BUNDLE CONSTRUCTIONS OF CALIBRATED SUBMANIFOLDS IN $\mathbb{R}^7$ AND $\mathbb{R}^8$

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Abstract. We construct calibrated submanifolds of $\mathbb{R}^7$ and $\mathbb{R}^8$ by viewing them as total spaces of vector bundles and taking appropriate sub-bundles which are naturally defined using certain surfaces in $\mathbb{R}^4$. We construct examples of associative and coassociative submanifolds of $\mathbb{R}^7$ and of Cayley submanifolds of $\mathbb{R}^8$. This construction is a generalization of the Harvey-Lawson bundle construction of special Lagrangian submanifolds of $\mathbb{R}^{2n}$.

1. Introduction

The study of calibrated geometries was first initiated by Harvey and Lawson 1982 in their seminal paper [9]. Because they are believed to play a crucial role in explaining the phenomenon of mirror symmetry [22], they have recently received much attention. There has been extensive research done on special Lagrangian submanifolds of $\mathbb{C}^n$, most notably by Joyce but see also [13] and the many references contained therein. Significatly less progress has been made in analyzing associative and coassociative submanifolds of $\mathbb{R}^7$ and Cayley submanifolds of $\mathbb{R}^8$, although the recent papers [16, 17] of Lotay presented some constructions analogous to earlier special Lagrangian constructions by Joyce. Needless to say, even less is known about calibrated submanifolds in more general Calabi-Yau, G_2 and Spin(7) manifolds, even in the non-compact case, although the examples in $\mathbb{R}^n$ serve as important local models, especially for studying the possible singularities that can occur.

In their original paper [9] Harvey and Lawson presented a construction of special Lagrangian submanifolds in $\mathbb{C}^n$ using bundles. In this paper, motivated by their work, we describe a similar bundle construction of associative and coassociative submanifolds of $\mathbb{R}^7$ and Cayley submanifolds of $\mathbb{R}^8$. The reader can consult [8, 9, 14] for background on these exceptional calibrations.

The Harvey-Lawson contruction involves viewing $\mathbb{C}^n$ as a vector bundle over $\mathbb{R}^n$, and taking an appropriate sub-bundle of the restriction of this bundle to a submanifold $M^p \subset \mathbb{R}^n$. In this case $\mathbb{C}^n = T^*(\mathbb{R}^n)$ and the subbundle is the conormal bundle $N^*(M^p)$. They find that the conormal bundle is special Lagrangian if and only in M^p is *austere* in $\mathbb{R}^n$, which is a condition which is in general much stronger than minimal. Their construction is reviewed in detail in Section 2.

It is well known that if one views $\mathbb{R}^7$ as the space of anti-self dual 2-forms on $\mathbb{R}^4$, and $\mathbb{R}^8$ as the negative spinor bundle of $\mathbb{R}^4$, there are naturally defined G_2 and

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Spin(7)-structures on them, respectively. See [12, 14, 15] for background on G_2, and Spin(7)-structures. We consider restricting these bundles to a surface $M^2 \subset \mathbb{R}^4$, and then take appropriate naturally defined sub-bundles of this restriction, the total spaces of which are candidates for associative, coassociative, and Cayley submanifolds. This is discussed in Section 3.

Since a calibrated submanifold is necessarily minimal, and since the vector bundle directions have trivial second fundamental form, the base manifold M^2 must be necessarily at least minimal in $\mathbb{R}^4$. (Just as austere submanifolds are at least minimal in the Harvey-Lawson construction.) In Theorem 3.2.1 we find that the naturally defined rank 2 sub-bundle of $\wedge^2_-(\mathbb{R}^4)|_{M^2}$ is coassociative iff the immersion of M^2 in $\mathbb{R}^4$ is a solution of exactly one half of the *real isotropic minimal* surface equation, sometimes also called superminimal. It is important that not all real isotropic minimal surfaces will work. Perhaps somewhat surprisingly, we find in Theorem 3.3.1 that the naturally defined rank 1 sub-bundle of $\wedge^2_-(\mathbb{R}^4)|_{M^2}$ is associative iff M^2 is just minimal in $\mathbb{R}^4$, with no extra conditions. Similarly in Theorem 3.5.1 we find two naturally defined rank 2 sub-bundles of $\not{S}_-(\mathbb{R}^4)|_{M^2}$ and each of them is Cayley iff M^2 is again just minimal.

The associative construction produces interesting new examples, while the coassociative construction actually produces examples which live in a $\mathbb{C}^3$ subspace of $\mathbb{R}^7$ and are complex submanifolds of $\mathbb{C}^3$. The Cayley construction produces submanifolds of $\mathbb{R}^8$ which are either of the form $\mathbb{R} \times L$ for an associative 3-fold L or are non-trivial coassociative submanifolds of $\mathbb{R}^7$.

It is perhaps interesting that special Lagrangian and coassociative submanifolds are harder to construct using these methods, requiring a base manifold which is more than just minimal. Special Lagrangian and coassociative submanifolds have a very nice, unobstructed local deformation theory [18], and the local moduli space is intrinsic to the submanifold. On the other hand, associative and Cayley submanifolds have a more complicated, non-intrinsic and obstructed deformation theory, and yet the bundle construction in these two cases is simpler, requiring only minimality.

Section 4 presents some explicit examples of these constructions. Appendix A summarizes some notations and calculations for submanifolds used repeatedly in the paper, while Appendix B contains an octonion multiplication table for the benefit of the reader, since this is also used often.

There exist examples of special holonomy metrics on non-compact manifolds which are bundles over a compact base, for example the Calabi-Yau metrics on $T^*(S^n)$, described in [21] and the G_2 holonomy metrics on $\wedge^2_-(S^4)$ and $\wedge^2_-(\mathbb{CP}^2)$ and the Spin(7) holonomy metrics on $\not{S}_-(S^4)$, described in [4, 6]. Similar constructions of calibrated submanifolds can be done in these cases, and this is the subject of a forthcoming paper [11].

We should remark that after this work was done, the authors found a similar although different statement, without proof, in an unpublished preprint by S.H. Wang [23]. His statement concerned the non-compact G_2 and Spin(7) manifolds first constructed by Bryant and Salamon [4] (we will deal with this case in [11]) and his claim is that superminimal is the required condition for all three constructions. We have proved that in the associative and Cayley cases, just minimal is enough, while in the coassociative case, we prove that only half of the superminimal surfaces work.

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2. The Harvey-Lawson Special Lagrangian Bundle Construction

In this section we review the bundle construction of Harvey and Lawson [9] of special Lagrangian submanifolds. The natural ambient manifold in which to consider Special Lagrangian submanifolds is a Calabi-Yau manifold, which is in particular a symplectic manifold. The simplest example of a symplectic manifold is the *cotangent bundle* $T^*(\mathbb{R}^n)$ of $\mathbb{R}^n$. This example is trivially Calabi-Yau, since $T^*(\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{C}^n$.

On $\mathbb{C}^n = T^*(\mathbb{R}^n)$ we have a Kähler form $\omega = \frac{i}{2} \sum dz^k \wedge d\bar{z}^k$ and a holomorphic $(n, 0)$ volume form $\Omega = \operatorname{Re}\Omega + i \operatorname{Im}\Omega = dz^1 \wedge \ldots \wedge dz^n$. An n-dimensional submanifold L^n of $\mathbb{C}^n$ is Special Lagrangian with phase θ (up to a possible change of orientation) if the following two independent conditions are satisfied:

$$\begin{aligned} \omega|_L &= 0 \\ \left(\operatorname{Im} e^{-i\theta}\Omega\right)|_L &= 0 \end{aligned}$$

The first condition simply says that L is Lagrangian, which involves only the symplectic structure ω of $\mathbb{C}^n$. The *special* condition is given by the second equation, which involves the Calabi-Yau metric structure.

Now it is a classical fact that if M^p is a p-dimensional submanifold of $\mathbb{R}^n$, then the *conormal bundle* $N^*(M^p)$ is a *Lagrangian* submanifold of the symplectic manifold $T^*(\mathbb{R}^n)$. (This will be shown below.) Motivated by this, Harvey and Lawson found conditions on the immersion $M^p \subset \mathbb{R}^n$ that makes $N^*(M^p)$ a special Lagrangian submanifold of $T^*(\mathbb{R}^n)$, in terms of the second fundamental form of the immersion. We reproduce their results here, to motivate the constructions in Section 3 and to fix our notation and conventions.

The canonical symplectic form ω on $T^*(\mathbb{R}^n)$ is a 2-form on the total space $T^*(\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$. An orthonormal coframe for $\mathbb{R}^n$ is given by $e^1, e^2, \ldots, e^n$. Hence an arbitrary element of the cotangent bundle can be written as

$$(\mathbf{x}, s_1 e^1 + s_2 e^2 + \ldots + s_n e^n)$$

where the s_i's are coordinates on the cotangent space. An orthonormal tangent frame for the total space is given by

$$(e_i, 0) \qquad i = 1, \ldots, n \qquad \text{and} \qquad (0, e^i) \qquad i = 1, \ldots, n$$

For notational simplicity, we will denote $(e_i, 0)$ by e_i and $(0, e^i)$ by α^i. The canonical symplectic form ω on $T^*(\mathbb{R}^n)$ is then given by

$$\omega = \sum_{k=1}^{n} e^k \wedge \alpha_k$$

Let $M^p \subset \mathbb{R}^n$. Observe that if we restrict the tangent bundle $T(\mathbb{R}^n)$ to M^p, we have

$$T(\mathbb{R}^n)|_M = T(M) \oplus N(M)$$

and dually the restriction of the cotangent bundle $T^*(\mathbb{R}^n)$ to M^p gives

$$T^*(\mathbb{R}^n)|_M = T^*(M) \oplus N^*(M)$$

Since M is p-dimensional, the total space of the conormal bundle has dimension $p + (n - p) = n$. It therefore makes sense to ask if $N^*(M)$ is Lagrangian.

We use the local coordinate notation described in Appendix A. An orthonormal coframe for $\mathbb{R}^n$ is given by $e^1, e^2, \ldots, e^p, \nu^1, \nu^2, \ldots, \nu^q$, where the e_i's are tangent to M^p and the ν_i's are normal to M^p. Then ω takes the form

$$\omega = \sum_{k=1}^{p} e^k \wedge \alpha_k + \sum_{l=1}^{q} \nu^l \wedge \beta_l \tag{2.1}$$

where as above $\nu_j = (\nu_j, 0)$ and $\beta^j = (0, \nu^j)$.

