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Scalar curvature rigidity of asymptotically hyperbolic spin manifolds*

Maung Min-Oo

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

1. Introduction

The rigidity statement of the positive mass theorem for Riemannian manifolds says that an asymptotically Euclidean spin manifold with everywhere non-negative scalar curvature and zero mass is isometric to Euclidean space. The positive mass conjecture in its various versions was first proved by Schoen and Yau in a series of papers [11–13] and then independently by Witten [15] who introduced a new and elegant method of proof using spinors. (See also [1, 10]).

In this paper we study the hyperbolic version of this result using Witten's method and show that a spin manifold which is (strongly) asymptotically hyperbolic cannot have scalar curvature $R \geqq -n(n-1)$ everywhere unless it is isometric to hyperbolic space.

We denote hyperbolic space with constant sectional curvature $K \equiv -1$ and scalar curvature $R \equiv -n(n-1)$ by $(H^n, \bar{g})$. We will think of hyperbolic space as $\mathbb{R}^n$ endowed with the metric $\bar{g} = ds^2 = dr^2 + (\sinh r)^2\, d\theta^2$ in polar coordinates. For $r > 0$ we denote by $H^n(r, \infty)$ the complement of a closed ball of radius r around the origin, i.e. $H^n(r, \infty) = H^n - \bar{B}_r(0)$.

On a complete Riemannian manifold (M^n, g), $L^{k,q}$ will denote the Sobolev space of all functions with derivatives of order $\leqq k$ in L^q, equipped with the usual Sobolev norm $\| \quad \|_{k,q}$. The same notation will also be used for the Sobolev space of tensors, spinors or other sections in vector bundles over M equipped with metrics where the derivatives are defined by some metric preserving connection. For tensors and spinors, which are intrinsically associated to the manifold, the derivative means covariant derivative with respect to the Levi-Civita connection, unless stated otherwise explicitly.

Definition. A smooth Riemannian manifold (M^n, g) is said to be *strongly asymptotically hyperbolic* (with one end) if there exists a compact subset $B \subset M$ and a diffeomorphism $\phi: M - B \xrightarrow{\sim} H^n(r_0, \infty)$ for some $r_0 > 0$, such that if we define

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the *gauge transformation* $A: T(M-B) \xrightarrow{\sim} T(M-B)$ by the equations:

$$\begin{aligned} &\text{(i)}\ \ g(Au, Av) = \phi^* g(u,v) = \bar{g}(d\phi(v), d\phi(v)) \\ &\text{(ii)}\ \ g(Au, v) = g(u, Av) \end{aligned} \tag{1.1}$$

then A satisfies the following two properties:

AH1 $\exists$ a uniform Lipschitz constant $C \geqq 1$ such that, for all $v \in T(M-B)$:

$$C^{-1} \leqq \min_{|v|=1} |Av| \leqq \max_{|v|=1} |Av| \leqq C$$

AH2 $\exp(\phi \circ r) \cdot (A - \mathrm{id}) \in L^{1,2}(T^*(M-B) \otimes T(M-B), g)$

We leave it to the reader to make the obvious extension of the above definition to the case of finitely many ends. The theorem which follows also holds for manifolds with finitely many asymptotically hyperbolic ends if we make the corresponding trivial modifications of the proof. We restrict ourselves therefore to the case of a connected end. If the dimension $n \geqq 3$, this end will also be *simply connected.* This fact however, is used essentially in the proof.

Condition AH1 assures, in particular, that the manifold is complete. The second condition AH2 is of course the main assumption on the asymptotic structure of M, although the exact Sobolev space $L^{1,2}$ and the weight e^r we have chosen here are certainly not optimal. We will see that a condition on the metric, with respect to a special coordinate chart is by no means the most natural restriction one can impose on the structure at ∞. Our proof shows that a more natural definition for a manifold to be asymptotically hyperbolic, or for that matter to be asymptotically like a symmetric space is given in terms of an almost parallel framing at infinity for an almost symmetric structure with a Cartan connection and our results would also be true with this more general definition of asymptotically hyperbolic. (See [7] for a discussion about almost symmetric spaces and Cartan connections). The formulation given above is similar to the definition of asymptotic flatness used in relativity.

We recall that a spin manifold is an oriented Riemannian manifold (M^n, g) together with a lift of the structural group $SO(n)$ of its principal bundle $SO(M^n, g)$ of all oriented orthonormal frames to its simply connected double cover $Spin(n)$. It is well known that M has a spin structure iff its 2[nd] Stiefel-Whitney class $w_2(M)$ vanishes.

The main result of this paper can now be stated as follows:

Theorem. *A strongly asymptotically hyperbolic spin manifold of dimension* $\geqq 3$, *whose scalar curvature satisfies* $R \geqq -n(n-1)$ *everywhere, is isometric to hyperbolic space.*

The basic tool used to prove the above theorem is a Weitzenböck formula for the Dirac operator associated to a Cartan connection of hyperbolic type. The analogous formula for the usual Dirac operator of the Levi-Civita connection of Riemannian geometry is the famous formula of A. Lichnerowicz, which is the basis of many investigations related to the scalar curvature. In particular we refer to the fundamental papers of Hitchin [5], and of Gromov and Lawson [2–4], where a twisted version of the Dirac operator is employed. The Cartan connection $\tilde{\nabla}$ used in this paper is a different connection for the base manifold itself and therefore our Dirac operator is not the usual Dirac operator twisted with some auxiliary coefficient bundle. It is however still a generalized Dirac operator as defined in Sect. 1 of [4].

