J. reine angew. Math. **415** (1991), 167–174

Journal für die reine und angewandte Mathematik

Vanishing theorems for the basic cohomology of Riemannian foliations

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1. Introduction

Since the Laplace operator plays such an important role in the study of the cohomology of Riemannian manifolds, it is natural to try to find a generalization of this operator to deal with the basic cohomology of Riemannian foliations. Several proposals for such an operator can be found in the literature, which all have some advantages and disadvantages. Reinhart's definition in [R2] is a straightforward generalization of the Laplace-Beltrami operator. However it is not presented as a differential operator on a differentiable manifold, and it is therefore not clear if the usual theorems apply. The definition by El Kacimi and Hector in [EH] is well suited to prove finite-dimensionality results. However, because the definition does not allow for an explicit local formula, the Bochner technique and the Weitzenböck formula do not apply. The operator used by Kamber and Tondeur in [KT1], [KT2] is well adapted to the usual set-up of global analysis as well as the Bochner technique. However, the assumption of a basic mean curvature form, as well as the need for assumptions on this form restrict the scope of its application.

In this paper we introduce an elliptic operator on the DeRham complex of the total space of a Riemannian foliation, such that its restriction to basic differential forms coincides with the Laplace operator introduced by Reinhart [R2]. The advantage of this set-up is that on the one hand the standard theory of partial differential equations applies, and on the other hand the local computations on basic forms in distinguished coordinate neighbourhoods are identical with the computations in the usual Riemannian case. In particular the vanishing theorems based on the positivity of certain curvature expressions can be proved essentially as in the special case of Riemannian manifolds.

The construction of the elliptic operator mentioned above is based on a connection $\tilde{\nabla}$ on M which is metric, but has non-vanishing torsion $\tilde{T}$. The connection $\tilde{\nabla}$ turns the leaves of $\mathscr{F}$ into totally geodesic submanifolds. The resulting differential operator $\tilde{\Delta}$ is not

This work was partially supported by N.S.E.R.C. Grant A7873 of Canada, and N.S.F. Grant DMS-8601282 of the USA. The hospitality of the University of Basel is gratefully acknowledged.

self-adjoint. As a consequence the space of harmonic forms may be larger than the corresponding cohomology. However, using the heat equation, we prove that if the curvature term in the Weitzenböck formula is strictly positive, then any closed basic form is exact. As illustrations, we present the case of positive transversal Ricci curvature due to Hebda [H], as well as the case of a positive transversal curvature operator which generalizes Gallot-Meyer [GM].

The authors would like to thank J. A. Alvarez López and the referee for helpful comments.

2. Definitions and results

Let $\mathscr{F}$ be a smooth foliation on a closed oriented manifold M. Let L be its tangent bundle and $Q = TM/L$ its normal bundle. Let g_M be a Riemannian metric on M, which decomposes TM as an orthogonal direct sum $L \oplus L^\perp$, and canonically identifies $L^\perp$ with Q. The metric g_M throughout this paper is assumed to be bundle-like [R1]. This means that the induced metric g_Q on Q satisfies the (infinitesimal) holonomy invariance condition $\theta_X g_Q = 0$ for all $X \in \Gamma L$, where θ_X denotes the Lie derivative with respect to X.

For a distinguished chart $U \subset M$ the leaves of $\mathscr{F}$ in U are given as the fibers of the Riemannian submersion $f\colon U \to B \subset N$ onto an open subset B of a model Riemannian manifold N. Let $p = \dim L$, $q = \dim Q$ and $n = p + q = \dim M$. Then $\dim N = q$. For overlapping charts $U_\alpha \cap U_\beta$ the corresponding local transition functions $\gamma_{\alpha\beta} = f_\beta \circ f_\alpha^{-1}$ on N are isometries.

