplectic form on the total space P. The maximal possible value ϵ_{max} is a symplectic invariant called the maximal weak coupling constant.

Using this interpretation of symplectic curvature, we can now define the symplectic K-area of a given *fixed* symplectic fibration over a compact symplectic as the inverse of the minimum possible curvature. First we define the norm:

$$\|\omega^\Gamma\| == \max_{v,w} \frac{|\omega^\Gamma(\bar{v},\bar{w})|}{|\omega_B(v,w)|}$$

where $\bar{v},\bar{w}$ denote the horizontal lifts and the maximum is taken over all pairs of vectors v, w in the base manifold B such that $\omega_B(v,w) \neq 0$.

The symplectic K-area is then defined to be:

Definition.

$$K_{symp}\text{-area}\,(P) = \sup_\Gamma \|\omega^\Gamma\|^{-1}$$

where the maximum is taken over all symplectic connections on the fixed fibration:$P \to M$.

In many situations we are dealing with symplectic fibrations that arise from linear vector bundles. An important case is when the fiber is a co-adjoint orbit $F = \mathcal{O} \subset \mathfrak{g}^*$ and the bundle is associated to a principle

G-bundle via the adjoint representation. G is here a finite dimesional Lie group, e.g. $U(n)$. An "ordinary" connection ∇ is then necessarily symplectic since G acts symplectically on the co-adjoint orbits with moment map given by the incluson: $F \subset \mathfrak{g}^*$. The symplectic curvature of ∇ is therefore simply the curvature $R^\nabla \in \mathfrak{g}$ thought of as a linear function on $\mathfrak{g}^*$ restricted to the co-adjoint orbit. It is clear that the symplectic K-area is an upper bound for the "ordinary" K-area (of the fixed bundle), since we are taking the smallest possible curvature among a larger class of connections. Optimistically, one might expect that for simple co-adjoint orbits, the two K-areas are equal, since the curvature of "ordinary" connection induces a simple linear Hamiltonian function of the fiber and so should be minimizing among all symplectic connections. That this is in fact true, was shown by Polterovich [P1] for the special case of certain complex projective bundles over S^2. Moreover, it is not hard to see that the symplectic K-area is bounded from above by the maximal ϵ_{max} for the weak coupling constant and Polterovich was able to show that all three invariants are equal in these special cases. This allows him also to conclude a sharp estimate for the Hofer norm of some loops in $U(n+1)$ acting on $\mathbb{C}P^n$. His proof uses the theory of J-holomorphic curves and Gromov-Witten invariants on the total space of the fibration. It would be intriguing to see whether there is a simple "spinorial" proof in the spirit of the last section of these results and also more generally investigate when symplectic K-area inequalities are sharp.

2.4 The Vafa-Witten inequality

In 1984 Vafa and Witten [VW] proved the following surprising fact about the spectrum of twisted Dirac operators on compact spin manifolds. These results do not hold for ordinary Laplacians on bundle valued forms.

Theorem 6. *Let $|\lambda_1| \leq |\lambda_2| \leq ...$ be the eigenvalues ordered by their absolute values of a twisted Dirac operator D_E defined on $\mathbb{S} \otimes E$ over a compact spin manifold $(M.g)$ of dimension n, where we use a connection ∇ for E. Then there exists a constant $C(M, g)$, depending only on the Riemannian manifold $(M.g)$ and independent of the twisting bundle E*

and the connection ∇ on E such that for all k we have the following universal bound:

$$\frac{1}{k}|\lambda_k|^n \leq C(M,g)$$

For odd-dimensional manifolds, there are stronger results (which do not hold in even dimensions) to the effect that every interval on the real line of a certain lenth C (depending only on (M,g) but not on the twisting bundle E nor on the connection ∇ used on E) contains an eigenvalue of D_E.

Sketch of proof

For the sake of clarity, we will restrict ouselves to the case $k=1$, i.e., the first eigenvalue and to even-dimensional manifolds first.