In Sect. 2 we define this basic operator $\tilde{D}$ and derive two Weitzenböck formulas for it. One of them (the L^2-version) states that $\tilde{D}^2 = D^2 + \frac{n^2}{4}$ where D is the usual Dirac operator. This formula proves that $\tilde{D}$ is invertible. The second formula is the analogue of the Lichnerowicz formula and relates $\tilde{D}^2$ to the rough Laplacian and the *hyperbolic scalar curvature* $\tilde{R} = R + n(n-1)$.

The Theorem is then proved, in Sect. 3, by solving for a spinor ψ satisfying $\tilde{D}\psi = 0$, and which is asymptotically almost parallel with respect to the Cartan connection $\tilde{\nabla}$. An application of Stokes' theorem to the second Weitzenböck formula then yields a boundary term which goes to zero at infinity if the manifold is strongly asymptotically hyperbolic. The curvature assumption $\tilde{R} \geqq 0$ now implies that ψ is parallel everywhere and so we obtain a global parallelization of the bundle of spinors, forcing the manifold to be isometric to hyperbolic space. The calculations rely on some of the elementary properties about generalized Dirac operators as can be found in Sects. 1 and 2 of the paper [4].

Finally we would like to mention the paper [14], where some related rigidity results involving the sectional curvature are proved.

2. The hyperbolic Dirac operator

For an oriented Riemannian manifold (M^n, g), we denote by $SO(M,g) \to M$, the principal $SO(n)$-bundle of oriented orthonormal frames and if M is a spin manifold, the principal Spin(n)-bundle defining the spin structure, which is a double cover of $SO(M,g)$, will be denoted by $P = \mathrm{Spin}(M,g) \to M$.

The Levi-Civita connection is an $\mathcal{O}(n)$-valued one form η defined on $SO(M,g)$, where $\mathcal{O}(n)$ denotes the Lie algebra of $SO(n)$. Since connections are local objects, η lifts to the double cover P defining the Levi-Civita connection for the spin bundle. Similarly the horizontal $\mathbb{R}^n$-valued canonical one form θ, sometimes called the soldering form, can be lifted to P.

The sum of these two 1-forms $\omega = \eta + \theta$ defines a parallelization for both principal bundles. ω is then an $\mathcal{O}(n) \oplus \mathbb{R}^n$-valued 1-form. We now define the Lie bracket structure of the Lie algebra $\mathcal{O}(1,n)$ of the hyperbolic group $SO(1,n)$ on this direct sum $\mathcal{O}(n) \oplus \mathbb{R}^n$. ω then becomes what is called a *Cartan connection* of hyperbolic type on M. If the structure group Spin(n) of P is extended to the hyperbolic spin group Spin$(1,n)$ via the standard imbedding $\mathrm{Spin}(n) \subset \mathrm{Spin}(1,n)$ we obtain the principal Spin$(1,n)$-bundle $\tilde{P} = P \underset{\mathrm{Spin}(n)}{\times} \mathrm{Spin}(1,n)$ and ω can be canonically extended to an $\mathcal{O}(1,n)$-valued 1-form on $\tilde{P}$ defining a connection in the usual sense of the word. Since $P \subset \tilde{P}$ is a reduction, there is a corresponding section σ for the fibre bundle $B = \tilde{P}(H^n)$ associated to $\tilde{P}$ by the action of Spin$(1,n)$ on the quotient $H^n = \mathrm{Spin}(1,n)/\mathrm{Spin}(n)$. The Cartan connection ω defines a parallel translation and a developing map, via the soldering section σ, in the hyperbolic bundle B on M.

The curvature of ω is defined to be the equivariant horizontal $\mathcal{O}(1,n)$-valued 2-form:

$$\Omega = d\omega + [\omega, \omega]. \tag{2.01}$$

Breaking up Ω into its $\mathcal{O}(n)$- and $\mathbb{R}^n$-valued components gives us the structure equations of hyperbolic geometry:

$$\begin{aligned} &\text{(i)}\ \Omega_1 = d\eta + [\eta,\eta] + [\theta,\theta] \qquad (\mathcal{O}(n)\textit{-component}) \\ &\text{(ii)}\ \Omega_2 = d\theta + [\eta,\theta] + [\theta,\eta] = 0 \quad (\mathbb{R}^n\textit{-component}) \end{aligned} \tag{2.02}$$

Ω_2 is the torsion of the connection η and if we express the hyperbolic curvature Ω_1 as a tensor $\tilde{R}$ of type $(1,3)$ on the manifold M, then $\tilde{R}$ is given by:

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \bar{R}(X,Y)Z \tag{2.03}$$

where $R(X,Y)Z$ is the usual Riemannian curvature tensor of the Levi-Civita connection and

$$\bar{R}(X,Y)Z = -(X \wedge Y)Z = -\langle Y,Z\rangle X + \langle X,Z\rangle Y \tag{2.04}$$

is the curvature tensor of the hyperbolic space of constant sect. curvature -1.

Hence Ω_1 measures the deviation from being hyperbolic and $\Omega_1 \equiv 0$ if and only if (M^n, g) is locally isometric to hyperbolic space.

Let ρ:Spin$(1,n) \to$ End(V) be a finite dimensional representation (over $\mathbb{R}$ or $\mathbb{C}$). Then we can form the associated vector bundle $E_\rho = \tilde{P} \times_\rho V$ over M. Since the structure group of $\tilde{P}$ is reducible to the maximal compact subgroup Spin(n) and we have, in fact, a fixed reduction $P \subset \tilde{P}$ we could also obtain the vector bundle E_ρ as $E = E_\rho = P \times_\rho V$ with a compact structure group, where ρ denotes now the restriction of the given representation to Spin(n). We could then put a positive definite metric on E invariant under the compact group Spin(n).