Let $\nabla = \nabla^M$ be the Levi-Civita connection on (M, g_M). We define as in [AT], [R3] a connection on M by

$$(2.1) \qquad \tilde{\nabla}_A B = \pi_L \nabla_A \pi_L B + \pi_Q \nabla_A \pi_Q B \quad \text{for} \quad A, B \in \Gamma TM,$$

where π_L and π_Q denote the orthogonal projections of $TM \cong L \oplus L^\perp$ onto L and $Q \cong L^\perp$, respectively. For $X, Y \in \Gamma L$ we have $\tilde{\nabla}_X Y = \pi_L \nabla_X Y = \nabla_X Y - \alpha(X, Y)$, where α is the second fundamental form of the leaves. Thus the leaves of $\mathscr{F}$ are totally geodesic with respect to the connection $\tilde{\nabla}$. For $V, W \in \Gamma Q$, we have $\tilde{\nabla}_V W = \nabla_V W - \beta(V, W)$, where

$$\beta(V, W) = \frac{1}{2} \cdot \pi_L[V, W]$$

is the O'Neill tensor of the local Riemannian submersion.

$\tilde{\nabla}$ is a metric connection, but has non-zero torsion $\tilde{T}$ given by

$$\tilde{T}(V, W) = -\pi_L[V, W],$$

for $V, W \in \Gamma Q$ and zero otherwise.

The main purpose of introducing the connection $\tilde{\nabla}$ is the following property.

2.2. Proposition. *Let $V, W \in \Gamma Q$ be projectable vector fields on a distinguished chart U, with $\bar{V} = f_* V$, $\bar{W} = f_* W$ vector fields on the local base space $f(U) = B \subset N$. Then $\tilde{\nabla}_V W$ is projectable and*

$$f_* \tilde{\nabla}_V W = \nabla^N_{\bar{V}} \bar{W}. \tag{2.3}$$

This follows from (2.1) and Lemma 1, no. 3 of [ON].

Using the connection $\tilde{\nabla}$, we define a codifferential $\tilde{\delta}$ acting on forms by the formula:

$$\begin{aligned} \tilde{\delta}\omega(A_2, \ldots, A_r) = &- \sum_{k=1}^{n} E_k(\omega(E_k, A_2, \ldots, A_r)) \\ &+ \sum_{k=1}^{n} \omega(\tilde{\nabla}_{E_k} E_k, A_2, \ldots, A_r) \\ &+ \sum_{k=1}^{n} \sum_{i=2}^{r} \omega(E_k, A_2, \ldots, \tilde{\nabla}_{E_k} A_i, \ldots, A_r) \end{aligned} \tag{2.4}$$

where $\omega \in \Omega^r(M)$; $A_2, \ldots, A_r \in \Gamma TM$ and $\{E_k\}_{k=1,\ldots,n}$ is a local orthonormal frame field of TM.

Obviously, $\tilde{\delta}\omega$ is independent of the choice of the orthonormal frame $\{E_k\}_{k=1,\ldots,n}$, because it is equal to the trace of $-\tilde{\nabla}\omega$ over the appropriate entries. The operator $\tilde{\delta}$ is not the formal adjoint δ of the exterior derivative d, but differs from it by a zero-th order term determined by the tensors α and β.

Nevertheless we can define the second order differential operator

$$\tilde{\Delta} = d\tilde{\delta} + \tilde{\delta} d, \tag{2.5}$$

which, although not self-adjoint, is still an elliptic operator with a self-adjoint principal symbol.

Next we discuss the effect of these operators on basic forms. These are the forms ω satisfying $i_X \omega = 0$ and $\theta_X \omega = 0$ for all $X \in \Gamma L$. On a distinguished chart U such a form ω can be canonically identified with a form $\bar{\omega}$ on the quotient $B = f(U)$ via the pull-back f^*. Let $\Omega_B(\mathscr{F}) \subset \Omega(M)$ be the subspace of all basic forms.

2.6. Proposition. (i) *$\tilde{\delta}$ restricted to $\Omega_B(\mathscr{F})$ maps $\Omega_B(\mathscr{F})$ into itself, and coincides on a distinguished chart U with the formal usual codifferential δ defined on the Riemannian quotient $B = f(U)$.*

(ii) *The operator $\tilde{\Delta}$ restricted to $\Omega_B(\mathscr{F})$ maps $\Omega_B(\mathscr{F})$ to itself, and coincides on a distinguished chart U with the operator $\Delta = d\delta + \delta d$ defined on the Riemannian quotient $B = f(U)$.*

The proof is given in Section 3.

Our results can now be stated as follows. Throughout let $\mathscr{F}$ be a Riemannian foliation on a closed oriented Riemannian manifold with a bundle-like metric.