If $\text{index}(D_E^+) \neq 0$, then there is nothing to prove since there is a harmonic spinor (zero eigenvalue) for D_E. The strategy is to show that any D_E is close (up to an algebraic zeroth order operator) to a twisted Dirac operator which has non-zero index. To be more precise we show that this is true for the twisted Dirac operator of some multiple $E \otimes \mathbb{C}^N$ of E which has the same spectrum, as D_E up to multiplicities.

This is done in two steps:

Step 1. Find a bundle F such that $\text{index}(D^+_{E\otimes F}) \neq 0$

Step 2. Find a complementary bundle $F^\perp$ such that $F \oplus F^\perp$ is trivial.

Step one is achieved in the usual fashion, by pulling back to M the spinor bundle $\mathbb{S}^+$ of S^{2m}, whose top Chern class is non-zero, using a map of degree one: $M^{2m} \to S^{2m}$. (We can just map a small ball onto the sphere punctured at a point and the rest of the manifold can be mapped to that point). This makes the index of $D^+_{E\otimes F}$ equal to $\dim(F)$, since we are assuming that $\text{index}(D_E^+) = 0$.

For step two we can choose the complementary bundle to be the pull back (under the same map as in step one) of the universal complementary bundle for $\mathbb{S}^+$ on the sphere (this is simply the bundle $\mathbb{S}^-$ on the sphere). Moreover we can use the connections that are also pull backs of the standard spinor connections on the round sphere. The difference of the connection on $E \otimes (F \oplus F^\perp) \cong E \otimes \mathbb{C}^N$ defined by these pull-backs to that induced by the original connection ∇ on the multiple $E \otimes \mathbb{C}^N$ now depends only on the universal bundles on the sphere and the map used to pull back the bundles and hence is independent of E and ∇. Therefore the spectrum of a multiple of the original Dirac operator D_E is close to another twisted Dirac operator which has a zero eigenvalue and the theorem is proved in this case.

To extend the theorem to higher eigenvalues (higher k) we need to use higher degree maps (using disjoint balls) to pull back the bundles from the sphere in order to get bigger indices ($=$ degree $\times$ $\dim(F)$).

To prove the odd dimensional case, we take the product of the odd dimensional manifold with S^1 and use a spectral flow argument. This also proves the more general result about the distribution of eigenvalues in an interval (not necessarily containing zero) of a definite length.

Although the proof seems rough and topological, the Vafa-Witten upper bounds are sharp in certain cases.

Examples

(i) For the spheres, the estimates are sharp and the method is in fact closely related to the proof of Llarull's theorem.

(ii) To get sharp estimates for $M = \mathbb{C}P^n$ (n odd) one should use the trivial bundle $(TM \oplus 1_{\mathbb{C}}) \otimes H$ where H is the Hopf bundle and $1_{\mathbb{C}}$ is the trivial complex line bundle.

The main idea used behind the Vafa-Witten proof can be formalized to the notoin of *K-length* of a Riemannian manifold. For a given bundle E with connection, we can measure its non-triviality by finding the minimum amount of "second fundamental form" required to imbed it inside a larger trivial bundle. The K-length is then the inverse of the smallest norm of such "second fundamental forms" over all homologically essential bundles. The K-length can be used to control spectral gaps. We refer again to [G] for more details.

3 Positive Mass Theorems

3.1 Description of Results

In general relativity, there is no satisfactory notion of total energy, since the energy of the gravitational field itself is described purely in terms of geometry and does not contribute directly to the local stress-energy-momentum tensor T_{ij} . However, in an asymptotically flat space time describing an isolated system like a star or a black hole, where the gravitational field approaches ordinary Newtonian gravity with respect to a background inertial coordinate system at infinity, one can define the total mass, or more relativistically, the total energy-momentum four-vector of the system by asymptotic comparison with Newtonian theory at large distances.