There are now two connections on E, one defined by the Cartan connection ω, and the other connection coming from the original Levi-Civita connection η. We will denote the corresponding covariant derivatives by $\tilde{\nabla}$ and ∇ respectively. $\tilde{\nabla}$ does *not* preserve a positive definite metric in E, but it would preserve a Lorentzian metric defined by Spin$(1,n)$.

In this paper, we are interested in the following two representations of Spin$(1,n)$: (i) the vector representation on Minkowski space $\mathbb{R}^{1,n}$, and (ii) the spinor representation on a complex vector space $\mathbb{S}$ of dimension s over $\mathbb{C}$, where $s = 2^m$ with $m = \left[\frac{n}{2}\right]$.

In case (ii) we note that the spinor representation restricted to the maximal compact subgroup Spin(n) is still the spin representation of Spin(n). In fact, the Clifford algebra of $\mathbb{R}^{1,n}$: $Cl_{1,n} \cong Cl_{-1} \hat{\otimes} Cl_n$ is isomorphic to $Cl_n \oplus Cl_n$. Thus elements of $\mathbb{S}$ are spinors for both groups. In case (i), we have of course the splitting $\mathbb{R}^{1,n} \cong \mathbb{R} \oplus \mathbb{R}^n$ under Spin(n).

These two representations therefore define two natural vector bundles over any spin manifold M^n, which we denote by:

(i) $\tilde{T}M = TM \oplus 1$ (1 is the trivial real line bundle over M) and

(ii) $\mathbb{S}(M)$ (or short just $\mathbb{S}$ for short).

The two covariant derivatives defined by ω and η on these vector bundles are then related by the following formulas:

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \langle X, Y \rangle e_0 \\ \tilde{\nabla}_X e_0 &= X \end{aligned} \tag{2.05}$$

where $X, Y \in \Gamma(TM)$, e_0 is the trivial section $e_0 : p \to (0, 1) \in \tilde{T}_p M = T_p M \oplus \mathbb{R}$, $\langle\ ,\ \rangle$ is the Riemannian metric and ∇ is the Levi-Civita connection on TM.

∇ and $g = \langle\ ,\ \rangle$ can be extended to $\tilde{T}M$ in the obvious way as an orthogonal direct sum so that g is positive-definite, $|e_0| = 1$, $e_0 \perp TM$, $\nabla e_0 = 0$. We will denote this extension by the same symbols g or $\langle\ ,\ \rangle$ and ∇. We have $\nabla g = 0$, but $\tilde{\nabla} g \neq 0$.

On the other hand, we get a Lorentzian metric denoted by $\tilde{g}$ on $\tilde{T}M$ by extending g so that $|e_0| = -1$. We have then $\tilde{\nabla}\tilde{g} = 0$ but $\nabla \tilde{g} \neq 0$. We point out here the obvious fact that in case M is hyperbolic space, the equations (2.05) are the Gauss equations for the standard imbedding $H^n \subset \mathbb{R}^{1,n}$.

In the case (ii) of the spinor bundle $\mathbb{S}(M)$ the Cartan connection ω is given by:

$$\tilde{\nabla}_X \psi = \nabla_X \psi + \tfrac{1}{2} e_0 . X . \psi \tag{2.06}$$

where $\psi \in \Gamma(\mathbb{S})$, $X \in TM$ and the vectors e_0, $X \in \Gamma(\tilde{T}M)$ are acting on the spinor ψ via Clifford multiplication, defined by the Lorentzian metric $\tilde{g}$. The factor $\frac{1}{2}$ comes from the fact that Spin(n) is the double cover of $SO(n)$.

As remarked above, spinors associated to the extended group Spin$(1, n)$ can be thought of as just "ordinary" spinors $\mathbb{S}$ for the Euclidean group Spin(n) where vectors $X \in TM$ act via Clifford multiplication as usual and where we have an extra element e_0 acting on $\mathbb{S}$ with the following properties:

(i) $\langle e_0 . \psi_1, \psi_2 \rangle = \langle \psi_1, e_0 . \psi_2 \rangle$;

(ii) $e_0 . e_0 . \psi = +\psi$

and (iii) $e_0 . X . \psi = -X . e_0 . \psi$ for any $X \in TM$

The Dirac operators corresponding to the two connections are defined by:

$$\text{(i)}\ D\psi = \sum_{k=1}^{n} e_k . \nabla_k \psi \tag{2.08}$$

and $$\text{(ii)}\ \tilde{D}\psi = \sum_{k=1}^{n} e_k . \tilde{\nabla}_k \psi$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal base for TM and ∇_k means $\tilde{\nabla}_{e_k}$.

We call $\tilde{D}$ the *hyperbolic Dirac operator* in contrast to the "Euclidean" Dirac operator D. Using the formula (2.07) and the anti-commutativity of $e_0 .$ and $e_k .$ then implies the following important relation between the two Dirac operators:

$$\tilde{D}\psi = D\psi + \tfrac{1}{2} n e_0 . \psi \quad \text{i.e.} \quad \tilde{D} = D + \tfrac{1}{2} n e_0 . \tag{2.09}$$

This shows, in particular, that $\tilde{D}$ is a *formally self-adjoint* operator on the Hilbert space of L^2 spinors, since D and Clifford multiplication by $e_0.$ are both self-adjoint with respect to the positive definite metric induced by g.