Theorem A. *The heat equation* $\dot{\omega} = -\tilde{\Delta}\omega$ *with initial condition* $\lim_{t \downarrow 0} \omega_t = \omega_0 \in \Omega(M)$ *has a solution for* $0 \leqq t < \infty$. *If* $\omega_0 \in \Omega_B(\mathscr{F})$, *then* $\omega_t \in \Omega_B(\mathscr{F})$ *for* $0 \leqq t < \infty$.

Theorem B. *Let* $\omega_0 \in \Omega_B(\mathscr{F})$ *satisfy* $d\omega_0 = 0$. *Then* $d\omega_t = 0$ *for* $0 \leqq t < \infty$ *and the basic cohomology class of* ω_t *is the same for all* $0 \leqq t < \infty$.

The basic cohomlogy $H_B(\mathscr{F})$ of $\mathscr{F}$ is the cohomology of the complex $(\Omega_B(\mathscr{F}), d)$. Transversal curvature properties below refer to curvature properties of the local Riemannian quotients. Applying the Bochner technique yields the following vanishing theorems.

Corollary C. *Assume the transversal Ricci curvature of* $\mathscr{F}$ *to be strictly positive. Then* $H_B^1(\mathscr{F}) = 0$.

This result is due to Hebda [H], proved by an entirely different method. The next application involves the transversal curvature operator $\mathscr{R}: \Lambda^2 Q \to \Lambda^2 Q$.

Corollary D. *Assume the transversal curvature operator to be positive. Then* $H_B^r(\mathscr{F}) = 0$ *for* $0 < r < q = \operatorname{codim} \mathscr{F}$.

3. Proof of Proposition 2.6

In the sequel we use a local orthonormal frame field $\{E_k\}_{k=1,\ldots,n}$ of TM such that $E_1, \ldots, E_p \in \Gamma L$; and $E_{p+1}, \ldots, E_n \in \Gamma L^\perp$ are projectable vector fields. Applying formula (2.4) to $\omega \in \Omega_B^r(\mathscr{F})$ yields then

$$\begin{aligned}\tilde{\delta}\omega(A_2, \ldots, A_r) = &- \sum_{k=p+1}^{n} E_k(\omega(E_k, A_2, \ldots, A_r)) \\ &+ \sum_{k=p+1}^{n} \omega(\tilde{\nabla}_{E_k} E_k, A_2, \ldots, A_r) \\ &+ \sum_{k=p+1}^{n} \sum_{i=2}^{r} \omega(E_k, \ldots, \tilde{\nabla}_{E_k} A_i, \ldots, A_r)\end{aligned}$$

since $\tilde{\nabla}_X Y \in \Gamma L$ for $X, Y \in \Gamma L$. The formula above shows that $\tilde{\delta}\omega$ is also basic for $\omega \in \Omega_B(\mathscr{F})$, because $\tilde{\nabla}_{E_k} A \in \Gamma L$ if $A \in \Gamma L$. Moreover, by Proposition 2.2, $\tilde{\nabla}_V W$ coincides with the Levi-Civita connection ∇^N of the local Riemannian quotient. This completes the proof of Proposition 2.6. □

4. Proof of Theorems A and B

To study the operator $\tilde{\Delta}$ on M we first observe the following. Let δ denote the usual codifferential (formal adjoint of d) on M. Then (2.4) together with the usual similar formula for $\delta\omega$ yields

$$(4.1)\quad (\tilde{\delta}\omega - \delta\omega)(A_2, \ldots, A_r)$$
$$= \sum_{k=1}^{n} \left\{ \omega\big(\gamma(E_k, E_k), A_2, \ldots, A_r\big) + \sum_{i=2}^{r} \omega\big(E_k, A_2, \ldots, \gamma(E_k, A_i), \ldots, A_r\big) \right\}$$

where

$$(4.2)\qquad \gamma(A, B) = \tilde{\nabla}_A B - \nabla_A B$$

is the difference tensor of the two connections.

