The Levy concentration phenomenon for special functions on rank-one symmetric spaces

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1 Introduction

Paul Levy discovered the surprising phenomenon of measure concentration of Lipschitz maps for certain sequences of spaces as the dimension goes to infinity. This has been explored further in a geometrical framework by M.Gromov and V.Milman [GM][G][MS]. The main objective of this paper is to study in some detail this measure concentration phenomenon for compact, rank-one symmetric spaces. In particular, we will investigate the role of zonal eigenfunctions of the Laplacian and show that (see Theorems 1 and 2) the push-forward measures of a wide class of observables associated with these functions exhibit a much stronger concentration than the estimate one gets from the isoperimetric inequality of Gromov and Levy [G][MS].

Let (M^n, g) be a compact, n-dimensional, C^∞ Riemannian manifold and $d\mu_n$ be the associated n-dimensional measure. We will denote the normalized probability measure $(vol(M^n, g))^{-1} d\mu_n$ by $\hat{d}\mu_n$. Given a map $f : M^n \to \mathbb{R}$, we define the Lipschitz semi-norm by:

$$\|f\|_{Lip} := \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

We say that a sequence of functions $f_n : M^n \to \mathbb{R}$ is in $Lip^1(M^n)$ (ie. f_n is 1-Lipschitz) if $f_n \in Lip(M^n)$ and there exists $\kappa > 0$ independent of the dimension, n, such that $\|f_n\|_{Lip} \leq \kappa$ for all n. In order to describe our results in greater detail, we start with a very simple observation:

Lemma 1 *Let (M^n, g) be a compact, C^∞ Riemannian manifold with normalized measure $\hat{d}\mu_n$ and let $f_n : M^n \to \mathbb{C}$ a Lipschitz map with $f_n \in Lip^1(M^n)$ and $\int_M f_n d\mu_n = 0$. Then, for any test function, $g \in C_0^\infty(\mathbb{R})$, we have:*

$$I_{f_n}(g) := \int_{\mathbb{R}} g\,(f_n)_* \hat{d}\mu_n = g(0) + E(n),$$

with error estimate:

$$|E(n)| \leq \frac{C(g)}{\lambda_1}$$

where λ_1 is the first non-zero eigenvalue of the Laplacian.

Proof: Make a second-order Taylor expansion:

$$g(f_n(x)) = g(0) + g'(0) f_n(x) + \mathcal{O}(1)\, f_n^2(x).$$

Since $\int_M f_n d\mu = 0$, it follows that

$$\int_M g(f_n(x)) \hat{d}\mu_n(x) = g(0) + \mathcal{O}(1) \int_M f_n^2 \hat{d}\mu_n.$$

The lemma then follows from the Poincaré inequality:

$$\int_M |f_n|^2 \hat{d}\mu_n \leq \frac{1}{\lambda_1} \int_M \|\nabla f_n\|^2 \hat{d}\mu_n \leq \frac{1}{\lambda_1}.$$

Suppose we now also assume that

$$Ric(M^n, g) \geq (n-1)k > 0. \tag{1}$$

Then, by the theorem of Lichnerowicz, $\lambda_1 \geq k\,n$, and combined with Lemma 1 this inturn implies that

$$\int_{\mathbb{R}} g\,(f_n)_* \hat{d}\mu_n = g(0) + \mathcal{O}(n^{-1}) \tag{2}$$

for an arbitrary 1-Lipschitz map, f_n. By choosing f_n to be the first 1-Lipschitz Laplace eigenfunction on $\mathbb{S}^n$ (see section 2), it is easily seen that the estimate in (2) cannot be improved for general observables. However, when f_n is a 1-Lipschitz eigenfunction of the Laplacian, $-\Delta$, with eigenvalue, λ_k, Green's Theorem implies that, for any $g \in C_0^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} g\,(f_n)_* \hat{d}\mu_n = g(0) + \mathcal{O}(\lambda_k^{-1}). \tag{3}$$

Thus, in particular, for large enough λ_k the estimate in (3) is better than (2).

Such dimensional concentration results are well-known (see [MS,GM,GLP]). Although Levy-Milman-Gromov results are mainly concerned with large deviation estimates, and we compute the second moment in Lemma 1, we will nevertheless refer to (2) as the LMG-estimate.