In fact, since (M,g) is complete, it follows from Thm. 1.17 of [4] that D and $\tilde{D}$, which are both *generalized Dirac operators* in the sense of Sect. 1 of [4] are in fact, *essentially self-adjoint*, with a unique closed self-adjoint extension defined on the common domain of definition: $\mathrm{Dom}(D) = \mathrm{Dom}(\tilde{D}) \subset L^{1,2}(\mathbb{S})$, where $L^{1,2}(\mathbb{S})$ is defined to be the completion of the space $C_c^\infty(\mathbb{S})$ of all compactly supported smooth sections of $\mathbb{S}$ with respect to the Sobolev norm:

$$\|\psi\|_{1,2} = \left(\int\left(|\nabla\psi|^2 + \frac{n}{4}|\psi|^2\right)\right)^{1/2}.$$

Using now the super commutativity of the operators D and $e_0.$:

$$\{D, e_0\} = De_0. + e_0.D = 0 \tag{2.10}$$

which follows from the fact that $\nabla e_0 = 0$ and $e_k.e_0 = -e_0.e_k$, we find, upon squaring the equation (2.07), the following fundamental relation:

$$\tilde{D}^2 = D^2 + \frac{n^2}{4}. \tag{2.11}$$

This is an L^2-Weitzenböck formula and we refer to [7] and [8] for some general background on the usefulness of these L^2-formulas in dealing with almost symmetric spaces of non-compact type. The term $\frac{n^2}{4}$ comes from an algebraic Laplacian, i.e. a Casimir operator.

Our next step is to derive the "ordinary" C^0-Weitzenböck formulas for $\tilde{D}^2$ and D^2. We introduce the *rough Laplacians* defined by:

$$\nabla^*\nabla = -tr\nabla^2 = -\sum_{k=1}^{n} \nabla_k\nabla_k, \quad \tilde{\nabla}^*\tilde{\nabla} = -tr\tilde{\nabla}^2 = -\sum_{k=1}^{n} \tilde{\nabla}_k\tilde{\nabla}_k \tag{2.12}$$

where $\{e_k\}_{k=1,\dots,n}$ is an orthonormal base for TM.

$\nabla^*\nabla$ and $\tilde{\nabla}^*\tilde{\nabla}$ are both essentially self-adjoint, non-negative elliptic operators (see for example Prop. 2.4 of [4]) and the relation between them is given by:

$$\tilde{\nabla}^*\tilde{\nabla} = \nabla^*\nabla + \frac{n}{4}. \tag{2.13}$$

This follows from the computation (we sum over the index k):

$$\tilde{\nabla}^*\tilde{\nabla} = (-\nabla_k + \tfrac{1}{2}e_0.e_k.)(\nabla_k + \tfrac{1}{2}e_0.e_k.) = -\nabla_k\nabla_k + \tfrac{1}{4}e_0.e_k.e_0.e_k = \nabla^*\nabla + \frac{n}{4}$$

(2.13) implies that in fact $\tilde{\nabla}^*\tilde{\nabla}$ is *positive-definite*, so that there are no *hyperbolic Killing spinors* satisfying the equation:

$$\tilde{\nabla}_X\psi = \nabla_X\psi + \tfrac{1}{2}e_0.X.\psi = 0 \tag{2.14}$$

which are in L^2 on any complete manifold.

The famous formula of Lichnerowicz for the ordinary Dirac operator is:

$$D^2 = \nabla^* \nabla + \frac{R}{4} \tag{2.15}$$

where R is the ordinary scalar curvature of the Riemannian manifold (M^n, g).

The corresponding Weitzenböck formula for the hyperbolic Dirac operator associated to a Cartan connection is therefore:

$$\tilde{D}^2 = \tilde{\nabla}^* \tilde{\nabla} + \frac{\tilde{R}}{4} \tag{2.16}$$

where $\tilde{R} = R + n(n-1) = \sum_{k=1}^{n} \sum_{j=1}^{n} \langle \tilde{R}(e_j, e_k)e_k, e_j \rangle$ is the *hyperbolic scalar curvature.*

(2.16) follows from (2.11), (2.13) and (2.15). It can, of course, also be derived in as direct manner, exactly as for the Lichnerowicz formula, by replacing the Riemannian connection and curvature terms by their hyperbolic analogues. We also record here a few more formulas that we need for the next section relating the basic operators D, $\tilde{D}$, ∇, $\tilde{\nabla}$ and e_0. (The metric used below is the positive definite Riemannian metric $g = \langle\ ,\ \rangle$)

$$\text{(i)}\ |\tilde{\nabla}\psi|^2 = |\nabla\psi|^2 + \frac{n}{4}|\psi|^2 + \langle e_0.D\psi, \psi\rangle$$

$$\text{(ii)}\ \int |\tilde{\nabla}\psi|^2 = \int |\nabla\psi|^2 + \frac{n}{4}\int |\psi|^2 = \|\psi\|_{1,2}^2 \tag{2.17}$$

$$\text{(i)}\ |\tilde{D}\psi|^2 = |D\psi|^2 + \frac{n^2}{4}|\psi|^2 + n\langle e_0 \cdot D\psi, \psi\rangle$$

$$\text{(ii)}\ \int |\tilde{D}\psi|^2 = \int |D\psi|^2 + \frac{n^2}{4}\int |\psi|^2 \tag{2.18}$$

$$\text{(i)}\ \tilde{D}^2 = \nabla^*\nabla + \frac{n}{4} + \frac{\tilde{R}}{4} = \nabla^*\nabla + \frac{n^2}{4} + \frac{R}{4}$$

$$\text{(ii)}\ \int |\tilde{D}\psi|^2 = \int |\nabla\psi|^2 + \frac{n}{4}\int |\psi|^2 + \int \frac{\tilde{R}}{4}|\psi|^2 = \|\psi\|_{1,2}^2 + \int \frac{\tilde{R}}{4}|\psi|^2. \tag{2.19}$$