Lemma 2.0.1. *The conormal bundle $N^*(M)$ is a Lagrangian submanifold of $T^*(\mathbb{R}^n)$.*

Proof. We show that every tangent space to $N^*(M)$ is a Lagrangian subspace of the corresponding tangent space to $T^*(\mathbb{R}^n)$. In local coordinates the immersion Ψ is given by

$$(u^1, u^2, \ldots, u^p, t_1, t_2, \ldots, t_q) \mapsto (x^1(\mathbf{u}), x^2(\mathbf{u}), \ldots, x^n(\mathbf{u}), t_1\nu^1 + t_2\nu^2 + \ldots + t_q\nu^q)$$

Hence the tangent space at $(\mathbf{x}(\mathbf{u}_0), t_1, t_2, \ldots, t_q)$ is spanned by the vectors

$$\begin{aligned} E_i &= \Psi_*\left(\frac{\partial}{\partial u^i}\right) = \left(e_i, \sum_{k=1}^{q} t_k \nabla_{e_i}(\nu^k)|_{\mathbf{x}_0}\right) \qquad i = 1, \ldots, p \\ F_j &= \Psi_*\left(\frac{\partial}{\partial t_j}\right) = (0, \nu^j) = \beta^j \qquad j = 1, \ldots, q \end{aligned}$$

Using Lemma A.0.1 we can write

$$\begin{aligned} E_i &= \left(e_i, \sum_{k=1}^{q}\sum_{l=1}^{p} t_k \langle e_l, A^{\nu_k}(e_i)\rangle e^l\right) \\ &= e_i + \sum_{l=1}^{p} \langle e_l, A^{\nu}(e_i)\rangle \alpha^l \end{aligned}$$

where we have used $(0, e^l) = \alpha^l$ and defined $\nu = \sum_{k=1}^{q} t_k \nu_k$. To check that the immersion is Lagrangian, we use (2.1) and compute

$$\omega(F_i, F_j) = \omega(\beta^i, \beta^j) = 0 \qquad \forall i, j = 1, \ldots, q$$

and

$$\omega(F_i, E_j) = \omega(\beta^i, e_j + \langle e_k, A^{\nu}(e_j)\rangle \alpha^k) = 0 \qquad \forall i = 1, \ldots, q \quad j = 1, \ldots, p$$

Finally we have

$$\begin{aligned} \omega(E_i, E_j) &= \omega(e_i + \langle e_l, A^{\nu}(e_i)\rangle \alpha^l, e_j + \langle e_k, A^{\nu}(e_j)\rangle \alpha^k) \\ &= \langle e_i, A^{\nu}(e_j)\rangle - \langle e_j, A^{\nu}(e_i)\rangle = 0 \end{aligned}$$

using the symmetry of A^ν. Hence ω restricts to zero on $N^*(M)$ and the conormal bundle is Lagrangian in $T^*(\mathbb{R}^n)$. □

Since $T^*(\mathbb{R}^n)$ is Calabi-Yau, we can further ask under what conditions the conormal bundle $N^*(M)$ is actually *special Lagrangian.* A basis for the $(1, 0)$ forms is given by $e^j + i\alpha_j$ for $j = 1, \ldots, p$ and $\nu^k + i\beta_k$ for $k = 1, \ldots, q$. Thus the holomorphic $(n, 0)$ form Ω can be written as

$$\Omega = (e^1 + i\alpha_1) \wedge \ldots \wedge (e^p + i\alpha_p) \wedge (\nu^1 + i\beta_1) \wedge \ldots \wedge (\nu^q + i\beta_q)$$

Proposition 2.0.2 (Harvey and Lawson, 1982 [9]). *The conormal bundle $N^*(M)$ is special Lagrangian in $T^*(\mathbb{R}^n)$ with respect to the calibration $\mathrm{Re}(i^{-q}\Omega)$ if and only if all the odd degree symmetric polynomials in the eigenvalues of A^ν vanish for all normal vector fields ν on M, where A^ν is the second fundamental form for the immersion of M in $\mathbb{R}^n$. Such a submanifold is called* austere.

Proof. From Lemma 2.0.1 we had a basis for the tangent space to the immersion of $N^*(M)$ at a point $(\mathbf{x}(\mathbf{u}_0), t_1, t_2, \ldots, t_k)$ was given by

$$E_i = e_i + \sum_{l=1}^{p} \langle e_l, A^\nu(e_i)\rangle \alpha^l \qquad i = 1, \ldots, p$$
$$F_j = \beta^j \qquad j = 1, \ldots, q$$

Without loss of generality we can assume that the tangent vector fields were chosen to diagonalize A^ν at $\mathbf{x}_0$. That is, $A^\nu(e_i) = \lambda_i e_i$ for $i = 1, \ldots, p$. We compute easily that

$$\begin{aligned}(e^j + i\alpha_j)(E_i) &= \delta_i^j + i\lambda_i\delta_j^i \\ (e^j + i\alpha_j)(F_i) &= 0 \\ (\nu^j + i\beta_j)(E_i) &= 0 \\ (\nu^j + i\beta_j)(F_i) &= i\delta_i^j\end{aligned}$$

and hence

$$\Omega(E_1, \ldots, E_p, F_1, \ldots, F_q) = i^q(1 + i\lambda_1)(1 + i\lambda_2)\cdots(1 + i\lambda_p)$$

If instead we consider the point $(\mathbf{x}(\mathbf{u}_0), ct_1, ct_2, \ldots, ct_q)$ then the eigenvalues of $A^{c\nu}$ are $c\lambda_i$ and thus $\mathrm{Im}(i^{-q}\Omega)$ restricts to zero on all these tangent spaces (for any c) if and only if all the odd degree symmetric polynomials in the eigenvalues vanish. □

Remark 2.0.3. The first symmetric polynomial is the trace, so the submanifold M^p is necessarily minimal, as expected. If $p = 1, 2$ this is the only condition, but for $p \geq 3$ the austere condition is much stronger than minimal.

Remark 2.0.4. It is interesting to note that we cannot construct special Lagrangian submanifolds in this way of arbitrary phase. The factor of i^{-q} means that the allowed phase (up to orientation) depends on the codimension q of the immersion.

3. Bundle Constructions for Exceptional Calibrations

Motivated by the results of Harvey and Lawson for constructing Special Lagrangian submanifolds using bundles, we look for a similar procedure which will produce exceptional calibrated submanifolds: associative and coassociative submanifolds of $\mathbb{R}^7$, and Cayley submanifolds of $\mathbb{R}^8$. The idea is as follows. There are natural ways to view $\mathbb{R}^7$ and $\mathbb{R}^8$ as total spaces of vector bundles over the base space $\mathbb{R}^4$, which are compatible with the canonical G_2 and Spin(7)-structures on $\mathbb{R}^7$ and $\mathbb{R}^8$. Specifically, the bundle of anti-self-dual 2-forms $\wedge^2_-(\mathbb{R}^4) \cong \mathbb{R}^7$ has a natural G_2-structure, and the negative spinor bundle $\slashed{S}_-(\mathbb{R}^4) \cong \mathbb{R}^8$ has a natural Spin(7)-structure.

Now we let M^p be a submanifold immersed in $\mathbb{R}^4$ and consider the restriction of these bundles to M^p. For the right choice of dimension p, this restriction breaks up naturally into the direct sum of bundles, which can have the correct dimension (as total spaces) to be candidates for calibrated submanifolds. Then we can find conditions on the second fundamental form of the immersion of M in $\mathbb{R}^4$ for this to actually happen. As discussed in Section 1, we know that the conditions on M must include (at least) being a minimal immersion.

3.1. **The space $\wedge^2_-(\mathbb{R}^4)$ as a manifold with a G_2-structure.** The space of anti-self-dual 2-forms $\wedge^2_-(T^*(\mathbb{R}^4))$ on $\mathbb{R}^4$ (which we will sometimes denote simply as $\wedge^2_-$ is naturally isomorphic to $\mathbb{R}^7$, with a natural G_2-structure which we will now describe. (See [4], for example.) Let e^1, e^2, e^3, e^4 be an oriented coframe of orthonormal covector fields on $\mathbb{R}^4$. Then a basis of sections for $\wedge^2_-$ is given by

$$\begin{aligned}\omega^1 &= e^1 \wedge e^2 - e^3 \wedge e^4 \\ \omega^2 &= e^1 \wedge e^3 - e^4 \wedge e^2 \\ \omega^3 &= e^1 \wedge e^4 - e^2 \wedge e^3\end{aligned}$$

The canonical G_2 form φ on $\wedge^2_-(\mathbb{R}^4)$ is a 3-form on the total space $\wedge^2_-(\mathbb{R}^4) = \mathbb{R}^4 \oplus \mathbb{R}^3$. An arbitrary element of $\wedge^2_-(\mathbb{R}^4)$ can be written as

$$\left(\mathbf{x}, t_1\omega^1 + t_2\omega^2 + t_3\omega^3\right)$$

An orthonormal tangent frame for the total space is given by

$$(e_i, 0) \qquad i = 1, \ldots, 4 \qquad \text{and} \qquad (0, \omega^i) \qquad i = 1, \ldots, 3$$

For notational simplicity, we will denote $(e_i, 0)$ by e_i and $(0, \omega^i)$ by η^i. The canonical 3-form ϕ on $\wedge^2_-(\mathbb{R}^4)$ is then given by

$$\varphi = \eta_1 \wedge \eta_2 \wedge \eta_3 + \eta_1 \wedge \omega^1 + \eta_2 \wedge \omega^2 + \eta_3 \wedge \omega^3 \tag{3.1}$$

Remark 3.1.1. Alternatively, we could consider the bundle $\wedge^2_+(X)$ of *self-dual* 2-forms and obtain a G_2-structure on this space where the three plus signs in (3.1) become minus signs, and the three minus signs become plus signs.

Let M^2 be a surface isometrically immersed in $\mathbb{R}^4$. If we restrict the cotangent bundle of $\mathbb{R}^4$ to M^2, we can write

$$T^*(\mathbb{R}^4)\big|_{M^2} = T^*(M^2) \oplus N^*(M^2)$$

as the sum of two rank 2 bundles over M^2. Thus the space of 2-forms $\wedge^2(\mathbb{R}^4)$ restricts to M as

$$\wedge^2(\mathbb{R}^4)\big|_M = \wedge^2(T^*(M)) \oplus (T^*(M) \otimes N^*(M)) \oplus \wedge^2(N^*(M))$$

As in Appendix A, we let e_1, e_2 be a local orthonormal frame of tangent vector fields to M and ν_1, ν_2 be a local orthonormal frame of normal vector fields to M. Then the dual covector fields e^1, e^2 and ν^1, ν^2 are local coframes for the cotangent and conormal bundles. Locally we can write

$$\begin{aligned}\wedge^2(T^*(M)) &= \operatorname{span}(e^1 \wedge e^2) \\ T^*(M) \otimes N^*(M) &= \operatorname{span}(e^1 \wedge \nu^1, e^1 \wedge \nu^2, e^2 \wedge \nu^1, e^2 \wedge \nu^2) \\ \wedge^2(N^*(M)) &= \operatorname{span}(\nu^1 \wedge \nu^2)\end{aligned}$$

Now the anti-self-dual 2-forms restrict to M as

$$\wedge^2_-(\mathbb{R}^4)\big|_M = \operatorname{span}(e^1 \wedge e^2 - \nu^1 \wedge \nu^2, e^1 \wedge \nu^1 - \nu^2 \wedge e^2, e^1 \wedge \nu^2 - e^2 \wedge \nu^1)$$

Define $\omega^1 = e^1 \wedge e^2 - \nu^1 \wedge \nu^2$, $\omega^2 = e^1 \wedge \nu^1 - \nu^2 \wedge e^2$, and $\omega^3 = e^1 \wedge \nu^2 - e^2 \wedge \nu^1$. Then ω^1 is globally defined on M independent of the choice of orthonormal tangent frames e_1, e_2 and normal frames ν_1, ν_2. Hence $\operatorname{span}(\omega^1)$ defines a rank 1 bundle E

over M^2 and its orthogonal complement (locally defined as span(ω^2, ω^3)) defines a rank 2 bundle F over M^2.

$$\wedge^2_-(\mathbb{R}^4)|_M = E \oplus F$$

The total spaces of E and F are 3 and 4-dimensional submanifolds of $\mathbb{R}^7$ and hence candidates for associative and coassociative submanifolds, respectively. Before proceeding to check when this happens, we develop some formulas that will be needed.