Comparing $\tilde{\Delta}$ with the usual Hodge-DeRham Laplacian $\Delta^M = d\delta + \delta d$, we have

$$(4.3)\qquad \tilde{\Delta} - \Delta^M = d(\tilde{\delta} - \delta) + (\tilde{\delta} - \delta)d$$

which is a differential operator of order one. It follows that $\tilde{\Delta}$ is an elliptic operator with self-adjoint symbol. Thus the heat equation

$$\dot{\omega} = -\tilde{\Delta}\omega \quad \text{with} \quad \lim_{t \to 0^+} \omega_t = \omega_0 \in \Omega(M)$$

has a smooth solution $\omega_t = e^{-t\Delta}\omega_0$ for all $t > 0$ and for any initial value ω_0. The basic ingredient is the following result.

4.5. Lemma (Gårding inequality).

$$\langle \tilde{\Delta}\omega, \omega \rangle \geqq \frac{1}{2} \|\nabla\omega\|^2 - c \cdot \|\omega\|^2 \quad \textit{for some constant } c.$$

The proof of the Gårding inequality is standard and is omitted.

By Proposition 2.6, the modified Laplacian $\tilde{\Delta}$ leaves $\Omega_B(\mathcal{F})$ invariant. Since the Gårding inequality holds a fortiori for basic forms, the operator $e^{-t\Delta}$ is a well defined smoothing operator on $L_2(\Omega(M))$ and $L_2(\Omega_B(\mathcal{F}))$ for all $t > 0$. This is because for $\mathrm{Re}(\lambda)$ large enough, for instance $\mathrm{Re}(\lambda) \geqq c + 2$, the operator $-(\tilde{\Delta} + \lambda I)$ is dissipative (see [P], p. 210), and hence generates a semigroup by a theorem of Lumer and Phillips (see [P], p. 14 and 15). The same is then true for the bounded perturbation $-\tilde{\Delta}$ of $-(\tilde{\Delta} + \lambda I)$ (see [P], p. 81, Cor. 2.2). Uniqueness implies that the semigroup generated by $-\tilde{\Delta}$ on $L_2(\Omega(M))$ restricts on $L_2(\Omega_B(\mathcal{F}))$ to the semigroup generated by $-\tilde{\Delta}/\Omega_B(\mathcal{F})$. This completes the proof of Theorem A.

To prove Theorem B, it suffices to observe that $d\tilde{\Delta} = \tilde{\Delta}d$. This implies $de^{-t\Delta}\omega_0 = e^{-t\Delta}d\omega_0$ for $\omega_0 \in \Omega_B(\mathcal{F})$, and the desired conclusion follows.

Remark. We wish to thank the referee for pointing out to us that the above technique suffices to prove the finite dimensionality of the basic cohomology (first established in [EHS]). Indeed, the heat operator (say for time t) is a compact operator and acts as the identity on the cohomology; the result follows.

5. Proof of Corollaries C and D

For a distinguished chart U, $f(U) = B \subset N$ the Weitzenböck formula for $\Delta\omega$, $\omega \in \Omega(B)$ (representing a basic form) reads

$$\Delta\omega = -\operatorname{trace}(\nabla^N)^2\omega + S_{R^N}(\omega).$$

For $r = 1$ the operator S_{R^N} is well known to be given as follows.

5.1. Lemma (Bochner). *Let* $\omega \in \Omega_B^1(\mathscr{F})$. *Then*

$$S_{R^N}(\omega) = \omega \circ \operatorname{Ric},$$

where Ric : $TB \to TB$ *is the Ricci curvature operator.*

For arbitrary $0 < r < q$ the self-adjoint operator S_{R^N} can be described as follows. Let $\bar{x} = f(x)$, $x \in U$. The symmetric curvature operator $R^N : \Lambda^2 T_{\bar{x}} B \to \Lambda^2 T_{\bar{x}} B$ has eigenvalues $\varrho_1, \ldots, \varrho_m \left(m = \frac{1}{2}q(q-1)\right)$ and corresponding eigenforms $A_1, \ldots, A_m$. These can be thought of as skew symmetric endomorphisms of $T_{\bar{x}} B$, and hence they operate naturally (and skew symmetrically) on r-forms $\Lambda^r T_{\bar{x}}^* B$. The action of A on any tensor τ at $\bar{x}$ is denoted $A \cdot \tau$.