The last formula above is of particular importance since it shows that, under the assumption: $\tilde{R} \geq 0$, $\tilde{D}: L^{1,2}(\mathbb{S}) \xrightarrow{\sim} L^2(\mathbb{S})$ is an isomorphism, by Corollary 2.10 of [4]. As explained in [4], strictly speaking, the technical assumption that $\tilde{R}$ is uniformly bounded would be needed to assure that the domain of definition of $\tilde{D}$ is exactly $L^{1,2}$. In general, if we do not assume an upper bound for R this domain of definition could be a proper subspace of $L^{1,2}$ and can be defined as

$$\operatorname{Dom}(\tilde{D}) = \left\{ \psi \in L^{1,2}(\mathbb{S}) \,\middle|\, \int \frac{\tilde{R}}{4}|\psi|^2 < \infty \right\}. \tag{2.20}$$

Under the assumption $\tilde{R} \geq 0$ (2.19) would then still imply an isomorphism $\tilde{D}: \operatorname{Dom}(\tilde{D}) \xrightarrow{\sim} L^2$.

Since we would be working on a non-compact manifold we need the following integrated version of the Weitzenböck formula (2.16) involving a boundary term:

$$\int_M |\tilde{D}\psi|^2 + \int_{\partial M} < \tilde{\nabla}_v\psi + v.\tilde{D}\psi, \psi > = \int_M \left(|\tilde{\nabla}\psi|^2 + \frac{\tilde{R}}{4}|\psi|^2 \right) \qquad (2.21)$$

where (M, g) is any spin manifold with boundary ∂M and v is the exterior normal of $\partial M \subset M$.

This formula is proved, as usual, by computing the divergence of a one form and applying Stokes' theorem. The specific one form α we need here is defined by:

$$\alpha(X) = \langle \tilde{\nabla}_x\psi + X.\tilde{D}\psi, \psi \rangle = \langle \nabla_X\psi + X.\psi, \psi \rangle + \frac{n-1}{2}\langle X.e_0.\psi, \psi \rangle \qquad (2.222)$$

for $X \in TM$ and for any *fixed* spinor field $\psi \in \Gamma(\mathbb{S})$. The divergence of α is computed to be:

$$\begin{aligned} -\delta\alpha &= |\nabla\psi|^2 - |D\psi|^2 + \frac{R}{4}|\psi|^2 + (1-n)\langle e_0.D\psi, \psi \rangle \\ &= |\tilde{\nabla}\psi|^2 - |\tilde{D}\psi|^2 + \frac{\tilde{R}}{4}|\psi|^2 \end{aligned}$$

where we freely used the formulas computed above, in particular (2.17), (2.18) and (2.19).

3. The proof

As explained in the introduction, we will study an elliptic boundary value problem on M with given boundary values at ∞. We will solve for a spinor ψ satisfying the first order elliptic equation $\tilde{D}\psi = 0$ on the manifold M which is asymptotically parallel in the sense that it approaches, in a suitable Sobolev norm, a spinor which would be parallel with respect to the hyperbolic metric.

Since the Cartan connection $\bar{\omega}$ is flat for the hyperbolic space $\bar{M} = H^n$ the spinor bundle $\mathbb{S}(\bar{M})$ is trivialized by globally parallel sections which we denote generically by $\bar{\psi}$. These spinors can be regarded as the parallel spinors of the Minkowskian vector space $\mathbb{R}^{1,n}$ restricted to H^n (imbedded in the standard way as the set of vectors of length $= -1$ with positive first component). Since the flat connection of $\mathbb{R}^{1,n}$ does not leave the Riemannian metric of H^n invariant, parallel spinors do not have constant length with respect to a positive-definite metric. Therefore we need to calculate first the rate of growth of the Euclidean length of a parallel spinor. If X is any unit tangent vector, $\bar{\nabla}$ is the Levi-Civita connection and $\bar{\psi}$ is spinor, parallel with respect to the Cartan connection $\tilde{\bar{\nabla}}$ then:

$$\begin{aligned} X(|\bar{\psi}|^2) &= 2\langle \bar{\nabla}_X\bar{\psi}, \bar{\psi} \rangle && \text{since} \quad \bar{\nabla}\langle\ ,\ \rangle = 0 \\ &= -\langle e_0.X.\bar{\psi}, \bar{\psi} \rangle && \text{since} \quad \tilde{\bar{\nabla}}_X\bar{\psi} = \bar{\nabla}_X\bar{\psi} + \tfrac{1}{2}e_0.X.\bar{\psi} = 0 \\ &= -\langle X.\bar{\psi}, e_0.\bar{\psi} \rangle \end{aligned}$$

and therefore

$$|X(|\bar{\psi}|^2)| \leqq |X.\bar{\psi}|.|e_0.\bar{\psi}| \leqq |\bar{\psi}|^2 \quad \text{since} \quad |X| = 1.$$

Choosing X to be the unit vector field $\frac{\partial}{\partial r}$ in the radial direction we have the estimate

$$\left|\frac{\partial}{\partial r}|\bar{\psi}|^2\right| \leqq |\bar{\psi}|^2 \tag{3.01}$$

for a spinor $\bar{\psi}$ on H^n which is parallel with respect to the flat Cartan connection.