Proposition 3.1.2. *Using the notation of Appendix A, we have the following expressions for the covariant derivatives of $\omega^1, \omega^2, \omega^3$ in the e_1, e_2 directions at the point $\mathbf{x}_0$.*

$$\begin{aligned}
\nabla_{e_i}\omega^1 &= \left(\langle A^{\nu_2}(e_i), e_1\rangle - \langle A^{\nu_1}(e_i), e_2\rangle\right)\omega^2 + \left(-\langle A^{\nu_1}(e_i), e_1\rangle - \langle A^{\nu_2}(e_i), e_2\rangle\right)\omega^3 \\
\nabla_{e_i}\omega^2 &= \left(\langle A^{\nu_1}(e_i), e_2\rangle - \langle A^{\nu_2}(e_i), e_1\rangle\right)\omega^1 \\
\nabla_{e_i}\omega^3 &= \left(\langle A^{\nu_2}(e_i), e_2\rangle + \langle A^{\nu_1}(e_i), e_1\rangle\right)\omega^1
\end{aligned}$$

Proof. We prove the second expression. We use Lemma A.0.1 and compute:

$$\begin{aligned}
\nabla_{e_i}\omega^2 &= \left(\nabla_{e_i}e^1\right)\wedge\nu^1 + e^1\wedge\left(\nabla_{e_i}\nu^1\right) - \left(\nabla_{e_i}\nu^2\right)\wedge e^2 - \nu^2\wedge\left(\nabla_{e_i}e^2\right) \\
&= \left(-\langle e_1, A^{\nu_1}(e_i)\rangle\nu^1 - \langle e_1, A^{\nu_2}(e_i)\rangle\nu^2\right)\wedge\nu^1 \\
&\quad + e^1\wedge\left(\langle e_1, A^{\nu_1}(e_i)\rangle e^1 + \langle e_2, A^{\nu_1}(e_i)\rangle e^2\right) \\
&\quad - \left(\langle e_1, A^{\nu_2}(e_i)\rangle e^1 + \langle e_2, A^{\nu_2}(e_i)\rangle e^2\right)\wedge e^2 \\
&\quad - \nu^2\wedge\left(-\langle e_2, A^{\nu_1}(e_i)\rangle\nu^1 - \langle e_2, A^{\nu_2}(e_i)\rangle\nu^2\right) \\
&= \left(\langle e_2, A^{\nu_1}(e_i)\rangle - \langle e_1, A^{\nu_2}(e_i)\rangle\right)e^1\wedge e^2 \\
&\quad - \left(\langle e_2, A^{\nu_1}(e_i)\rangle - \langle e_1, A^{\nu_2}(e_i)\rangle\right)\nu^1\wedge\nu^2 \\
&= \left(\langle e_2, A^{\nu_1}(e_i)\rangle - \langle e_1, A^{\nu_2}(e_i)\rangle\right)\omega^1
\end{aligned}$$

The other two are obtained similarly. □

3.2. **Coassociative Submanifolds of $\wedge^2_-(\mathbb{R}^4)$.** We are now ready to determine conditions on the immersion $M^2 \subset \mathbb{R}^4$ so that the total space of the bundle F over M is a coassociative submanifold. A 4-manifold L^4 is coassociative [9, 8] if and only if $\varphi|_{L^4} = 0$ where φ is the 3-form defining the G_2-structure. Equivalently, one can check that the *coassociator* vanishes. We will use the 3-form φ.

In anticipation of our results, we need to first make some definitions. A rank 2 real vector bundle which is both oriented and possesses a Riemannian metric on each fibre comes equipped with a natural almost complex structure J defined as follows. If v_1, v_2 is an oriented orthonormal basis in a fixed fibre, we define $Jv_1 = v_2$ and $Jv_2 = -v_1$. If we change the orientation, we change J to $-J$. In our setting, both $T(M)$ and $N(M)$ are rank 2 vector bundles with induced Riemannian metrics coming from the isometric immersion of M into $\mathbb{R}^4$. Since $\mathbb{R}^4$ is taken to be oriented, even if M is not oriented, a choice of orientation on $T(M)$ induces an orientation on $N(M)$ and vice-versa.

Theorem 3.2.1. *The total space of the rank 2 bundle F over M is a coassociative submanifold of $\wedge^2_-(\mathbb{R}^4)$ if and only if the second fundamental form A^ν of the immersion $M \subset \mathbb{R}^4$ satisfies*

$$A^{J\nu} = -JA^\nu \tag{3.2}$$

for all normal vector fields ν. In this equation the J on the left hand side corresponds to the natural almost complex structure on $N(M)$ while the J on the right hand side *corresponds to the natural almost complex structure on $T(M)$. Explicitly, $A^{J\nu}(w) = -J(A^{\nu}(w))$ for all tangent vectors w. Note that this condition is independent of the choice of orientation of $T(M)$, since it determines the orientation on $N(M)$ and changing J to $-J$ in both sides of this equation leaves it invariant.*

Proof. We show that every tangent space to F is a coassociative subspace of the corresponding tangent space to $\wedge^2_-$. In local coordinates the immersion Ψ is given by

$$(u^1, u^2, t_2, t_3) \mapsto (x^1(u^1, u^2), x^2(u^1, u^2), t_2\omega^2 + t_3\omega^3)$$

Hence the tangent space at $(\mathbf{x}(\mathbf{u}_0), t_2, t_3)$ is spanned by the vectors

$$\begin{aligned} E_i &= \Psi_* \left(\frac{\partial}{\partial u^i}\right) = \left(e_i, t_2 \nabla_{e_i}(\omega^2)\big|_{\mathbf{x}_0} + t_3 \nabla_{e_i}(\omega^3)\big|_{\mathbf{x}_0}\right) \qquad i = 1, 2 \\ F_j &= \Psi_* \left(\frac{\partial}{\partial t_j}\right) = (0, \omega^j) = \eta^j \qquad j = 2, 3 \end{aligned}$$

Using Proposition 3.1.2 we can write

$$\begin{aligned} E_1 &= e_1 + \left(t_2 \left(\langle A^{\nu_1}(e_1), e_2\rangle - \langle A^{\nu_2}(e_1), e_1\rangle\right) + t_3 \left(\langle A^{\nu_2}(e_1), e_2\rangle + \langle A^{\nu_1}(e_1), e_1\rangle\right)\right)\eta^1 \\ E_2 &= e_2 + \left(t_2 \left(\langle A^{\nu_1}(e_2), e_2\rangle - \langle A^{\nu_2}(e_2), e_1\rangle\right) + t_3 \left(\langle A^{\nu_2}(e_2), e_2\rangle + \langle A^{\nu_1}(e_2), e_1\rangle\right)\right)\eta^1 \end{aligned}$$

where we have used $\omega^j = \eta^j$. If we define $\nu = t_2\nu_1 + t_3\nu_2$ and $\nu^\perp = -t_3\nu_1 + t_2\nu_2$, which are orthogonal normal vectors, then the expressions for E_1, E_2 simplify to

$$\begin{aligned} E_1 &= e_1 + \Big(\langle A^{\nu}(e_1), e_2\rangle - \langle A^{\nu^\perp}(e_1), e_1\rangle\Big)\eta^1 \\ E_2 &= e_2 + \Big(\langle A^{\nu}(e_2), e_2\rangle - \langle A^{\nu^\perp}(e_2), e_1\rangle\Big)\eta^1 \end{aligned}$$

Now since we have

$$\begin{aligned} \varphi &= \eta_1 \wedge \eta_2 \wedge \eta_3 + \eta_1 \wedge (e^1 \wedge e^2 - \nu^1 \wedge \nu^2) \\ &\quad + \eta_2 \wedge (e^1 \wedge \nu^1 - \nu^2 \wedge e^2) + \eta_3 \wedge (e^1 \wedge \nu^2 - e^2 \wedge \nu^1) \end{aligned}$$

we can check when the immersion is coassociative by finding when φ restricts to zero on each of these tangent spaces. It is easy to compute that

$$\begin{aligned} \varphi(E_1, E_2, \cdot) &= E_2 \lrcorner E_1 \lrcorner \varphi \\ &= \eta_1 + (\cdots)\, e^1 + (\cdots)\, e^2 \end{aligned}$$

and hence since $F_j = \eta^j$ we see that $\varphi(E_1, E_2, F_2) = \varphi(E_1, E_2, F_3) = 0$ always. It remains to check when $\varphi(F_2, F_3, E_j) = 0$ for $j = 1, 2$. Since $\varphi(F_2, F_3, \cdot) = \eta_1$ these become the conditions

$$\begin{aligned} \langle A^{\nu}(e_1), e_2\rangle - \langle A^{\nu^\perp}(e_1), e_1\rangle &= 0 \\ \langle A^{\nu}(e_2), e_2\rangle - \langle A^{\nu^\perp}(e_2), e_1\rangle &= 0 \end{aligned}$$

for the tangent space at $(\mathbf{x}_0, t_2, t_3)$ to be coassociative. We get two more equations that must be satisfied by demanding that the tangent space at $(\mathbf{x}_0, -t_3, t_2)$ also be coassociative. This corresponds to changing $t_2 \mapsto -t_3$ and $t_3 \mapsto t_2$ in the above equations, which is equivalent to $\nu \mapsto \nu^\perp$ and $\nu^\perp \mapsto -\nu$. This gives

$$\begin{aligned} \langle A^{\nu^\perp}(e_1), e_2\rangle + \langle A^{\nu}(e_1), e_1\rangle &= 0 \\ \langle A^{\nu^\perp}(e_2), e_2\rangle + \langle A^{\nu}(e_2), e_1\rangle &= 0 \end{aligned}$$

Thus we see that at each point $\mathbf{x}(\mathbf{u}_0)$ on the surface M^2, the matrix $A^{\nu^\perp}$ is determined by A^ν for all normal vector fields ν. In the basis (e_1, e_2), we can write

$$A^\nu = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \qquad \text{and} \qquad A^{\nu^\perp} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}$$

Then the above four equations are

$$b_{11} = a_{12} \qquad b_{12} = a_{22} \qquad b_{12} = -a_{11} \qquad b_{22} = -a_{12} \tag{3.3}$$

and we can combine them in the following matrix equation:

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{12} & a_{22} \\ -a_{11} & -a_{12} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

which says $A^{\nu^\perp} = A^{J\nu} = -JA^\nu$ for $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which is the natural almost complex structure described above. □

Remark 3.2.2. Note (3.3) implies that $a_{11} + a_{22} = b_{11} + b_{22} = 0$. Since ν and $\nu^\perp$ are a basis for the normal space at every point, we see that $\mathrm{Tr}(A) = 0$ and M^2 is necessarily *minimal* in $\mathbb{R}^4$, as expected. However the condition $A^{J\nu} = -JA^\nu$ is actually stronger than minimal, just as the austere condition in Proposition 2.0.2 was stronger than minimal. These surfaces are well known, and are sometimes called *superminimal* although following the suggestion of R.L. Bryant we prefer to call them *real isotropic minimal surfaces*. (See Section 4.2 for more information on these surfaces.) They have been intensively studied by many, and the interested reader can refer to [1, 5, 19] and the references contained therein.