The plan of the paper is as follows: In sections 2 and 3, we show that (see Theorem 1) for Lipschitz-normalized, zonal eigenfunctions of the Laplacian on rank-one symmetric spaces, the LMG estimate can be improved significantly. This is consistent with (3), but the precise estimates involve a detailed study of the second moments appearing in the error terms and in fact, turn out to be much better than the naive prediction in (3).

In section 4, we show that (see Theorem 2) the error term in the LMG estimate can be improved for a large class of 1-Lipschitz functions including lots of Laplace eigenfunctions that are not radial (Corollary 1).

Although we will not pursue this here, we should point out that this dimensional concentration phenomena for zonals is really a problem in semiclassical analysis. In particular, the eigenfunction equations (5), (13) and (20) are all $\hbar$-differential equations where $\hbar^{-1} = n$, the dimensional parameter of the *base* rank-one space. Thus, the content of Theorem 1 is an estimate for the second moments of Lipschitz-normalized, semi-excited states associated with these equations. The question we are dealing with here is dual to the problem one usually studies in semiclassical analysis; namely, we consider sequences of 1-Lipschitz zonal eigenfunctions, where the *base* space (ie. the rank-one manifold) varies in dimension. This connection with semiclassical analysis seems to extend to more general situations.

One can also naturally consider dimensional asymptotics for compact, semi-simple Lie groups and higher-rank symmetric spaces. It would be of interest to establish sharp estimates for the dimensional measure concentration of special observables such as spherical functions (the analogues of the zonals) on such spaces. We hope to pursue all these questions in future work.

2 Explicit calculations for the first eigenfunction

Before we investigate the dimensional asymptotics of general zonal eigenfunctions of the Laplacian for rank one symmetric spaces, $M^n = G^n/K^n$, we will begin with the first non-trivial zonal eigenfunction. The higher zonal functions are polynomials in the first zonal eigenfunction and we will treat them in the next section. In this section, we will compute all moments of the push-forward measure by the first non-trivial eigenfunction for compact, rank-one symmetric spaces.

Case 0.

S^n with the round metric of sectional curvature $K = 1$. The volume form in geodesic polar coordinates is given by [BGM][H]:

$$d\mu_n = (\sin r)^{n-1} dr d\theta \tag{4}$$

where $d\theta$ is the volume form of the unit sphere S^{n-1} and $0 \leq r \leq \pi$. Since the logarithmic derivative of $(\sin r)^{n-1}$ is $(n-1)\cot r$, we have the following expression for the Laplacian acting on a radial function $\phi(r)$:

$$\Delta\phi(r) = \phi''(r) + (n-1)\cot r\, \phi'(r). \tag{5}$$

It follows that $\phi_1(r) = -\cos r$ satisfies $-\Delta\phi_1(r) = n\phi_1(r)$ and is a zonal eigenfunction for the lowest eigenvalue n which is 1-Lipschitz. The volume form can be expressed in terms of ϕ_1 as:

$$d\mu_n = (1-\phi_1^2)^{\frac{n}{2}-1}\, d\phi_1 d\theta \tag{6}$$

where ϕ_1 ranges in the interval $[-1, +1]$. Therefore, if we normalize the push-forward measure $(\phi_1)_* d\mu_n$, we get the following measure supported in the interval $[-1, +1] \subset \mathbb{R}$:

$$\hat{d\mu_1} = \frac{1}{I_n}(1-\phi_1^2)^{\frac{n}{2}-1}\, d\phi_1 \tag{7}$$

where $I_n = \int_{-1}^{+1}(1-x^2)^{\frac{n}{2}-1}dx = B(\frac{n}{2}, \frac{1}{2})$. This is a probability measure on $\mathbb{R}$ with mean zero and variance:

$$\frac{1}{I_n}\int_{-1}^{+1} x^2(1-x^2)^{\frac{n}{2}-1}dx = \frac{1}{n+1} \tag{8}$$

which is of order $\mathcal{O}(n^{-1})$.

The estimate in (8) holds for any eigenfunction in the first eigenspace, since, for the sphere, they are all zonal with respect to a suitable pole.