This shows that $|\bar{\psi}|^2$ grows at most exponentially with r and if we normalize $\bar{\psi}$ to be of unit length at the origin, then on the sphere of radius r, we have

$$|\bar{\psi}(r)|^2 \leqq e^r. \tag{3.02}$$

Using our hyperbolic coordinates at ∞, $\phi: M - B \xrightarrow{\sim} H^n(r_0, \infty)$ we now transplant these parallel spinors to each end of an asymptotically hyperbolic manifold M.

We first identify the end $M_\infty = M - B$ with $H^n(r_0, \infty)$ via ϕ, and we will simply denote the induced hyperbolic metric $\phi^*\bar{g}$ on M_∞ by $\bar{g}$ and call it the background metric. We then write the metric g as:

$$g(u,v) = \bar{g}(A^{-1}u, A^{-1}v); \quad \bar{g}(u,v) = g(Au, Av) \tag{3.03}$$

where the gauge transformation $A: TM_\infty \to TM_\infty$ is assumed to be symmetric with respect to g. (A^{-1} is then symmetric w.r.t. $\bar{g}$).

If $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$, then the gauge-transformed connection $\hat{\nabla} = (A^{-1})^*\bar{\nabla}$ as defined below would leave the metric $g = (A^{-1})^*g$ invariant.

$$\hat{\nabla}_X Y = A(\bar{\nabla}_X(A^{-1}Y)). \tag{3.04}$$

However $\hat{\nabla}$ would have non-zero torsion $\hat{T}$, computed to be:

$$\hat{T}(X,Y) = A(d^{\bar{\nabla}}A^{-1}(X,Y)) \tag{3.05}$$

and the difference $\hat{\nabla}_X - \nabla_X = B_X$ to the Levi-Civita connection ∇ of g is given by:

$$2\langle B_X Y, Z\rangle = \langle \hat{T}(X,Y), Z\rangle - \langle \hat{T}(Y,Z), X\rangle + \langle \hat{T}(Z,X), Y\rangle. \tag{3.5)}$$

For technical reasons—since our assumption AH2 is in terms of ∇A instead of $\bar{\nabla}A^{-1}$ – we prefer a formula for B in terms of ∇A. A little computation gives:

$$-2\langle B_X AY, AZ\rangle = \langle d^\nabla A(X,Y), AZ\rangle - \langle d^\nabla A(Y,Z), AX\rangle + \langle d^\nabla A(Z,X), AY\rangle. \tag{3.06}$$

On a spinor ψ, defined with respect to the Spin structure of the g-metric: $\mathrm{Spin}(M_\infty, g) \to M_\infty$, the connections $\hat{\nabla}$ and ∇ are related by:

$$\hat{\nabla}_i\psi = \nabla_i\psi + \tfrac{1}{2}B^k_{ij}e_j.e_k.\psi \tag{3.07}$$

where we sum over repeated indices and $\{e^k\}$ is an orthonormal base for g.

The curvature of $\hat{\nabla}$ is given by:

$$\hat{R}(X,Y) = A \circ \bar{R}(X,Y) \circ A^{-1} \tag{3.08}$$

where $\bar{R}$ is the curvature of hyperbolic space defined in (2.04).

The gauge transformation A is a bundle isomorphism $A: SO(M_\infty, \bar{g}) \xrightarrow{\sim} SO(M_\infty, g)$ mapping a $\bar{g}$-orthonormal frame $\{\bar{e}_k\}_{k=1,\dots,n}$ over M_∞ to the g-orthonormal frame

$\{A\bar{e}_k\}_{k=1,\dots,n}$. This bundle map can be lifted uniquely (up to a choice of base points) to an isomorphism of the corresponding spin bundles, still to be denoted by $A:\mathrm{Spin}(M_\infty,\bar{g})\tilde{\to}\mathrm{Spin}(M_\infty,g)$. For the uniqueness of the lift we use the important fact that the end M_∞ is *simply connected*, since the dimension of M is >2.

A spinor ψ with respect to a spin structure $\mathrm{Spin}(M,g)$ can be regarded as an $\mathrm{Spin}(n)$-equivariant map $\psi:\mathrm{Spin}(M,g)\to\mathbb{C}^s$, where $\mathbb{C}^s$ is the spin representation space, and therefore, by composing with the gauge transformation A we can transform a spinor $\bar{\psi}$ defined by the metric $\bar{g}$ on M_∞ into a spinor $\hat{\psi}=A^*\bar{\psi}=\bar{\psi}\circ A^{-1}$ with respect to the metric $g=A^*\bar{g}$, and if ψ is parallel w.r.t. the flat hyperbolic Cartan connection $\bar{\omega}$ associated to the $\bar{g}$-spin structure $\mathrm{Spin}(M_\infty,\bar{g})$, then $\hat{\psi}$ is parallel with respect to the gauge transformed Cartan connection $\hat{\omega}=A^*(\bar{\omega})$ for $\mathrm{Spin}(M_\infty,g)$, which is of course also *flat*. The covariant derivative of this connection on the extended vector bundle $\tilde{T}M=TM\oplus 1$ is given by:

$$\begin{aligned}\tilde{\hat{\nabla}}_X Y &= \hat{\nabla}_X Y + g(AX,Y).e_0\\ \tilde{\hat{\nabla}}_X e_0 &= AX\end{aligned} \tag{3.09}$$

and the formula for the spinor bundle $\mathbb{S}(M)$ is:

$$\tilde{\hat{\nabla}}_X\hat{\psi}=\hat{\nabla}_X\hat{\psi}+\tfrac{1}{2}e_0.AX.\hat{\psi} \tag{3.10}$$

where the Clifford multiplication is defined by the Lorentzian metric induced by g on $\tilde{T}M$.