Remark 3.2.3. It is easy to check that if we assume that the second fundmental forms A^ν and $A^{\nu^\perp}$ are *simultaneously diagonalizable* and satisfy $A^{J\nu} = -JA^\nu$ then necessarily $A^\nu = A^{\nu^\perp} = 0$ and M^2 is *totally geodesic* in $\mathbb{R}^4$, and hence is a plane. The constructed coassociative submanifold is then a 4-plane.

Suppose a surface $M^2 \subset \mathbb{R}^4$ satisfies $A^{J_2\nu} = -J_1A^\nu$, where J_1 and J_2 are the natural almost complex structures on the tangent and normal spaces, respectively. (These were both referred to as J above but now we distinguish them explicitly for clarity.) We can define an almost complex structure $\widetilde{J}$ on the rank 4 vector bundle $T^*(\mathbb{R}^4)|_M$ over M as follows:

$$\widetilde{J} = \begin{pmatrix} J_1 & 0 \\ 0 & -J_2 \end{pmatrix}$$

acting diagonally on the tangent and normal spaces. In this notation, the condition (3.2) becomes $A^{\widetilde{J}\nu} = \widetilde{J}A^\nu$. This is equivalent to

$$(\overline{\nabla}_X(\widetilde{J}\nu))^T = \widetilde{J}(\overline{\nabla}_X\nu)^T \tag{3.4}$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\mathbb{R}^4$, X is a tangent vector field to M, and ν is a normal vector field to M.

Proposition 3.2.4. *If* (3.2) *holds, then the almost complex structure* $\widetilde{J}$ *defined above satisfies*

$$\overline{\nabla}_X\widetilde{J} = 0$$

for all tangent vector fields X to M.

Proof. Let X and Y be tangent vector fields to M. Using the fact that $\widetilde{J}$ is orthogonal and also preserves the tangent and normal spaces, we can use (3.4) to compute

$$\begin{aligned} \langle (\overline{\nabla}_X(\widetilde{J}\nu))^T, Y\rangle &= \langle \widetilde{J}(\overline{\nabla}_X \nu)^T, Y\rangle \\ -\langle \widetilde{J}\nu, \overline{\nabla}_X Y\rangle &= -\langle \overline{\nabla}_X \nu, \widetilde{J}Y\rangle \\ \langle \nu, \widetilde{J}(\overline{\nabla}_X Y)^N\rangle &= \langle \nu, (\overline{\nabla}_X \widetilde{J}Y)^N\rangle \end{aligned}$$

which holds for all normal vector fields ν, and hence

$$(\overline{\nabla}_X(\widetilde{J}Y))^N = \widetilde{J}(\overline{\nabla}_X Y)^N \tag{3.5}$$

Let ∇ denote the Levi-Civita connection on M^2 from the induced metric, we have

$$\begin{aligned} \overline{\nabla}_X(\widetilde{J})Y &= \overline{\nabla}_X(\widetilde{J}Y) - \widetilde{J}(\overline{\nabla}_X Y) \\ &= \nabla_X(\widetilde{J}Y) + (\overline{\nabla}_X(\widetilde{J}Y))^N - \widetilde{J}(\nabla_X Y + \left(\overline{\nabla}_X Y\right)^N) \\ &= \nabla_X(J_1 Y) - J_1(\nabla_X Y) = \nabla_X(J_1)Y = 0 \end{aligned}$$

where we have used (3.5) in the third line and the last equality is due to the fact that any almost complex structure on a rank 2 bundle is necessarily parallel. In the same way (3.4) can be used to show

$$\overline{\nabla}_X(\widetilde{J})\nu = \nabla_X(J_2)\nu = 0$$

and the result now follows. □

Unfortunately, Proposition 3.2.4 means that all the coassociative submanifolds of $\mathbb{R}^7$ thus constructed are everywhere orthogonal to a parallel direction, given by $\omega_1 = e^1 \wedge e^2 - \nu^1 \wedge \nu^2$, and actually live in an $\mathbb{R}^6$ subspace of $\mathbb{R}^7$. A coassociative submanifold of $\mathbb{R}^7$ which misses one direction is actually a complex dimension 2 complex submanifold of $\mathbb{C}^3 = \mathbb{R}^6$ (up to a possible change of orientation). It is interesting to note, however, that precisely which $\mathbb{C}^3$ sitting in $\mathbb{R}^7$ contains this complex submanifold depends on the immersion of the surface M^2 in $\mathbb{R}^4$.

Remark 3.2.5. In Section 3.5, during our search for Cayley submanifolds of $\mathbb{R}^8$, we will obtain non-trivial coassociative submanifolds of $\mathbb{R}^7$ which are not contained in a strictly smaller subspace.

Remark 3.2.6. On more general non-compact manifolds with holonomy G_2 such as $\wedge^2_-(S^4)$ and $\wedge^2_-(\mathbb{CP}^2)$ (see [4, 6]), this construction will produce more interesting coassociative submanifolds. This is discussed in [11].

3.3. **Associative Submanifolds of $\wedge^2_-(\mathbb{R}^4)$.** Similarly we can determine conditions on the immersion $M^2 \subset \mathbb{R}^4$ so that the total space of the bundle E over M is an associative submanifold. A 3-manifold L^3 is associative [9, 8] if and only if its tangent space at every point x is an associative subspace of $T_x(\wedge^2_-(\mathbb{R}^4)) \cong \mathbb{R}^7$. Here we identify $\mathbb{R}^7 \cong \operatorname{Im}\mathbb{O}$, the imaginary octonions.

Theorem 3.3.1. *The total space of the rank* 1 *bundle E over M is an associative submanifold of $\wedge^2_-(\mathbb{R}^4)$ if and only if the immersion $M \subset \mathbb{R}^4$ is* minimal. *That is,* $\operatorname{Tr} A^\nu = 0$ *for all normal vector fields ν.*

Proof. We show every tangent space to E is an associative subspace of the corresponding tangent space to $\wedge^2_-(\mathbb{R}^4)$. In local coordinates the immersion Ψ is

$$(u^1, u^2, t_1) \mapsto (x^1(u^1, u^2), x^2(u^1, u^2), t_1\omega^1)$$

Hence the tangent space at $(\mathbf{x}(\mathbf{u}_0), t_1)$ is spanned by the vectors

$$\begin{aligned} E_i &= \Psi_* \left(\frac{\partial}{\partial u^i} \right) = \left(e_i, t_1 \nabla_{e_i} (\omega^1)|_{\mathbf{x}_0} \right) \qquad i = 1, 2 \\ F_1 &= \Psi_* \left(\frac{\partial}{\partial t_1} \right) = (0, \omega^1) = \eta^1 \end{aligned}$$

From Proposition 3.1.2 we have

$$\begin{aligned} E_i &= e_i + t_1 \left(\langle A^{\nu_2}(e_i), e_1 \rangle - \langle A^{\nu_1}(e_i), e_2 \rangle \right) \eta^2 \\ &\quad + t_1 \left(-\langle A^{\nu_1}(e_i), e_1 \rangle - \langle A^{\nu_2}(e_i), e_2 \rangle \right) \eta^3 \end{aligned}$$

where we have used $\omega^j = \eta^j$. To simplify the notation, write

$$A^{\nu_1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \qquad \text{and} \qquad A^{\nu_2} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}$$

then we have

$$\begin{aligned} E_1 &= e_1 + t_1 \left((b_{11} - a_{12})\eta^2 + (-a_{11} - b_{12})\eta^3 \right) \\ E_2 &= e_2 + t_1 \left((b_{12} - a_{22})\eta^2 + (-a_{12} - b_{22})\eta^3 \right) \end{aligned}$$

To check that the tangent space at $(\mathbf{x}_0, t_1)$ is associative, we need to verify that the associator $[E_1, E_2, F_1] = (E_1 E_2) F_1 - E_1 (E_2 F_1)$ vanishes. Without loss of generality, at a point we can take the following explicit identification $T_x(\wedge^2_-(\mathbb{R}^4)) \cong \operatorname{Im} \mathbb{O}$:

$$\begin{pmatrix} \eta^1 & \eta^2 & \eta^3 & e_1 & e_2 & \nu_1 & \nu_2 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{e} & \mathbf{ie} & \mathbf{je} & \mathbf{ke} \end{pmatrix}$$

and hence

$$\begin{aligned} E_1 &= \mathbf{e} + t_1 \left((b_{11} - a_{12})\mathbf{j} + (-a_{11} - b_{12})\mathbf{k} \right) \\ E_2 &= \mathbf{ie} + t_1 \left((b_{12} - a_{22})\mathbf{j} + (-a_{12} - b_{22})\mathbf{k} \right) \\ F_1 &= \mathbf{i} \end{aligned}$$

Now we can compute the associator (see the octonion multiplication table in Appendix B), with the result being

$$\begin{aligned} [E_1, E_2, F_1] &= (E_1 E_2) F_1 - E_1 (E_2 F_1) \\ &= (-2b_{11} - 2b_{22})\mathbf{je} + (2a_{11} + 2a_{22})\mathbf{ke} \end{aligned}$$

which vanishes if and only if $\operatorname{Tr} A^{\nu_1} = \operatorname{Tr} A^{\nu_2} = 0$. □

3.4. **The space $\not{S}_-(\mathbb{R}^4)$ as a manifold with a** Spin(7)**-structure.** The simplest Spin(7)-structure on the total space of a bundle is the negative spinor bundle $\not{S}_-(\mathbb{R}^4)$ of $\mathbb{R}^4$. We will now explain how to see this. Over each point $x \in \mathbb{R}^4$, the fibre of spinors over x is isomorphic to two copies of the quaternions $\not{S}_+ \oplus \not{S}_- = \mathbb{H} \oplus \mathbb{H}$. The one-forms (covectors) at x are a subset of the Clifford algebra over x, and hence act on the spinor space. A good reference for spin representations is the book of Harvey [7]. If e^1, e^2, e^3, e^4 is an orthonormal basis of 1-forms at x, then the Clifford algebra relations are

$$e^i \cdot e^j + e^j \cdot e^i = -2\delta^{ij}$$

where the $\cdot$ denotes the Clifford product. Clifford multiplication by 1-forms interchanges the two spaces $\not{S}_\pm$. We identify the spinor space with the octonions, $\not{S}_+ \oplus \not{S}_- \cong \mathbb{H}\mathbf{e} \oplus \mathbb{H} \cong \mathbb{O}$. Octonionic multiplication by elements of $\mathbb{H}\mathbf{e}$ interchanges