If we compute the higher moments, we find that the odd moments vanish and the $2p$-th moment is:

$$\frac{1}{I_n}\int_{-1}^{+1} x^{2p}(1-x^2)^{\frac{n}{2}-1}dx = \frac{1\cdot 3\cdots(2p-1)}{(n+1)(n+3)\cdots(n+2p-1)} \tag{9}$$

which is of order $\mathcal{O}(n^{-p})$.

Case 1.

$\mathbb{CP}^n$ with the Fubini-Study metric with sectional curvature satisfying $1 \le K \le 4$. The volume form of this metric in geodesic polar coordinates is given by [BGM][H]:

$$d\mu_n = \frac{1}{2}(\sin r)^{2n-2}\sin 2r\, dr d\theta = (\sin r)^{2n-1}\cos r\, dr d\theta \tag{10}$$

where $d\theta$ is the volume form of the unit standard sphere and $0 \le r \le \frac{\pi}{2}$. Since the logarithmic derivative of $(\sin r)^{2n-2}\sin 2r$ is $(2n-2)\cot r + 2\cot 2r$, we have the following expression for the Laplacian acting on a radial function $\phi(r)$:

$$-\Delta\phi(r) = \phi''(r) + ((2n-2)\cot r + 2\cot 2r)\phi'(r). \tag{11}$$

It follows now by a direct calculation that

$$\phi_1(r) = \frac{1}{n+1} - (\cos r)^2 = (\sin r)^2 - \frac{n}{n+1} = -(\frac{1}{2}\cos 2r + \frac{n-1}{n+1}) \tag{12}$$

satisfies

$$-\Delta\phi_1(r) = 4(n+1)\phi_1(r) \tag{13}$$

and hence is a zonal eigenfunction for the lowest eigenvalue. Note that, we have normalized the eigenfunction ϕ_1 to be 1-Lipschitz.

Since $\sin^2 r = \phi_1 + \frac{n}{n+1}$ and $d\phi_1 = 2\sin r\, \cos r$, the volume form can be expressed as:

$$d\mu_n = \frac{1}{2}\left(\phi_1 + \frac{n}{n+1}\right)^{n-1} d\phi_1 d\theta \tag{14}$$

where ϕ_1 ranges in the interval $[-\frac{n}{n+1}, \frac{1}{n+1}]$. Therefore, if we normalize the push-forward measure $(\phi_1)_* d\mu_n$ to have unit total mass, we get the following measure supported in the interval $[-\frac{n}{n+1}, \frac{1}{n+1}] \subset \mathbb{R}$:

$$\hat{d\mu_1} = n\,\left(\phi_1 + \frac{n}{n+1}\right)^{n-1} d\phi_1. \tag{15}$$

This is a probability measure on $\mathbb{R}$ with mean zero, since $\int \phi_1 d\mu_n = 0$. The variance is:

$$n\int_{-\frac{n}{n+1}}^{\frac{1}{n+1}} \phi_1^2\,\left(\phi_1 + \frac{n}{n+1}\right)^{n-1} d\phi_1 = \frac{n}{(n+1)^2(n+2)}. \tag{16}$$

So, the variance is of order $\mathcal{O}(n^{-2})$ and is much sharper than the case of the sphere where it is $\mathcal{O}(n^{-1})$, even for the first eigenfunction. If we compute the higher moments we find that the p^{th} moment is:

$$n\int_{-\frac{n}{n+1}}^{\frac{1}{n+1}} \phi_1^p\,\left(\phi_1 + \frac{n}{n+1}\right)^{n-1} d\phi_1 = n\int_0^1 \left(x - \frac{n}{n+1}\right)^p x^{n-1}dx$$

which is a rational expression in n of order $\mathcal{O}(n^{-p})$.

At this point, we would like to remark that the large deviation estimate for this pushforward measure trivially satisfies the Levy-type inequality:

$$\int_0^{1-\epsilon} d(x^n) = (1-\epsilon)^n \leq e^{-n\epsilon} \tag{17}$$

However, ϵ is measured in terms of the values of the first eigenfunction ϕ_1 and not with respect to the distance function r.

Case 2.