5.2. Lemma (Gallot-Meyer).

$$S_{R^N}(\omega) = -\sum_{k=1}^{m} \varrho_k A_k \cdot A_k \cdot \omega.$$

Proof. See [GM].

5.3. Corollary. *If the curvature operator* R^N *is positive, then* S_{R^N} *on* $\Lambda^r T^* B$ *is positive definite for* $0 < r < q = \dim B$.

Proof. If $\varrho_1 > 0$ is the smallest eigenvalue, then

$$\begin{aligned} g_Q(S_{R^N}(\omega), \omega) &= -g_Q\left(\sum_{k=1}^{m} \varrho_k A_k \cdot A_k \cdot \omega, \omega\right) \\ &= \sum_{k=1}^{m} \varrho_k g_Q(A_k \cdot \omega, A_k \cdot \omega) \geqq \varrho_1 \sum_{k=1}^{m} |A_k \cdot \omega|^2 . \end{aligned}$$

The sum $\sum_{k=1}^{m} |A_k \cdot \omega|^2$ is nonzero for nonzero ω, because the representation of the Lie algebra $\mathfrak{o}(m)$ on r-forms is irreducible for $r \neq 0, q$. In fact, $\sum_{k=1}^{m} |A_k \cdot \omega|^2 = r(q-r)|\omega|^2$ ([GM], Cor. 2.6). Hence

$$F(\omega) \equiv g_Q(S_{R^N}(\omega), \omega) \geqq \varrho \cdot |\omega|^2 \tag{5.4}$$

with $\varrho = r(q-r) \cdot \varrho_1$. □

In the case of 1-forms, we have by Bochner's formula (Lemma 5.1) the same inequality (5.4), except that now ϱ is the minimum eigenvalue of the Ricci curvature.

We consider now the solution $\omega_t = e^{-t\Delta}\omega_0$ of the heat equation $\dot{\omega} = -\tilde{\Delta}\omega$ with initial condition $\lim_{t \downarrow 0} \omega_t = \omega_0$. By Theorems A and B, for a closed basic form ω_0, the forms ω_t are also closed basic forms for $t \geqq 0$. We compute the pointwise derivative

$$\frac{1}{2}\frac{\partial}{\partial t}|\omega_t|^2 = -g_Q(\tilde{\Delta}\omega_t, \omega_t).$$

By Proposition 2.6 and the pointwise Weitzenböck formula in the local model we have then

$$g_Q(\tilde{\Delta}\omega_t, \omega_t) = g_Q(\Delta\omega_t, \omega_t) = \frac{1}{2}\Delta|\omega_t|^2 + |\nabla\omega_t|^2 + F(\omega_t).$$

Note that $|\omega_t|^2$ for basic ω_t is a basic function. It follows that

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right)|\omega_t|^2 &= -|\nabla\omega_t|^2 - F(\omega_t)\\ &\leqq -\varrho|\omega_t|^2 \quad \text{with} \quad \varrho > 0.\end{aligned}$$

At a point $x_0 \in M$ where $|\omega_t|^2$ attains its maximum value, the standard DeRham-Hodge Laplacian Δ^M of M acting on the basic function $f = |\omega_t|^2$ coincides with the Laplacian Δ of the local model acting on f. In fact Δf can by Proposition 2.6 be evaluated as $\tilde{\Delta}f = \tilde{\delta}df$. It differs from $\Delta^M f = \delta df$ by the expression given by the right-hand side of (4.1) applied to $\omega = df$, which vanishes at the critical point x_0 of $f = |\omega_t|^2$.

The maximum principle now implies that $e^{\varrho t} \cdot \max|\omega_t|^2$ is a nonincreasing function of t, hence

$$\max|\omega_t|^2 \leqq e^{-\varrho t} \cdot \max|\omega_0|^2 \quad \text{for all} \quad t \geqq 0.$$

This proves that $\lim_{t \to \infty} \omega_t = 0$ in the C^0-norm. By the smoothing properties of the heat semigroup $e^{-t\Delta}$, it follows that $\lim_{t \to \infty} \omega_t = 0$ in the C^∞-norm. Since by Theorem B the basic cohomology class of ω_t is the same for all $t \geqq 0$, the proof of Corollaries C and D is now complete.

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Eingegangen 10. März 1990