More precisely, we define an asymptotically Euclidean space-like hypersurface to be a 3-dimensional oriented Riemannian manifold (M, g) isometrically imbedded in 4-dimensional space-time whose first and second fundamental forms g_{ij} and h_{ij} satisfy the following asymptotic conditions:

(A) There is a compact set $K \subset M$ so that $M \setminus K$ is a finite disjoint union of ends, each diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$ and using the standard coordinates given by this diffeomorphism, g and h have the asymptotic behaviour:

$$\partial_\alpha(g_{ij} - \delta_{ij}) \in O(r^{-1-|\alpha|}) \quad \text{for} \quad |\alpha| \leq 2 \tag{2}$$

and

$$\partial_\beta(h_{ij}) \in O(r^{-2-|\beta|}) \quad \text{for} \quad |\beta| \leq 1 \tag{3}$$

These are not the optimal decay rates and we refer to [B] for refinements and also for a discussion of the independence of the ADM-mass, from the choice of the coordinate system at infinity. ADM stands for Arnowitt, Deser and Misner and their definition of the total energy-momentum (E, P_j) of an asymptotically Euclidean space-like slice is:

$$E = \frac{1}{16\pi G} \lim_{r\to\infty} \oint_{S(r)} (\partial_k g_{ik} - \partial_i g_{kk}) d\sigma^i \tag{4}$$

$$P_j = \frac{1}{8\pi G} \lim_{r\to\infty} \oint_{S(r)} (h_{ij} - \delta_{ij} h_{kk}) d\sigma^i \tag{5}$$

where $S(r)$ is the Euclidean sphere of radius r and the integrals are defined for each end. For the important prototypical example of the Schwarzschild metric, this definition of course recovers the usual mass that appears in the metric and $P = 0$.

The next important physical assumption assumption is the following dominant energy condition for the local mass density T:

(B) For each time like vector e_0 transversal to M, $T(e_0, e_0) \geq 0$ and $T(e_0, \ )$ ia a non-space-like covector. This implies that for any adapted orthonormal frame (e_0, e_1, e_2, e_3) with e_0 normal and e_1, e_2, e_3 tangential to M, we have the inequalities:

$$T^{00} \geq |T^{\mu\nu}| \text{ for all } 0 \leq \mu, \nu \leq 3 \tag{6}$$

and

$$T^{00} \geq (-T_{0k} T^{0k})^{1/2} \tag{7}$$

We of course also assume that space time satisfies Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{8}$$

Given these assumptions, the positive energy theorem states that

Theorem 7. *An asymptotically Euclidean space-like hypersurface in a space-time satisfying Eistein's equation and the dominant energy condition has non-negative total energy in the sense that $E \geq |P|$ for each end. Moreover if $E = 0$ for some end, then there is exactly one end and M is isometric to flat Euclidean space.*

After several attempts by relativists, who established several special cases, the first complete proof of this result was achieved by Schoen and Yau [SY 2,3,4], using minimal surface techniques. Subsequently, Witten [W] found a completely different proof using harmonic spinors. It should be noted that some analytical subtleties in Witten's paper were clarified later by Parker and Taubes [PT].

In the positive mass theorem, if we ignore the fact that the space-like slice (M^3, g) is imbedded in space-time, the assumption to be asymptotically Euclidean is well defined by (1) and the formula (3) for the energy (or mass) E still makes sense. In this Riemannian situation, the appropriate assumption that replaces the dominant energy condition (B) is that the scalar curvature of (M^3, g) is non-negative. In fact, if the space-like slice has zero mean curvature zero ($tr(h) = 0$), then by the Gauss-Codazzi equations, assumption (B) would imply that the scalar curvature of M^3 is non-negative. More generally, one would like to pose the problem whether, for all dimensions n, an asymptotically Euclidean Riemannian manifold (M^n, g) with non-negative scalar curvature has non-negative mass . This is also pertinent to physics, if one works in a more general framework than classical relativity. To be exact, one needs to modify the decay rate in definition (1), depending on n. We assume for simplicity:

$$\partial_\alpha (g_{ij} - \delta_{ij}) \in O(r^{-n+2-|\alpha|}) \quad \text{for } |\alpha| \leq 2 \tag{9}$$

With this definition of asymptotically Euclidean we have the following result.