$\mathbb{HP}^n$ with the symmetric metric of sectional curvature $1 \leq K \leq 4$. The volume form of this metric in geodesic polar coordinates is given by [BGM][H]:

$$d\mu_n = \frac{1}{8}(\sin r)^{4n-4}(\sin 2r)^3 dr d\theta = (\sin r)^{4n-1}(\cos r)^3 dr d\theta \tag{18}$$

where $d\theta$ is the volume form of the unit sphere S^{4n-1} and $0 \leq r \leq \frac{\pi}{2}$. The Laplacian acting on a radial function $\phi(r)$ is given by:

$$-\Delta\phi(r) = \phi''(r) + ((4n-4)\cot r + 6\cot 2r)\phi'(r). \tag{19}$$

Again, by a direct calculation we find that

$$\phi_1(r) = (\cos r)^2 - \frac{1}{n+1} = \frac{1}{2}\cos 2r + \frac{n-1}{n+1}$$

satisfies

$$-\Delta\phi_1(r) = 8(n+1)\phi_1(r), \tag{20}$$

and hence, is a 1-Lipschitz zonal eigenfunction. Since $\sin^2 r = \phi_1 + \frac{n}{n+1}$ and $d\phi_1 = 2\sin r\, \cos r$, the volume form can be expressed as:

$$d\mu_n = \frac{1}{2}\left[\left(\phi_1 + \frac{n}{n+1}\right)^{2n-1} - \left(\phi_1 + \frac{n}{n+1}\right)^{2n}\right] d\phi_1 d\theta$$

where ϕ_1 ranges in the interval $[-\frac{n}{n+1}, \frac{1}{n+1}]$. Therefore, the normalized push-forward probability measure is given by:

$$\hat{d}\mu_1 = 2n(2n+1)\left[\left(\phi_1 + \frac{n}{n+1}\right)^{2n-1} - \left(\phi_1 + \frac{n}{n+1}\right)^{2n}\right] d\phi_1 \tag{21}$$

where ϕ_1 is supported in the interval $[-\frac{n}{n+1}, \frac{1}{n+1}] \subset \mathbb{R}$. Here, $\hat{d}\mu_1$ is a probability measure on $\mathbb{R}$ with mean zero and variance:

$$\int_{\mathbb{R}} \phi_1^2\, \hat{d}\mu_1 = 2n(2n+1)\int_0^1 x^2(x^{2n-1} - x^{2n})\, dx - \frac{n^2}{(n+1)^2} = \frac{n}{(n+1)^2(2n+3)}. \tag{22}$$

The variance is again of order $\mathcal{O}(n^{-2})$, sharper than the case of the sphere. As in the case of $\mathbb{CP}^n$, we find that the p^{th} moment:

$$\int_{\mathbb{R}} \phi_1^p\, \hat{d}\mu_1 = 2n(2n+1)\int_0^1 \left(x - \frac{n}{n+1}\right)^p \left(x^{2n-1} - x^{2n}\right) dx$$

is a rational expression in n of order $\mathcal{O}(n^{-p})$.

3 Dimensional asymptotics for higher zonal eigenfunctions

Let M be a rank-one symmetric space G/K where K is the isotropy group at the point $p_0 \in M$. Let K_O be the identity component of K and V be a finite-dimensional subspace of $C^\infty(M)$ invariant under $l(g)$, where

$$l(g)f(x) := f(g^{-1}x)$$

for all $g \in G$ and $x \in M$. The space of *zonal* functions [H] centered at p_0 is then defined to be

$$\mathcal{Z}(V) = \{v \in V;\, l(k)v = v \text{ for all}\, k \in K_0\}.$$

In particular, one can take V to be the eigenspace

$$V_\lambda = \{u \in C^\infty(M); \Delta u = -\lambda u\}.$$

It is well-known that the action

$$l(g) : V_\lambda \to V_\lambda$$

is irreducible and moreover, the assumption that M is a rank-one symmetric space ensures that

$$\dim \mathcal{Z}(V_\lambda) = 1.$$

Furthermore, it follows readily from the definition of $\mathcal{Z}(V_\lambda)$ that all functions $\phi \in \mathcal{Z}(V_\lambda)$ are radial; that is $\phi(x) = \phi(|x|)$ where, $|x| = d(x, p_0)$.