$\mathbb{H}\mathbf{e}$ and $\mathbb{H}$ (see Appendix B). Also, we have the following identities for octonionic multiplication [9]:

$$\begin{aligned} a(ax) &= a^2 x \\ a_1(\bar{a_2}x) &= -\bar{a_1}(a_2 x) \qquad \text{for } a_1, a_2 \text{ orthogonal} \end{aligned}$$

If we take $a_i \in \mathbb{H}\mathbf{e}$, then $\bar{a_i} = -a_i$ and hence if e^1, e^2, e^3, e^4 is an orthonormal basis of $\mathbb{H}\mathbf{e}$, these relations become

$$e^i(e^j x) + e^j(e^i x) = -2\delta^{ij} x$$

Thus we see we obtain the spin representation at each point from octonionic multiplication by identifying $\not{S}_+ \oplus \not{S}_- \cong \mathbb{H}\mathbf{e} \oplus \mathbb{H}$ and the 1-forms with $\mathbb{H}\mathbf{e}$. We will only require this representation for Clifford products of 1-forms and it will be written

$$\begin{aligned} \gamma \quad &: \quad T^* \to \operatorname{End}(\not{S}_+ \oplus \not{S}_-) \\ \gamma(\alpha)(s) &= \alpha s \end{aligned}$$

where α is a 1-form, $s \in \not{S}_+ \oplus \not{S}_-$ and the product αs is octonionic multiplication. Note that since $\mathbb{O}$ is *not associative*, we have to be careful when composing two elements of this representation:

$$\begin{aligned} (\gamma(\alpha_1)\gamma(\alpha_2))(s) &= \gamma(\alpha_1)(\gamma(\alpha_2)(s)) \\ &= \gamma(\alpha_1)(\alpha_2 s) \\ &= \alpha_1(\alpha_2 s) \end{aligned}$$

which in general is *not* the same as $(\alpha_1\alpha_2)s$.

Now a manifold has a Spin(7)-structure if at every point its tangent space can be naturally identified with $\mathbb{O}$. With the identifications we have made, the total space of $\not{S}_-(\mathbb{R}^4)$ has a tangent space (at a point) isomorphic to $T(\mathbb{R}^4) \oplus \not{S}_- \cong T^*(\mathbb{R}^4) \oplus \not{S}_- \cong \mathbb{H}\mathbf{e} \oplus \mathbb{H} \cong \mathbb{O}$.

Proceeding as before, we now isometrically immerse a submanifold M^p in $\mathbb{R}^4$ so that the restriction $\not{S}_-(\mathbb{R}^4)|_{M^p}$ splits naturally into pieces, and hope to obtain Cayley submanifolds in this way. Once again, the only natural choice occurs when $p = 2$, the case of a surface. If we let e^1, e^2 be a local orthonormal coframe for M^2, and ν^1, ν^2 a local orthonormal basis for the conormal bundle, then we can consider the operations on the fibre $\not{S}_-$ of Clifford multiplication with $\gamma(e^1)\gamma(e^2)$ or $\gamma(\nu^1)\gamma(\nu^2)$. Two remarks are in order. First, since multiplication by $\gamma(\alpha)$ interchanges $\not{S}_+$ and $\not{S}_-$, we need to consider the composition of two such multiplications to stay in $\not{S}_-$. Second, up to a sign (corresponding to a choice of orientation for M^2) these operators are independent of the choice of e^1, e^2 or ν^1, ν^2 since, for example $\gamma(e^1)\gamma(e^2) = \gamma(e^1 \cdot e^2) = \gamma(e^1 \wedge e^2)$ because e^1 and e^2 are orthonormal.

The spinor space $\not{S}_-$ can be given the structure of a complex 2-dimensional vector space in many ways. One can check that if $a, b, p, q \in \mathbb{H}$, then

$$(a\mathbf{e})((b\mathbf{e})(pq)) = p((a\mathbf{e})((b\mathbf{e})q))$$

That is, left multiplication by ordinary quaternions $\mathbb{H}$ commutes with the composition of *two* left multiplications by elements of $\mathbb{H}\mathbf{e}$. Now left multiplication by a unit imaginary quaternion is a complex structure on $\mathbb{H}$, so $\not{S}_-$ has an S^2 family of complex structures with respect to operators of the form $\gamma(a\mathbf{e})\gamma(b\mathbf{e}) : \not{S}_- \to \not{S}_-$. Since we have a surface M^2 immersed in $\mathbb{R}^4$, this determines a canonical complex structure $\mathbf{j}_M$ on $\not{S}_-$ as follws. If $e^1 = a\mathbf{e}$ and $e^2 = b\mathbf{e}$ are an orthonormal basis of tangent vectors to M, then $\mathbf{j}_M$ is defined by

$$\mathbf{j}_M = e^1 e^2 = (a\mathbf{e})(b\mathbf{e}) = -\bar{b}a$$

It is easy to check that $\mathbf{j}_M$ is purely imaginary, and of unit length, so $\mathbf{j}_M^2 = -1$. Alternatively, if we had used an orthonormal basis of the normal space ν^1 and ν^2 and multiplied them together as elements of $\mathbb{H}\mathbf{e}$, we would have obtained $-\mathbf{j}_M$. Either choice will produce the same results below.

Lemma 3.4.1. *The operator $r_T = \gamma(e^1)\gamma(e^2)$ satisfies $r_T^2 = -1$ and hence decomposes the space $\not{S}_-$ into two 2-dimensional eigenspaces $V_{\pm\mathbf{j}_M}$ of eigenvalues $\pm\mathbf{j}_M$. Further, the operator $r_N = \gamma(\nu^1)\gamma(\nu^2)$ is equal to r_T.*

Proof. We compute $(r_T)^2 = \gamma(e^1\cdot e^2\cdot e^1\cdot e^2) = -\gamma(1) = -1$ using the fact that $e^1\cdot e^2 = -e^2\cdot e^1$ and $e^i\cdot e^i = -1$. The eigenspace decomposition now follows. Also, $\gamma(e^1)\gamma(e^2)\gamma(\nu^1)\gamma(\nu^2) = \gamma(e^1\cdot e^2\cdot \nu^1\cdot \nu^2) = \gamma(\mathrm{vol})$ where vol is the volume form, and the spinor spaces $\not{S}_\pm$ are defined as ± 1 eigenspaces of Clifford multiplication with vol: $\mathrm{vol}\cdot\not{S}_\pm = \pm\not{S}_\pm$. Thus $r_T r_N$ is minus the identity on $\not{S}_-$ and since r_T and r_N commute (and hence are simultaneously diagonalizable), it is easy to see that we must have $r_T = r_N$. □

We will henceforth denote $r_T = r$. Note that since we are only interested in the eigenspaces $V_{\pm\mathbf{j}_M}$, it does not matter which orientation we choose for M^2. In fact, we can identify these eigenspaces exactly. The octonion multiplication rules show

$$\begin{aligned}(a\mathbf{e})((b\mathbf{e})q) &= -\overline{(b\bar{q})}a \\ &= -q\bar{b}a = q\mathbf{j}_M\end{aligned}$$

and so the operator r is exactly right multiplication by $\mathbf{j}_M$. Thus the $+\mathbf{j}_M$ eigenspace of r is span$\{\mathbf{1}, \mathbf{j}_M\}$ and the $-\mathbf{j}_M$ eigenspace is the orthogonal complement of this.

3.5. **Cayley Submanifolds of $\not{S}_-(\mathbb{R}^4)$.** We have described the natural splitting

$$\not{S}_-(\mathbb{R}^4)|_{M^2} = V_{+\mathbf{j}_M} \oplus V_{-\mathbf{j}_M}$$

into two rank 2 bundles over the base surface M^2. The total space of either of these bundles is 4-dimensional and is a candidate for being a Cayley submanifold.

Theorem 3.5.1. *The total space of either rank 2 bundle $V_{\pm\mathbf{j}_M}$ over M is a Cayley submanifold of $\not{S}_-(\mathbb{R}^4)$ if and only the immersion $M \subset \mathbb{R}^4$ is* minimal. *That is,* $\mathrm{Tr}\, A^\nu = 0$ *for all normal vector fields ν.*

Proof. We show every tangent space to the total space of $V_{+\mathbf{j}_M}$ is a Cayley subspace of the corresponding tangent space to $\not{S}_-(\mathbb{R}^4)$. The proof for $V_{-\mathbf{j}_M}$ is identical. In local coordinates the immersion Ψ is

$$(u^1, u^2, t_1, t_2) \mapsto (x^1(u^1,u^2), x^2(u^1,u^2), t_1 q_1(u^1,u^2) + t_2 q_2(u^1,u^2))$$

where q_1 and q_2 are an orthonormal basis of $V_{+\mathbf{j}_M}$ and hence satisfy $rq_k = \mathbf{j}_M q_k$. The tangent space at $(\mathbf{x}(\mathbf{u}_0), t_1, t_2)$ is spanned by the vectors

$$\begin{aligned} E_k &= \Psi_*\left(\frac{\partial}{\partial u^k}\right) = e_k + \nabla_{e_k}(t_1 q_1 + t_2 q_2)|_{\mathbf{x}_0} \qquad k = 1,2 \\ F_k &= \Psi_*\left(\frac{\partial}{\partial t_k}\right) = q_k \qquad k = 1,2 \end{aligned}$$

We now derive an expression for $\nabla_{e_k} q_j|_{\mathbf{x}_0}$. To simplify notation we will use a dot to denote $\nabla_{e_k}|_{\mathbf{x}_0}$. Since $r^2 = -1$, we can differentiate to obtain

$$r\dot{r} + \dot{r}r = 0$$

Hence since r and $\dot{r}$ anti-commute, $r(\dot{r}q_j) = -\dot{r}(rq_j) = -\mathbf{j}_M \dot{r} q_j$ and thus $\dot{r}q_j \in V_{-\mathbf{j}_M}$. Now differentiating the equation $rq_j = \mathbf{j}_M q_j$, we have

$$\begin{aligned} \dot{r}q_j + r\dot{q}_j &= \mathbf{j}_M \dot{q}_j \\ (r - \mathbf{j}_M)\dot{q}_j &= -\dot{r}q_j \end{aligned}$$

The right hand side is in $V_{-\mathbf{j}_M}$, and on this space $r = -\mathbf{j}_M$, so $r - \mathbf{j}_M = -2\mathbf{j}_M$ on $V_{-\mathbf{j}_M}$ and we have

$$(r - \mathbf{j}_M)^{-1}(r - \mathbf{j}_M)\dot{q}_j = \dot{q}_j = \frac{-1}{2}(-\mathbf{j}_M)(-\dot{r}q_j) = -\frac{\mathbf{j}_M}{2}\dot{r}q_j$$