Theorem 8. *An asymptotically Euclidean spin manifold M with non-negative scalar curvature everywhere has positive total mass E. Moreover $E = 0$ if and only if M is isometric to flat Euclidean space.*

Both Schoen-Yau and Witten established this theorem in dimension 3 in the course of their proof of the positive mass conjecture. For general n, this was first proved by Bartnik [B]. We note that for a compactly supported perturbation of the flat metric, the above theorem would be a simple consequence of the corresponding result for the torus.

It is a natural question to ask for similar results with different asymptotic background geometries and in [M1], I established, using Witten's method, the following hyperbolic version of the rigidity part of the last theorem.

Theorem 9. *A strongly asymptotically hyperbolic spin manifold of dimension > 2 , whose scalar curvature satisfies $R \geq -n(n-1)$ everywhere, is isometric to hyperbolic space.*

There is a technical mistake in the definition of strongly asymptotically hyperbolic in my paper [M1] and I would like to redefine it here.

Definition: A Riemannian manifold (M^n, g) is said to be *strongly asymptotically hyperbolic* (with one end) if there exists a compact subset $B \subset M$ and a diffeomorphism $\phi : M \setminus B \to H^n \setminus \bar{B}(r_0)$ for some $r_0 > 0$ such that, if we define the gauge transformation $A : T(M \backslash B) \to T(M \backslash B)$ by the equations: (i) $g(Au, Av) = \phi^* g(u, v)$, (ii) $g(Au, v) = g(u, Av)$, then A satisfies the following properties:

AH1: There exists a uniform Lipschitz constant C such that

$$C^{-1} \leq \inf_{|v|=1} |Av| \leq \sup_{|v|=1} |Av| \leq C$$

AH2: $\exp(\phi \circ r)(A - \mathrm{id}) \in L^{1,1} \cap L^{1,2}(T^* \otimes T(M \setminus B)$.

Before I sketch the main idea behind the spinorial proofs of these theorems here are some remarks:

1. There are also Lorentzian versions of these results for asymptotically Anti-deSitter spaces. See for example: [RT],[WO].

2. X. Zhang [Z] proves a generalization of theorem 8 and there is also a version established by M. Herzlich [HE1], where the background metric is that of a complex hyperbolic space.

3. There is a much stronger version of the positive mass conjecture known as the Penrose conjecture for black holes where the mass is bounded from below by the area of the event horizon. The Riemannian version of this conjecture has been now established by Huisken-Ilmanen [HI] and H.Bray [BR]. The proofs are not spinorial and use what is known as the inverse mean curvature flow. For a spinorial proof of a related"Penrose-like inequality" see [HE2].

4. There are some recent new positive energy conjectures formulated by by Horowitz and Myers [HM], in their attempt to deal with stability problems arising from the AdS/CFT correspondence. For example, they conjecture a positive energy theorem and a corresponding rigidity result for metrics on four dimensional manifolds which asymptotically look like:

$$ds^2 = \frac{r^2}{l^2}\Big[\big(1 - \frac{r_0^4}{r^4}\big)d\theta^2 + (dx^1)^2 + (dx^2)^2\Big] + \Big(\frac{r^2}{l^2}\big(1 - \frac{r_0^4}{r^4}\big)\Big)^{-1} dr^2$$

where $r \geq r_0$ and $\theta \in S^1$ with period $\pi\, l^2/r_0$. This the Euclidean version of their conjecture and energy has to appropriately defined.

3.2 Sketch of proofs

The first basic step in proving these theorems is to solve for harmonic spinors which have the correct behaviour at infinity and then apply the integrated version of the Lichnerowicz formula, using Stokes' theorem. The boundary integrals, in the limit, are then identified with the "mass".

If now integrate the Lichnerwicz formula for the ordinary Dirac operator:

$$D^2 = \nabla^* \nabla + \frac{R}{4}$$

on a manifold with boundary, we obtain:

$$\int_M (|\nabla \psi|^2 + \frac{R}{4}\,|\psi|^2) + \int_M |D\psi|^2 = \int_{\partial M} \langle \nabla_\nu \psi + \nu \cdot D\psi, \psi \rangle$$

where ψ is a spinor and ν is the unit outer normal vector of the boundary.