Our objective is to study in some detail the LMG estimate (2) for rank-one symmetric spaces, $M^n = G^n/K^n$, where the observable is a zonal eigenfunction of the Laplacian. To wit, let $\phi_m(r)$ be the 1-Lipschitz zonal eigenfunction of the Laplacian, $-\Delta$, with eigenvalue, λ_m. We will now show that these functions have much sharper concentration properties in the dimensional limit than the general LMG estimate predicts.

We begin with the case of $\mathbb{S}^n$ normalized so that $K = 1$. Recall, the Poisson kernel for the unit ball $B = \{x \in \mathbb{R}^{n+1};\, |x|^2 \leq 1\}$ is:

$$P(x, y) = \frac{1}{\mathrm{vol}(\mathbb{S}^n)} \frac{1 - |x|^2}{|x - y|^{n+1}}. \tag{23}$$

Let $\mathcal{Z}_m(x, y)$ be the m-th zonal harmonic on $\mathbb{S}^n$ centered at $y \in \mathbb{S}^n$. The reproducing property [F] of these functions implies that:

$$P(x, y) = \sum_{m=0}^{\infty} |x|^m \, \mathcal{Z}_m\left(\frac{x}{|x|}, y\right). \tag{24}$$

By expanding $P(x, y)$, we get the following explicit formulas for the zonal harmonic of degree m centered at $y_0 = (0, , , 0, 1) \in \mathbb{R}^{n+1}$:

$$\mathcal{Z}_m(x, y_0) = \sum_{k=0}^{[\frac{m}{2}]} c_k(n)\, x_{n+1}^{m-2k}\, |x|^{2k}. \tag{25}$$

Here, $(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}$ and

$$c_k(n) = (-1)^k \frac{(n+2m-1)(n+1)(n+3)\cdots(n+2m-2k-3)}{2^k k!(2m-k)!}. \tag{26}$$

We henceforth assume that m is fixed independent of the dimension, n. Then,

$$c_k(n) \sim n^{m-k}.$$

Let $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ denote geodesic polar coordinates centered at $(0, ..., 0, 1) \in \mathbb{R}^{n+1}$. Since $x_{n+1} = \cos r = \phi_1(r)$, it then follows that the Lipschitz-1 eigenfunction, $\phi_m(r)$, corresponding to $\mathcal{Z}_m(x, y_0)$ is of the form:

$$\phi_m(r) = C_m(n)\left[(\phi_1(r))^m + \frac{c_1(n)}{c_0(n)}(\phi_1(r))^{m-2} + ... + \frac{c_m(n)}{c_0(n)}(\phi_1(r))^{m-2[\frac{m}{2}]}\right], \tag{27}$$

where, $C_m(n) = C_m + \mathcal{O}(n^{-1})$ and

$$\frac{c_k(n)}{c_0(n)} = \mathcal{O}(n^{-k}). \tag{28}$$

Since $\int_{\mathbb{S}^n} \phi_m d\mu_n = 0$, we must compute the second moment of ϕ_m. To do this, note that by our moment computations for ϕ_1 in section 2, we have for $0 \leq k, l \leq [\frac{m}{2}]$,

$$\left(\frac{c_k(n)}{c_0(n)}\right)\left(\frac{c_l(n)}{c_0(n)}\right)\int_{\mathbb{R}} (\phi_1)^{m-2k} (\phi_1)^{m-2l} \hat{d}\mu_1 = \mathcal{O}(n^{-m}). \tag{29}$$

It follows that:

$$\int_{\mathbb{R}} \phi_m^2 \hat{d}\mu_1 = \mathcal{O}(n^{-m}). \tag{30}$$

Now consider complex projective space, $\mathbb{CP}^n$, normalized so that $1 \leq K \leq 4$. Let $\mathbb{B} = \{z \in \mathbb{C}^{n+1}; |z|^2 \leq 1\}$ and denote the space of harmonic polynomials in $\mathbb{B}$ of degree p in z and degree q in $\overline{z}$ by $\mathcal{H}^{p,q}$. The zonal eigenfunction in this vector space will be denoted by $H^{p,q}$. Since we are interested in $\mathbb{CP}^n$, we need only consider those zonals which are invariant under the circle action:

$$e^{i\theta} \cdot (z_1, ..., z_{n+1}) = (e^{i\theta} z_1, ..., e^{i\theta} z_{n+1}).$$