Explicitly, at the point $\mathbf{x}_0$, we have

$$\nabla_{e_k} q_j = -\frac{\mathbf{j}_M}{2}\left(\gamma(\nabla_{e_k} e^1)\gamma(e^2) + \gamma(e^1)\gamma(\nabla_{e_k} e^2)\right) q_j$$

From Lemma A.0.1 this can be written as

$$\begin{aligned} \nabla_{e_1} q_j &= \frac{\mathbf{j}_M}{2}\left(a_{11}\gamma(\nu^1)\gamma(e^2) + b_{11}\gamma(\nu^2)\gamma(e^2) + a_{12}\gamma(e^1)\gamma(\nu^1) + b_{12}\gamma(e^1)\gamma(\nu^2)\right) q_j \\ \nabla_{e_2} q_j &= \frac{\mathbf{j}_M}{2}\left(a_{12}\gamma(\nu^1)\gamma(e^2) + b_{12}\gamma(\nu^2)\gamma(e^2) + a_{22}\gamma(e^1)\gamma(\nu^1) + b_{22}\gamma(e^1)\gamma(\nu^2)\right) q_j \end{aligned}$$

where we have used the notation $a_{ij} = \langle e_i, A^{\nu_1}(e_j)\rangle$ and $b_{ij} = \langle e_i, A^{\nu_2}(e_j)\rangle$. Note that the operators $\gamma(e^i)\gamma(\nu^j)$ all anti-commute with $r = \gamma(e^1)\gamma(e^2)$ and hence map $V_{+\mathbf{j}_M} \to V_{-\mathbf{j}_M}$. Therefore $\nabla_{e_k} q_j \in V_{-\mathbf{j}_M}$. To check that the tangent space at $(\mathbf{x}_0, t_1, t_2)$ is Cayley, we need to verify that the imaginary part of the 4-fold octonion "cross" product $\mathrm{Im}(E_1 \times E_2 \times F_1 \times F_2)$ vanishes. (We put "cross" in quotation marks because this 4-fold cross product is not orthogonal to its arguments. See [9] for details.) Without loss of generality we can assume that at the point $\mathbf{x}_0$, we have chosen our coordinates so that $e^1 = \mathbf{e}$ and $e^2 = \mathbf{ie}$ with respect to the identification $T_x(\not{S}_-(\mathbb{R}^4)) \cong \mathbb{O}$, where $T(\mathbb{R}^4)|_M \cong \mathbb{H}\mathbf{e}$ and the spinor space $\not{S}_- \cong \mathbb{H}$. From this choice it follows that $\mathbf{j}_M = \mathbf{e}(\mathbf{ie}) = \mathbf{i}$. Then the orthonormal basis for $V_{+\mathbf{j}_M}$ is just $q_1 = \mathbf{1}, q_2 = \mathbf{i}$. Now we compute (using the octonion multiplication table):

$$\begin{aligned} \gamma(e^1)\gamma(\nu^1)q_1 &= \mathbf{j} & \qquad \gamma(e^1)\gamma(\nu^1)q_2 &= \mathbf{k} \\ \gamma(e^1)\gamma(\nu^2)q_1 &= \mathbf{k} & \qquad \gamma(e^1)\gamma(\nu^2)q_2 &= -\mathbf{j} \\ \gamma(\nu^1)\gamma(e^2)q_1 &= \mathbf{k} & \qquad \gamma(\nu^1)\gamma(e^2)q_2 &= -\mathbf{j} \\ \gamma(\nu^2)\gamma(e^2)q_1 &= -\mathbf{j} & \qquad \gamma(\nu^2)\gamma(e^2)q_2 &= -\mathbf{k} \end{aligned}$$

Therefore the tangent vectors to the immersion at $(\mathbf{x}_0, t_1, t_2)$ are given by

$$\begin{aligned} E_1 &= \mathbf{e} + \frac{t_1}{2}\mathbf{i}\left((a_{12} - b_{11})\mathbf{j} + (a_{11} + b_{12})\mathbf{k}\right) + \frac{t_2}{2}\mathbf{i}\left((-a_{11} - b_{12})\mathbf{j} + (a_{12} - b_{11})\mathbf{k}\right) \\ E_2 &= \mathbf{ie} + \frac{t_1}{2}\mathbf{i}\left((a_{22} - b_{12})\mathbf{j} + (a_{12} + b_{22})\mathbf{k}\right) + \frac{t_2}{2}\mathbf{i}\left((-a_{12} - b_{22})\mathbf{j} + (a_{22} - b_{12})\mathbf{k}\right) \\ F_1 &= \mathbf{1} \\ F_2 &= \mathbf{i} \end{aligned}$$

Now we can compute $\mathrm{Im}(E_1 \times E_2 \times F_1 \times F_2)$, with the result being

$$\left(\frac{t_1}{2}(a_{11}+a_{22}) - \frac{t_2}{2}(b_{11}+b_{22})\right)\mathbf{je} + \left(\frac{t_1}{2}(b_{11}+b_{22}) + \frac{t_2}{2}(a_{11}+a_{22})\right)\mathbf{ke}$$

which vanishes for all t_1, t_2 if and only if $\mathrm{Tr}\, A^{\nu_1} = \mathrm{Tr}\, A^{\nu_2} = 0$. □

Although this construction does produce two distinct Cayley submanifolds of $\mathbb{R}^8$ for each minimal surface M^2 in $\mathbb{R}^4$, they are in a sense degenerate examples. Note that when the global identification of $\mathbb{R}^8 = \mathbb{O}$ has been made, then no matter what surface M we choose, the octonion $\mathbf{1}$ will be in $V_{+\mathbf{j}_M}$ and the space $V_{-\mathbf{j}_M}$ will be orthogonal to $\mathbf{1}$. Therefore the $V_{+\mathbf{j}_M}$ Cayley submanifold will always be of the form $\mathbb{R}\mathbf{1} \times L^3$ for some 3-manifold L^3 which therefore must be associative in $\mathrm{Im}(\mathbb{O}) = \mathbb{R}^7$. Similarly the $V_{-\mathbf{j}_M}$ Cayley submanifold will have zero projection onto the $\mathbf{1}$ component, and thus is actually a coassociative submanifold of $\mathbb{R}^7$. Note however that this does indeed give coassociative submanifolds which are not contained in a strictly smaller subspace of $\mathbb{R}^7$, which we were unable to find in Section 3.2. We present some explicit examples in Section 4.

Remark 3.5.2. On more general non-compact manifolds of holonomy Spin(7), like $\not{S}_-(S^4)$ (see [4, 6]), this construction does produce interesting Cayley submanifolds. This is discussed in [11].

3.6. **The space $\not{S}(\mathbb{R}^3)$ as a manifold with a G_2-structure.** A G_2-structure can similarly be placed on the spinor bundle $\not{S}(\mathbb{R}^3) \cong \mathbb{R}^7$ of $\mathbb{R}^3$. (See [4] for details.) In this case we do not have positive and negative spinor bundles. The fibre (spinor space) at each point is again isomorphic to the quaternions $\mathbb{H}$. In fact we have

$$\not{S}(\mathbb{R}^3) = \not{S}_\pm(\mathbb{R}^4)\big|_{\mathbb{R}^3}$$

Explicitly, if e^0, e^1, e^2, e^3 is a basis for the Clifford algebra of $\mathbb{R}^4$, then the Clifford products $e^0 \cdot e^1$, $e^0 \cdot e^2$, $e^0 \cdot e^3$ are a basis for the Clifford algebra of $\mathbb{R}^3$. We can take a surface $M^2 \subset \mathbb{R}^3$ with orthonormal cotangent frame e^1, e^2 and conormal vector $\nu = e^3$ and again consider the eigenspaces $V_{\pm\mathbf{j}_M}$ of the operator $r = \gamma(e^1)\gamma(e^2) = \pm\gamma(e^0)\gamma(e^3)$ where the sign depends on the choice of orientation and does not affect the eigenspaces. Then we can take the total spaces of $V_{\pm\mathbf{j}_M}$ over M^2 as 4-manifolds which can be coassociative in $\mathbb{R}^7$.

Proposition 3.6.1. *The total spaces of* $V_{\pm\mathbf{j}_M}$ *over* M^2 *are coassociative in* $\mathbb{R}^7$ *iff* $M^2 \subset \mathbb{R}^3$ *is* minimal.

Proof. Since being coassociative in $\mathbb{R}^7$ is equivalent to being Cayley in $\mathbb{R}^8$, Theorem 3.5.1 says that M^2 must be minimal in $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$. But since M^2 sits in $\mathbb{R}^3 \subset \mathbb{R}^4$, this is equivalent to being minimal in $\mathbb{R}^3$. □

Similarly we can try to take a curve $C^1 \subset \mathbb{R}^3$ and decompose the spinor space $\not{S}$ into eigenspaces of $r = \gamma(e^0)\gamma(e^1) = \pm\gamma(\nu^1)\gamma(\nu^2)$, where e^1 is a unit cotangent vector to C^1 and ν^1, ν^2 are an orthonormal basis of conormal vector fields. Then the total spaces of the bundles over C^1 would be 3-manifolds which could be associative. But since C^1 would have to be minimal, it is a straight line and this construction only produces associative 3-planes in $\mathbb{R}^7$.

4. Some Explicit Examples

4.1. Some Explicit Minimal Surfaces in $\mathbb{R}^4$. For the convenience of the reader, we present some explicit examples of minimal surfaces in $\mathbb{R}^4$ which are used to construct examples of calibrated submanifolds of $\mathbb{R}^7$ and $\mathbb{R}^8$ in Section 4.3. If we consider a *graph* of the form

$$\left(x^1, x^2, f^1(x^1, x^2), f^2(x^1, x^2)\right)$$

then the tangent vectors to this immersion are

$$\begin{aligned} e_1 &= \left(1, 0, f_1^1, f_1^2\right) \\ e_2 &= \left(0, 1, f_2^1, f_2^2\right) \end{aligned}$$

where the subscript k denotes partial differentiation with respect to x^k. The induced metric is $g_{ij} = e_i \cdot e_j$. The minimal surface equations in these coordinates are

$$g_{22} f_{11}^k + g_{11} f_{22}^k - 2 g_{12} f_{12}^k = 0 \qquad k = 1, 2 \tag{4.1}$$

They are a pair of second order, quasi-linear PDE's in which the second order derivatives are uncoupled.

Let us identify $\mathbb{R}^4 = \mathbb{C}^2$ with complex coordinates $z = x^1 + ix^2$ and $w = f^1 + if^2$. It is well known (and trivial to check) that the image of a holomorphic or anti-holomorphic map $w = f(z)$ is a minimal surface. These satisfy the Cauchy-Riemann equations $f_1^1 = f_2^2$ and $f_2^1 = -f_1^2$ in the holomorphic case and $f_1^1 = -f_2^2$ and $f_2^1 = f_1^2$ in the anti-holomorphic case.