The formula can also be proved by computing the divergence of a one form and applying Stokes' theorem. The specific one form α we use here is defined by:

$$\alpha(v) = \langle \nabla_v \psi + v \cdot D\psi, \psi \rangle$$

for $v \in TM$ and for a *fixed* spinor field ψ. The divergence of α is computed to be:

$$-\delta\alpha = |\nabla\, \psi|^2 - |D\, \psi|^2 + \frac{R}{4}\,|\psi|^2$$

For a harmonic spinor satisfying $D\psi = 0$, the boundary integral on the right-hand-side will be non-negative, provided the scalar curvature is non-negative. Moreover it can and vanish if and only if ψ is globally parallel. To prove Theorem 7, we prove first the existence of a harmonic spinor which is asymptotically parallel in the sense that it approaches a parallel spinor (with respect to the background flat metric) sufficiently fast. The limiting value of the boundary integral is then shown to be the mass ($= E$) when the boundary spheres go off to infinity. Rigidity follows from the fact that if the mass vanishes, we get a trivialization of the manifold by parallel spinors, since we get one for each asymptotic value.

To prove Theorem 8, one needs a connection $\widetilde{\nabla}$ that is flat for the standard hyperbolic space. There is a natural one, which I called a hyperbolic Cartan connection in [M1], which comes from imbedding hyperbolic space in Minkowski space and restricting the flat vector space parallelism. This can be done "virtually" for any Riemannian manifold on the stabilized tangent bundle $TM \oplus 1$, except that the connection would not be flat unless the manifold is hyperbolic. For spin manifolds, we also obtain induced connections on associated spinor bundles. The modified Dirac operator $\widetilde{D}$, is then defined using these Cartan connections. We refer to [M1] for details. This is very similar to Witten's proof of the Lorentzian version (Theorem 6), where he also used the Levi-Civita connection of the surrounding space-time restricted to the space-like slice to define a modified Dirac operator. The curvature terms that appear in the Lichnerowicz formula for the square of Witten's Dirac operator involve more than just the scalar curvature. However, the dominant energy condition together with Einstein's equation is exactly what is needed to prove that the integrand is non-negative. The boundary integrals are identified, in the limit, with the total energy-momentum vector.

In the case of the Dirac operator $\widetilde{D}$, defined by a hyperbolic Cartan connection the analogue of Lichnerowicz' formula is obtained simply by replacing the Riemannian connection and curvature terms by their hyperbolic analogues.

$$\widetilde{D}^2 = \widetilde{\nabla}^*\widetilde{\nabla} + \frac{\widetilde{R}}{4}$$

where $\widetilde{R} = R + n(n-1)$ is the *hyperbolic scalar curvature*.

The proof of Theorem 8 now proceeds exactly as in the Euclidean case. First solve for a hyperbolically harmonic spinor with good asymptotics and then integrate by parts.

The above proofs for positive mass theorems do not involve index arguments in contrast to results from the last section on compact manifolds. It would be interesting to have versions of these theorems for manifolds with (finite) boundary involving relative versions of the K-area and the index theorem. In fact, we express the boundary integral purely in terms of a self-adjoint tangential Green's operator acting on the boundary:

$$\nabla_\nu + \nu D = \nu \widehat{D} - \frac{H}{2}$$

where H is the mean curvature (with respect to the inner normal ν) of the boundary, and $\nu\widehat{D}$ is a tangential self adjoint boundary operator. We have then the following *fundamental formula* for a harmonic spinor:

$$\int_M \left(|\nabla\psi|^2 + \frac{R}{4}|\psi|^2 \right) = -\int_{\partial M} \langle \nu\widehat{D}\psi, \psi\rangle + \int_{\partial M} \frac{H}{2}|\psi|^2 \ .$$

One would then impose appropriate boundary conditions to control the sign of the boundary integral.

3.3 Some mathematical aspects of the AdS/CFT correspondence

There are many physical aspects about the AdS/CFT correspondence, more generally known as the holographic principle. I will only be able to describe some special mathematical features and my description is necessarily very limited.