The following computation for the curvature of $\tilde{\hat{\nabla}}$ is, strictly speaking, not necessary, since $\hat{\omega}$ is a gauge transform of the flat Cartan connection $\bar{\omega}$. However, it is a good check for the correctness of our covariant derivative formulas:

$$\begin{aligned}&\tilde{\hat{R}}(X,Y)Z\\ &=\hat{R}(X,Y)Z+g(AY,Z)AX-g(AX,Z)AY\\ &\quad+g(\hat{\nabla}_X(AY)-\hat{\nabla}_Y(AX)-A([X,Y]),Z)\\ &=A(\bar{R}(X,Y)A^{-1}Z)+\bar{g}(Y,A^{-1}Z)AX-\bar{g}(X,A^{-1}Z)AY\\ &\quad+g(A(\bar{\nabla}_X Y-\bar{\nabla}_Y X-[X,Y]),Z)\\ &=A(\bar{R}(X,Y)A^{-1}Z)+A(\bar{g}(Y,A^{-1}Z)X-\bar{g}(X,A^{-1}Z)Y)\\ &=0 \qquad \text{by}(2.04).\end{aligned}$$

A parallel spinor $\bar{\psi}$ on hyperbolic space therefore defines via the asymptotic coordinates ϕ and the induced gauge transformation A, a spinor $\hat{\psi}=\bar{\psi}\circ A$ which is parallel w.r.t. the connection $\hat{\omega}=A^*(\bar{\omega})$ on the end M_∞. In other words, by choosing a basis of $\bar{\omega}$-parallel spinors for H^n, we obtain a trivialization of the spin bundle $\mathrm{Spin}(M,g)$ restricted to the end M_∞ by $\hat{\omega}$-parallel spinors.

Moreover, if we make the assumption that the Cartan connection ω induced by the Levi-Civita connection ∇ of g is close to the *flat* Cartan connection $\hat{\omega}=A^*(\bar{\omega})$ defined by the asymptotic coordinates ϕ near ∞, then this trivialization is almost parallel w.r.t. ω. In this sense, as we mentioned in Sect. 1, the natural condition one should impose at infinity is that there is an almost flat Cartan connection together with an almost parallel framing (section of the principal bundle) at each end.

We now extend these transplanted spinors $\hat{\psi}$ to all of M by multiplying with a suitable cut-off function χ supported near ∞. To be specific we choose a radial function χ with:

$$\begin{aligned} \chi(r) &\equiv 1 \quad \text{for} \quad r \geqq 2\rho > 2r_0; \\ \chi(r) &\equiv 0 \quad \text{for} \quad r \leqq \rho \end{aligned} \tag{3.11}$$

and $0 \leqq \chi' \leqq \rho^{-1}$ with ρ sufficiently large.

If we now set

$$\tilde{\psi} = \chi.\hat{\psi} = \chi.\bar{\psi} \circ A \tag{3.12}$$

then $\tilde{\psi}$ vanishes on a compact set and $\tilde{\nabla}\tilde{\psi} \to 0$ at ∞, since $\tilde{\hat{\nabla}}\hat{\psi} = 0$ for large values of r and $\omega \to \hat{\omega}$ as $r \to \infty$. More precisely we have:

$$\begin{aligned} \hat{\omega}_X - \omega_X &= \hat{\nabla}_X - \nabla_X + \tfrac{1}{2}e_0.(AX - X). \\ &= B_X + \tfrac{1}{2}e_0.(AX - X), \end{aligned}$$

where we are using the spin structure of g for the Clifford multiplication.

Therefore since $\tilde{\nabla}_v\tilde{\psi} = (\nabla_v - \tilde{\hat{\nabla}}_v).\hat{\psi}$ and $\hat{\psi} = \bar{\psi} \circ A$ for larger r, we have, using (3.02) and our assumption AH1:

$$\begin{aligned} |\tilde{\nabla}_v\tilde{\psi}| &\leqq C(n).(|B_v| + |A - id|)|\bar{\psi}| \\ &\leqq C(n).|B|.e^{r/2} \end{aligned}$$

for large r, where $C(n)$ denotes from now on any generic constant depending only on the constant C of our assumption AH1 and the dimension n.

(3.06) together with the assumption AH1, also gives us the estimate:

$$|B| \leqq C(n).|\nabla A|. \tag{3.15}$$

The second condition AH2 for asymptotic hyperbolicity: $e^r.(A - id) \in L^{1,2}$ now implies, first of all, that $\tilde{\nabla}\tilde{\psi}$ and hence also $\tilde{D}\tilde{\psi} \in L^2(\mathbb{S}, g)$. It also implies that if $S(r)$ denotes the sphere $\phi^{-1}(S(r))$ for large r, then

$$\lim_{r \to \infty} e^r \int_{S(r)} |\nabla A| = 0. \tag{3.16}$$

We now solve for a spinor $\varepsilon \in \operatorname{Dom}(\tilde{D}) \subset L^{1,2}$ satisfying:

$$\tilde{D}\varepsilon = -\tilde{D}\tilde{\psi}. \tag{3.17}$$

Then if we set $\psi = \tilde{\psi} + \varepsilon$, ψ would satisfy $\tilde{D}\psi = 0$ and $\psi \to \tilde{\psi}$ at ∞. We are therefore not solving for an L^2 harmonic spinor ψ. In fact there are no L^2 harmonic spinors on M because of our fundamental equation: $\tilde{D}^2 = D^2 + \dfrac{n^2}{4}$.