Alternatively we can instead choose complex coordinates $z = x^1 + if^1$ and $w = x^2 + if^2$. Then a special Lagrangian graph is an example of a minimal surface in $\mathbb{R}^4 = \mathbb{C}^2$. In this case $f^k = \frac{\partial F}{\partial x^k}$ for some potential function $F(x^1, x^2)$ and the special Lagrangian differential equation with phase θ is

$$\begin{aligned} F_{11} + F_{22} &= 0 \qquad \text{for phase } \theta = 0 \\ F_{11} F_{22} - F_{12}^2 &= 1 \qquad \text{for phase } \theta = \frac{\pi}{2} \end{aligned} \tag{4.2}$$

We can also look for minimal surfaces which are not of these special types. Our first example is a generalization of the holomorphic example $f^1 = e^u \cos(v)$, $f^2 = e^u \sin(v)$, which corresponds to the holomorphic function e^z where we are now writing $z = u + iv$. We can ask for the most general minimal surface of the form

$$(u, v, f(u)\cos(v), f(u)\sin(v))$$

for some function $f(u)$. Substitution into (4.1) yields the following non-linear ODE for $f(u)$:

$$f(1 + (f')^2) = f''(1 + f^2)$$

This can be explicitly integrated to give the general solution

$$f(u) = \frac{C}{2} e^{Ku} + \frac{1 - K^2}{2CK^2} e^{-Ku}$$

for two constants of integration C and K. Note that $K = 1$ corresponds to the holomorphic solution e^u. In Section 4.3 we use this minimal surface with $C = 2$ and $K = \frac{1}{2}$:

$$\left(u, v, \left(e^{\frac{u}{2}} + \frac{3}{4} e^{-\frac{u}{2}}\right)\cos(v), \left(e^{\frac{u}{2}} + \frac{3}{4} e^{-\frac{u}{2}}\right)\sin(v)\right) \tag{4.3}$$

Another explicit example can be obtained by considering graphs which are rotationally symmetric:

$$\big(u, v, f(u^2+v^2), g(u^2+v^2)\big)$$

This time substitution into (4.1) yields the following system of non-linear ODE's, where we have denoted $t = u^2 + v^2$:

$$\begin{aligned} tf'' + f' + 2tf'\left((f')^2 + (g')^2\right) &= 0 \\ tg'' + g' + 2tg'\left((f')^2 + (g')^2\right) &= 0 \end{aligned}$$

These can also be integrated explicitly to obtain

$$\begin{aligned} f(t) &= \frac{2K}{\sqrt{L}} \log\left(\sqrt{t} + \sqrt{t - \frac{4(1+K^2)}{L}}\right) \\ g(t) &= \frac{2}{\sqrt{L}} \log\left(\sqrt{t} + \sqrt{t - \frac{4(1+K^2)}{L}}\right) \end{aligned}$$

for two constants of integration K and L. Note that this example is only defined outside a circle in the u, v plane. We use this minimal surface in Section 4.3 with $K = 1$ and $L = 4$:

$$\left(u, v, \log\left(\sqrt{u^2+v^2} + \sqrt{u^2+v^2-2}\right), \log\left(\sqrt{u^2+v^2} + \sqrt{u^2+v^2-2}\right)\right) \tag{4.4}$$

4.2. **Real Isotropic Minimal Surfaces in $\mathbb{R}^4$.** In this section we give a brief introduction to real isotropic minimal surfaces in $\mathbb{R}^4$, sometimes also called *superminimal* surfaces. See [1, 5, 19] and the references contained therein for more details.

For a surface M^2 isometrically immersed in $\mathbb{R}^4$, the second fundamental form A can be viewed as a symmetric tensor on M^2 with values in the normal bundle $N(M^2)$. That is,

$$A = A_{11}e^1 \otimes e^1 + A_{12}e^1 \otimes e^2 + A_{21}e^2 \otimes e^1 + A_{22}e^2 \otimes e^2$$

where $A_{ij} = A_{ji} = A^{\nu_1}(e_i, e_j)\nu_1 + A^{\nu_2}(e_i, e_j)\nu_2$ and e_1, e_2 and ν_1, ν_2 are oriented orthonormal tangent and normal frames for M^2, respectively. We have a natural almost complex structure J on $T(M)$ given by $Je_1 = e_2$, and $Je_2 = -e_1$. Therefore we can consider the 1-forms $\beta = e^1 + ie^2$ and $\bar{\beta} = e^1 - ie^2$, which are of type $(1,0)$ and $(0,1)$ respectively. We can rewrite A in this basis as follows:

$$A = \frac{\mathbf{W}}{4}\beta \otimes \beta + \frac{\mathbf{H}}{4}(\beta \otimes \bar{\beta} + \bar{\beta} \otimes \beta) + \frac{\overline{\mathbf{W}}}{4}\bar{\beta} \otimes \bar{\beta}$$

where $\mathbf{H} = A^{\nu_1}_{11}\nu_1 + A^{\nu_2}_{11}\nu_2 + A^{\nu_1}_{22}\nu_1 + A^{\nu_2}_{22}\nu_2$ is the mean curvature vector of the immersion and $\mathbf{W}$ is the complex valued normal vector

$$\mathbf{W} = \mathbf{W}_1 + i\mathbf{W}_2 = (A^{\nu_1}_{11}\nu_1 + A^{\nu_2}_{11}\nu_2 - A^{\nu_1}_{22}\nu_1 - A^{\nu_2}_{22}\nu_2) + i\,(-2A^{\nu_1}_{12}\nu_1 - 2A^{\nu_2}_{12}\nu_2)$$

If M is minimal in $\mathbb{R}^4$, then $\mathbf{H} = 0$ and the $(1,1)$ term in A vanishes. The $(2,0)$ and $(0,2)$ terms are conjugates of each other. We use the real inner product on $\mathbb{R}^4$ to define the *complex quartic form* Q of the minimal surface M to be

$$Q = \mathbf{W} \cdot \mathbf{W} = (\mathbf{W}_1 \cdot \mathbf{W}_1 - \mathbf{W}_2 \cdot \mathbf{W}_2) + 2i\,(\mathbf{W}_1 \cdot \mathbf{W}_2)$$

The minimal surface M is called *real isotropic* if $Q = 0$. If we denote

$$A^{\nu_1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \qquad\qquad A^{\nu_2} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}$$

then it is easy to check that real isotropic is equivalent (using the fact that M is already minimal) to the equations

$$a_{11} = \pm b_{12} \qquad a_{22} = \mp b_{12} \qquad b_{11} = \mp a_{12} \qquad b_{22} = \pm a_{12}$$

which can be written concisely as

$$A^{J\nu} = \pm J A^{\nu} \tag{4.5}$$

for any normal vector field ν. The equation obtained in Section 3.2 for constructing coassociative submanifolds in $\wedge^2_-(\mathbb{R}^4)$ was this equation with *only* the minus sign. (And it appears with only the plus sign if we are considering $\wedge^2_+(\mathbb{R}^4)$.) Hence only half of the real isotropic minimal surfaces in $\mathbb{R}^4$ can be used in the construction.

We can write down the system of PDE's explicitly in the case of a graph

$$\left(u^1, u^2, f^1(u^1, u^2), f^2(u^1, u^2)\right)$$

since (non-orthonormal) bases for the tangent and normal spaces are given by

$$\begin{aligned} \tilde{e}_1 &= \left(1, 0, f^1_1, f^2_1\right) & \qquad \tilde{\nu}_1 &= \left(-f^1_1, -f^1_2, 1, 0\right) \\ \tilde{e}_2 &= \left(0, 1, f^1_2, f^2_2\right) & \qquad \tilde{\nu}_2 &= \left(-f^2_1, -f^2_2, 0, 1\right) \end{aligned}$$

The induced metric on M is $g_{ij} = \tilde{e}_i \cdot \tilde{e}_j$ and let $h_{ij} = \tilde{\nu}_i \cdot \tilde{\nu}_j$ denote the metric on the normal space. Using the Gram-Schmidt process to obtain orthonormal bases, one can check that (4.5) becomes the following quasi-linear second order system:

$$\begin{aligned} g_{11} f^1_{12} - g_{12} f^1_{11} &= \pm\left(h_{12} f^1_{11} - h_{11} f^2_{11}\right) \\ g_{11} f^2_{12} - g_{12} f^2_{11} &= \pm\left(-h_{12} f^2_{11} + h_{22} f^1_{11}\right) \\ g_{22} f^2_{12} - g_{12} f^2_{22} &= \pm\left(h_{12} f^2_{22} - h_{22} f^1_{22}\right) \\ g_{22} f^1_{12} - g_{12} f^1_{22} &= \pm\left(-h_{12} f^1_{22} + h_{11} f^2_{22}\right) \end{aligned}$$

This is again a quasi-linear second order system, however this time the second order derivatives of f^1 and f^2 are no longer uncoupled. Solutions to this system are a subset of solutions to the minimal surface equation. For example, the minimal surfaces in (4.3) and (4.4) are *not* real isotropic. If we identify $\mathbb{R}^4 = \mathbb{C}^2$ then holomorphic maps satisfy (4.5) with the plus sign, while anti-holomorphic maps satisfy it with the minus sign. Minimal surfaces in $\mathbb{C}^2$ which are special Lagrangian with phase 0 or $\frac{\pi}{2}$ (satisfying (4.2)) are real isotropic with the minus sign. This can also be seen from the fact that special Lagrangian submanifolds of $\mathbb{C}^2$ are complex submanifolds with respect to a different complex structure, where the orientation is changed. (See [9]).

4.3. **Examples of Calibrated Submanifolds.** We now apply the constructions described in Section 3 to some explicit examples. Our surfaces M^2 will all be given as graphs $(u, v, f^1(u, v), f^2(u, v))$.

Recall from Section 4.2 that anti-holomorphic surfaces (or equivalently special Lagrangian surfaces of any phase) satisfied the real isotropic minimal surface equation (with the minus sign) from Theorem 3.2.1 that was required to construct coassociative submanifolds. One can check that in these cases the constructed 4-fold is simply a product $\mathbb{R}^2 \times M^2$. Similarly a product 3-manifold $\mathbb{R} \times M^2$ is obtained when using these minimal surfaces to construct associative submanifolds using Theorem 3.3.1.