First of all AdS stands for Anti-deSitter space which is the Lorentzian analogue of hyperbolic space in Riemannian Geometry. CFT stands for conformal field theory and the correspondence is between supergravity (or string theory) of the bulk manifold (say hyperbolic space) and conformal field theory on its boundary (which in the case of hyperbolic space is the conformal sphere).

To be more specific, one studies a complete Riemannian Einstein manifold M^{n+1} with negative Ricci curvature, which has a conformal compactification in the following sense: M is the interior of a compact manifold with boundary $\overline{M}$ and the metric of M can be written near the boundary as:

$$ds^2 = \frac{1}{t^2}(dt^2 + g_{ij}(t,x)dx^i dx^j)$$

where t is a smooth function which is positive on M and has a zero of first order at $N = \partial\overline{M}$ (i.e. $dt \neq 0$ near N).

Although the induced metric on the boundary depends on the choice of the defining function t, the conformal class is independent of the choice and so conformal invariants on N are well defined. In particular, one is interested in expressing classical action functionals for (super)-gravity in terms of correlation functions of the boundary values at infinity. As a simple example, for scalar functions on hyperbolic space H^{n+1}, one can solve the Laplace equation for functions with any given prescribed boundary values. The resulting harmonic function is classically given by the Poisson kernel and so the Dirichlet functional (action) for harmonic functions can easily be written as a boundary integral on the conformal sphere S^n. (This is a two point correlation function for the boundary values). More generally one would study this correspondence for more elaborate functionals arising from supergravity and gauge theory on say, asymptotically hyperbolic spaces.

If the conformal class of the metric on the boundary contains a metric of positive scalar curvature (i.e. if the Yamabe invariant is positive), then Witten and Yau [WY] has shown that the conformal boundary N is connected and also the nth homology group of $\overline{M}$ vanishes. This is a basic result, since the non-connectedness would have unnatural physical implications.

Somewhat more general results were then established by M.Cai and G.Galloway [CG]. The results can be formulated in a more geometrical fashion and the proofs are also based on more traditional comparison methods of Riemannian geometry.

Theorem 10. *Let M^{n+1} be a complete Riemannian manifold with compact boundary and suppose that the Ricci curvature of M satisfies $Ric(g) \geq -n\,g$ everywhere and suppose that the boundary has mean curvature $H > n$. Then M is compact.*

Theorem 11. *Let M^{n+1} be a complete Riemannian manifold admitting a coformal compactification $\overline{M}$, with boundary N^n, and suppose that the Ricci curvature of M satisfies $Ric(g) \geq -n\,g$ everywhere and such that $Ric(g) \to -n\,g$ sufficiently fast near the conformal boundary. Assume also that N has a component with a metric of non-negative scalar curvature. Then the following properties hold:*

(i) N is connected.

(ii) If M is orientable then $H_n(\overline{M}, \mathbb{Z}) = 0$.

(iii) The map $i_* : \pi_1(N) \longrightarrow \pi_1(M)$ induced by the inclusion is onto.

The Witten-Yau proof uses variational and comparison methods for minimal surfaces and related functionals for branes whereas Cai and Galloway use only variational and comparison methods for geodesics.

The proof of Theorem 9 is quite simple and follows from the fact that the condition $H > n$ on the boundary forces the geodesics starting perpendicular to the boundary to focus more strongly than the negative Ricci curvature of the bulk manifold and hence these geodesics will have conjugate points within a finite distance. This is, of course, very reminiscent of the classical Bonnet-Myers argument. Theorem 10 is a little bit harder to prove, but still is based on classical methods of Riemannian geometry using comparison arguments for Busemann functions. It would be nice to see a spinorial approach and maybe obtain stronger results using only lower bounds on the scalar curvature.

Finally, I would like to remark that there might be a connection of these type of problems with the entropy rigidity result of Besson-Courtois-Gallot [BCG]. In one of their proofs they used the fact that the imbedding of the manifold in the Hilbert space of all probability measures on the boundary at infinity given by the square root of the Poisson kernel calibrates the volume form. It is a homothety for the standard hyperbolic space. It would be interesting to extend this idea to other harmonic propagators, involving the Dirac operator, spinors, scalar curvature, K-area and the index theorem.

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