The basic fact about the Cartan Dirac operator $\tilde{D}$ that we are using here, in order to solve (3.17) is the following analytical

Lemma

$$\tilde{D}: \operatorname{Dom}(\tilde{D}) \subset L_1^2(M) \xrightarrow{\sim} L^2(M) \quad \textit{is an isomorphism.} \tag{3.18}$$

This Lemma is a consequence of our fundamental L^2-Weitzenböck formula (2.11) together with some standard elliptic estimates, since on a complete manifold (see Thm. 1.23 in [4]) $\tilde{D}^2\psi = 0$ iff $\tilde{D}\psi = 0$ for $\psi \in L^2$. We refer to Sect. 1, Sect. 2 of [4] for basic facts about generalized Dirac operators and vanishing theorems. In particular, applying Thm. 2.8, Cor. 2.9 and Thm. 2.11 in [4] to our Weitzenböck formula (2.19) gives a proof of the Lemma. As remarked earlier, Dom$(\tilde{D})$ would coincide with $L^{1,2}$ if we assume a uniform bound on the scalar curvature since then multiplication with the term $\frac{\tilde{R}}{4}$ in (2.19) is a bounded operator on L^2.

Substituting now the harmonic spinor ψ into the integrated Weitzenböck formula (2.20)

$$\int_M |\tilde{D}\psi|^2 + \int_{\partial M} \langle \tilde{\nabla}_v\psi + v.\tilde{D}\psi, \psi\rangle = \int_M \left(|\tilde{\nabla}\psi|^2 + \frac{\tilde{R}}{4}|\psi|^2 \right)$$

applied to the manifold $M_r = M - \phi^{-1}(H(r, \infty))$ with $r \to \infty$ would then give us a boundary term which will be shown below to go to zero as the boundary sphere approaches ∞. The curvature assumption $\tilde{R} \geqq 0$ would then imply that ψ is in fact everywhere parallel: $\tilde{\nabla}\psi = 0$.

The operator $\tilde{\nabla}_v + v.\tilde{D}$ appearing in the integrand of the boundary term of the Weitzenböck formula (2.20) is self-adjoint with respect to the L^2-inner product when restricted to the boundary, since all the other terms appearing in the formula have this property and hence:

$$\oint_r \langle \tilde{\nabla}_v\psi + v.\tilde{D}\psi, \psi\rangle$$
$$= \oint_r (\langle \tilde{\nabla}_v\bar{\psi} + v.\tilde{D}\bar{\psi}, \bar{\psi}\rangle + 2\langle \tilde{\nabla}_v\bar{\psi} + v.\tilde{D}\bar{\psi}, \varepsilon\rangle + \langle \tilde{\nabla}_v\varepsilon + v.\tilde{D}\varepsilon, \varepsilon\rangle)$$

where $\oint_r$ denotes the boundary integral on the sphere $S_r = \partial M_r = \phi^{-1}(S(r))$ for large r.

This boundary integral can be estimated as follows:

(i) Since $\varepsilon \in L_1^2$, we have $\tilde{\nabla}_v\varepsilon$, $\tilde{D}\varepsilon \in L^2$ and therefore $\oint_r \langle \tilde{\nabla}_v\varepsilon + v.\tilde{D}\varepsilon, \varepsilon\rangle$ tends to zero as $r \to \infty$.

(ii) On a large sphere $S_r = \partial M_r$, $\chi = 1$, $\chi' = 0$, $\bar{\psi} = \hat{\psi} = A^*\psi$ and since $\hat{\psi}$ is parallel w.r.t. the connection $\hat{\omega}$ we have by (3.14)

$$|\langle \tilde{\nabla}_v\bar{\psi}, \bar{\psi}\rangle| \leqq C(n).|\nabla A|e^r \quad \text{and} \quad |\tilde{\nabla}_v\bar{\psi}| \leqq C(n).|\nabla A|e^{r/2}.$$

Since $\tilde{D}\bar{\psi} = e_k.\tilde{\nabla}e_k\bar{\psi}$ we also have:

$$|\langle v.\tilde{D}\bar{\psi}, \bar{\psi}\rangle| \leqq C(n).|\nabla A|e^r \quad \text{and} \quad |v.\tilde{D}\bar{\psi}| \leqq C(n).|\nabla A|e^{r/2}.$$

Substituting these estimates and using the fact that $\varepsilon \in L^2$, we find:

$$\oint_r |\langle \tilde{\nabla}_v\bar{\psi} + v.\tilde{D}\bar{\psi}, \bar{\psi}\rangle + 2\langle \tilde{\nabla}_v\bar{\psi} + v.\tilde{D}\bar{\psi}, \varepsilon\rangle| \leqq C(n).e^r.\oint_r |\nabla A| \to 0 \quad \text{as} \quad r \to \infty \text{ by (3.16).}$$

We therefore have extended any given asymptotically parallel boundary value $\bar{\psi}$ to a global parallel spinor ψ defined on all of M. Choosing a basis of parallel spinors at ∞ induced by the asymptotically hyperbolic structure and extending them harmonically, we obtain a global basis of parallel spinors trivializing the bundle $\mathbb{S}(M)$, which implies that the Cartan connection ω defining $\tilde{V}$ is flat, thus proving that (M, g) is locally and hence globally isometric to hyperbolic space $(H^n, \bar{g})$. Global isometry follows from local isometry because M is diffeomorphic to H^n at infinity and $n > 2$.

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