However, we can also try holomorphic surfaces (which are still minimal) in the associative case. (Recall that these satisfy the real isotropic equation with the plus sign, and cannot be used to construct coassociative submanifolds. They would work in $\wedge^2_+(\mathbb{R}^4)$, but would produce product manifolds there.) Consider the holomorphic surface $(x, y, u(x, y), v(x, y))$ in $\mathbb{R}^4$ where the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied. Then one can construct the vector $e^1 \wedge e^2 - \nu^1 \wedge \nu^2$ in $\wedge^2_-$ and it turns out to be (using the Cauchy-Riemann equations to simplify):

$$\left(\frac{1-|\nabla u|^2}{1+|\nabla u|^2}, \frac{2u_y}{1+|\nabla u|^2}, \frac{2u_x}{1+|\nabla u|^2}\right)$$

Hence Theorem 3.3.1 gives the following associative submanifold of $\mathbb{R}^7$:

$$\left(t\frac{1-|\nabla u|^2}{1+|\nabla u|^2}, t\frac{2u_y}{1+|\nabla u|^2}, t\frac{2u_x}{1+|\nabla u|^2}, x, y, u(x,y), v(x,y)\right)$$

For an explicit example, we can take $u = e^x \cos(y)$ and $v = e^x \sin(y)$ to obtain

$$\left(t\frac{\sinh(x)}{\cosh(x)}, t\frac{\sin(y)}{\cosh(x)}, -t\frac{\cos(y)}{\cosh(x)}, x, y, e^x\cos(y), e^x\sin(y)\right)$$

If we take instead the minimal surface in (4.3) we obtain, after rescaling the fibre direction basis vector to simplify the expression, the following non-trivial associative submanifold of $\mathbb{R}^7$,:

$$\left(t\frac{4e^x-9}{12e^{\frac{1}{2}x}}, t\sin(y), -t\cos(y), x, y, \left(e^{\frac{x}{2}}+\frac{3}{4}e^{-\frac{x}{2}}\right)\cos(y), \left(e^{\frac{x}{2}}+\frac{3}{4}e^{-\frac{x}{2}}\right)\sin(y)\right)$$

Finally, the minimal surface in (4.4) yields the following associative submanifold of $\mathbb{R}^7$ (defined for $x^2 + y^2 > 2$):

$$((y-x)h_1h_2, y-x, x+y, x, y, \log(h_1+h_2), \log(h_1+h_2))$$

where $h_1(x, y) = \sqrt{x^2+y^2}$ and $h_2(x, y) = \sqrt{x^2+y^2-2}$.

Recall from the remarks made at the end of Section 3.5 that the Cayley construction actually produces Cayley submanifolds which are either a line cross an associative submanifold of $\mathbb{R}^7$ or a coassociative submanifold of $\mathbb{R}^7$. Thus they can be used to provide non-trivial examples of coassociative submanifolds which are not contained in a strictly smaller subspace of $\mathbb{R}^7$, by taking the $V_{-\mathbf{j}_M}$ eigenspace. Taking a holomorphic surface $(x, y, u(x, y), v(x, y))$ in $\mathbb{R}^4$, one can compute that the $-\mathbf{j}_M$ eigenspace is spanned by

$$(0, -2u_y, 1-|\nabla u|^2, 0) \qquad \text{and} \qquad (0, -2u_x, 0, 1-|\nabla u|^2)$$

Thus Theorem 3.5.1 gives the following coassociative submanifold of $\mathbb{R}^7$:

$$\left(-2(t_1u_y+t_2u_x), t_1(1-|\nabla u|^2), t_2(1-|\nabla u|^2), x, y, u(x,y), v(x,y)\right)$$

The example of $u = e^x \cos(y)$ and $v = e^x \sin(y)$ gives

$$\left(2e^x(t_2\cos(y)-t_1\sin(y)), t_1e^{2x}, t_2e^{2x}, x, y, e^x\cos(y), e^x\sin(y)\right)$$

as a coassociative submanifold of $\mathbb{R}^7$. One can similarly use (4.3) or (4.4) and Theorem 3.5.1 to produce explicit coassociative submanifolds of $\mathbb{R}^7$. The expressions tend to be extremely complicated in these cases.

Appendix A. Local Computations for Immersions $M^p \subset \mathbb{R}^n$

In this appendix we present some local computations for an isometric immersion of a p-dimensional submanifold M^p immersed in $\mathbb{R}^n$ that are used repeatedly in the paper.

Take $(x^1, x^2, \ldots, x^n)$ to be coordinates on $\mathbb{R}^n$, and denote the immersion $M \subset \mathbb{R}^n$ by $x^i = x^i(u^1, u^2, \ldots, u^p)$, for $1 \leq i \leq n$ where $(u^1, u^2, \ldots, u^p)$ are local coordinates on M^p. Consider a point $\mathbf{u}_0$ in M with coordinates $(u_0^1, u_0^2, \ldots, u_0^p)$ and corresponding to the point $\mathbf{x}_0 = \mathbf{x}(\mathbf{u_0})$ in X with coordinates $(x_0^1, x_0^2, \ldots, x_0^n)$. Near $\mathbf{x}_0$ let $e_1, e_2, \ldots, e_p$ be a local orthonormal frame of tangent vector fields to M and let $\nu_1, \nu_2, \ldots, \nu_q$ be a local orthonormal frame of normal vector fields to M, where $q = n - p$. Without loss of generality we can assume that these local vector fields have been chosen so that at the point $\mathbf{x}_0$, we have

$$(\nabla_{e_i} e_j)|^T_{\mathbf{x_0}} = 0 \qquad \text{and} \qquad (\nabla_{e_i} \nu_j)|^N_{\mathbf{x_0}} = 0 \tag{A.1}$$

Here ∇ denotes the Levi-Civita connection on $\mathbb{R}^n$ and $()^T$ and $()^N$ denote the orthogonal projections onto the tangent and normal bundles of M in $\mathbb{R}^n$. Equivalently, we are saying that the e_i's are parallel with respect to the induced Levi-Civita connection on M and the ν_j's are parallel with respect to the induced normal connection on the bundle $N(M)$. This can be done by choosing an orthonormal tangent frame and an orthonormal normal frame at the point $\mathbf{x}_0$ and then parallel transporting via the two respective connections.

For ν any normal vector field, we can define the second fundamental form A^ν as the linear operator

$$\begin{aligned} A^\nu &: T(M) \to T(M) \\ A^\nu &: w \mapsto A^\nu(w) = (\nabla_w \nu)^T \end{aligned}$$

A couple of remarks are in order. First, this is sometimes called the *shape operator* and the associated bilinear form $B(v, w) = \langle v, A^\nu w \rangle$ is often called the second fundamental form. Second, we are following the sign convention of Harvey and Lawson, which differs from most definitions. The statements of all results in this paper are independent of the choice of sign for the definition of A^ν. The important property of A^ν is that it is a symmetric operator, and hence diagonalizable. This follows from

$$\begin{aligned} & \langle e_i, A^\nu(e_j) \rangle = \langle e_i, \left(\nabla_{e_j} \nu\right)^T \rangle = \langle e_i, \nabla_{e_j} \nu \rangle = -\langle \nabla_{e_j} e_i, \nu \rangle \\ = \quad & -\langle \nabla_{e_i} e_j, \nu \rangle + \langle [e_i, e_j], \nu \rangle = -\langle \nabla_{e_i} e_j, \nu \rangle = \langle e_j, A^\nu(e_i) \rangle \end{aligned}$$

where we have used $[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i$ and the fact that $[e_i, e_j]$ is orthogonal to ν since the bracket of two tangent vector fields on M is again a tangent vector field on M.

We also have the dual coframe of orthonormal cotangent vector fields $e^1, e^2, \ldots, e^p$ and the orthonormal conormal vector fields $\nu^1, \nu^2, \ldots, \nu^q$. These satisfy

$$e^i(e_j) = \delta^i_j \qquad \nu^i(\nu_j) = \delta^i_j \qquad e^i(\nu_j) = 0 \qquad \nu^i(e_j) = 0 \tag{A.2}$$

The following lemma is used repeatedly in the paper.

Lemma A.0.1. *Under the hypotheses of* (A.1), *we have the following expressions for the covariant derivatives of the e^i's and the ν^j's at the point $\mathbf{x}_0$.*

$$\nabla_{e_i} e^j = -\sum_{k=1}^{q} \langle e_j, A^{\nu_k}(e_i)\rangle \nu^k \tag{A.3}$$

$$\nabla_{e_i} \nu^j = \sum_{k=1}^{p} \langle e_k, A^{\nu_j}(e_i)\rangle e^k \tag{A.4}$$

Proof. From (A.2), we have $\left(\nabla_{e_i} e^j\right)(e_k) = -e^j\left(\nabla_{e_i} e_k\right)$. We can write $\nabla_{e_i} e^j = \sum_{k=1}^{p} a_k e^k + \sum_{l=1}^{q} b_l \nu^l$, and hence

$$\begin{aligned} a_k = e_k\left(\nabla_{e_i} e^j\right) = -e^j\left(\nabla_{e_i} e_k\right) &= -e^j\left(\left(\nabla_{e_i} e_k\right)^T + \left(\nabla_{e_i} e_k\right)^N\right) \\ &= -e^j \left(\nabla_{e_i} e_k\right)^T = 0 \end{aligned}$$

where we have used (A.1). Similarly we have

$$\begin{aligned} b_k = \nu_k\left(\nabla_{e_i} e^j\right) = -e^j\left(\nabla_{e_i} \nu_k\right) &= -e^j\left(\left(\nabla_{e_i} \nu_k\right)^T + \left(\nabla_{e_i} \nu_k\right)^N\right) \\ &= -e^j \left(\nabla_{e_i} \nu_k\right)^T = -\langle e_j, A^{\nu_k}(e_i)\rangle \end{aligned}$$

which proves (A.3).

To prove (A.4), we write $\nabla_{e_i} \nu^j = \sum_{k=1}^{p} c_k e^k + \sum_{l=1}^{q} d_l \nu^l$, and compute

$$\begin{aligned} d_k = \nu_k\left(\nabla_{e_i} \nu^j\right) = -\nu^j\left(\nabla_{e_i} \nu_k\right) &= -\nu^j\left(\left(\nabla_{e_i} \nu_k\right)^T + \left(\nabla_{e_i} \nu_k\right)^N\right) \\ &= -\nu^j \left(\nabla_{e_i} \nu_k\right)^N = 0 \end{aligned}$$

from (A.1) and also

$$\begin{aligned} c_k = e_k\left(\nabla_{e_i} \nu^j\right) = -\nu^j\left(\nabla_{e_i} e_k\right) &= -\nu^j\left(\left(\nabla_{e_i} e_k\right)^T + \left(\nabla_{e_i} e_k\right)^N\right) \\ &= -\nu^j \left(\nabla_{e_i} e_k\right)^N = -\langle \nabla_{e_i} e_k, \nu_j\rangle \\ &= \langle e_k, \nabla_{e_i} \nu_j\rangle = \langle e_k, A^{\nu_j}(e_i)\rangle \end{aligned}$$

which completes the proof. □

Appendix B. Octonion Multiplication Table

The following is a multiplication table for the octonions $\mathbb{O}$. The table corresponds to multiplying the element in the corresponding row on the left of the element in the corresponding column. For example $\mathbf{i} \cdot \mathbf{j} = \mathbf{k}$.

	1	**i**	**j**	**k**	**e**	**ie**	**je**	**ke**
1	**1**	**i**	**j**	**k**	**e**	**ie**	**je**	**ke**
i	**i**	**-1**	**k**	**-j**	**ie**	**-e**	**-ke**	**je**
j	**j**	**-k**	**-1**	**i**	**je**	**ke**	**-e**	**-ie**
k	**k**	**j**	**-i**	**-1**	**ke**	**-je**	**ie**	**-e**
e	**e**	**-ie**	**-je**	**-ke**	**-1**	**i**	**j**	**k**
ie	**ie**	**e**	**-ke**	**je**	**-i**	**-1**	**-k**	**j**
je	**je**	**ke**	**e**	**-ie**	**-j**	**k**	**-1**	**- i**
ke	**ke**	**-je**	**ie**	**e**	**-k**	**-j**	**i**	**-1**

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