Consequently, for m fixed independent of the dimension, n, we put $p = q = m$. Then, for fixed $\zeta \in \partial\mathbb{B}$, consider $H^{m,m}(z \cdot \overline{\zeta})$. It is readily verified that [Ko]:

$$H^{m,m}(z \cdot \overline{\zeta}) = C(m, n) \cdot P_m^{(\frac{n}{2}-1,0)}(2|z \cdot \overline{\zeta}|^2 - 1). \tag{31}$$

Here, $C(m, n)$ is a constant and $P_m^{(\alpha,\beta)}$ denotes the Jacobi polynomial [AS] of degree m with indices (α, β):

$$P_m^{(\alpha,\beta)}(t) = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m+\alpha}{k}\binom{m+\beta}{m-k}(t-1)^{m-k}(1+t)^k. \tag{32}$$

Fix $\zeta = (0, 0, ..., 0, 1) \in \mathbb{C}^{n+1}$ and consider geodesic polar coordinates $(r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^{n-1}$ centered at ζ. In terms of these coordinates,

$$|z_{n+1}|^2 = (\cos r)^2.$$

So, the m-th zonal eigenfunction in (31) is, up to multiplicative constant, the function

$$P_m^{(\frac{n}{2}-1,0)}(2\cos^2 r - 1). \tag{33}$$

Next, we must Lipschitz-normalize this function. To do this, note that the expansion in (32) gives:

$$P_m^{(\frac{n}{2}-1,0)}(2\cos^2 r - 1) = \sum_{k=0}^{m} c_k(n)(\sin^2 r)^{m-k} (\cos^2 r)^k, \tag{34}$$

where, $c_k(n) \sim n^k$ as $n \to \infty$. So, to Lipschitz normalize this function, we divide by $c_0(n) \sim n^m$. Consequently, the m-th, 1-Lipschitz zonal eigenfunction is of the form:

$$\phi_m(r) = \sum_{k=0}^{m} \frac{c_k(n)}{c_0(n)} (\sin^2 r)^{m-k} (\cos^2 r)^k. \tag{35}$$

Again, we wish to compute the second moment $\int_{\mathbb{R}} \phi_m^2 \hat{d}\mu_1$. To do this, note that, for $0 \leq k, l \leq m$:

$$\left(\frac{c_k(n)}{c_0(n)}\right) \left(\frac{c_l(n)}{c_0(n)}\right) \int_{\mathbb{R}} (\sin r)^{2(2m-k-l)} (\cos r)^{2(k+l)} \, \hat{d}\nu \tag{36}$$

$$\leq \left(\frac{c_k(n)}{c_0(n)}\right) \left(\frac{c_l(n)}{c_0(n)}\right) \int_{\mathbb{R}} \left(\phi_1 - \frac{1}{n+1}\right)^{2(k+l)} \hat{d}\mu_1 = \mathcal{O}(n^{-2m}). \tag{37}$$

The last line (37) follows readily from the estimate $|\sin r| \leq 1$, the binomial expansion and our estimates for the moments of ϕ_1 in section 2. So, we have shown that:

$$\int_{\mathbb{R}} \phi_m^2 \, \hat{d}\mu_1 = \mathcal{O}(n^{-2m}). \tag{38}$$

Finally, we turn to the case of the quaternionic projective spaces, $\mathbb{HP}^n$. Here, one can make a direct change of variables in the eigenfunction equation [Ko] to transform it to a canonical equation for orhogonal polynomials. The end result is that

$$\phi_m(r) = C(m, n) \, P_m^{(\frac{n}{2}-1,1)}(2\cos^2 r - 1).$$

As the estimates are very similar to the case of $\mathbb{CP}^n$, we leave the details to the reader.

Taking into account the Taylor expansion in Lemma 1, we have now proved:

Theorem 1 *Let $\phi_m(r)$ be the m-th, 1-Lipschitz, zonal eigenfunction of the Laplacian on the rank-one symmetric space $M^n = G^n/K^n$ and let $g \in C_0^\infty(\mathbb{R})$. Suppose m is arbitrarily large but fixed independent of the dimension, n. Then, for $M^n = \mathbb{CP}^n, \mathbb{HP}^n$:*

$$\int_{\mathbb{R}} g(t)(\phi_m)_* \hat{d}\mu_n(t) = g(0) + \mathcal{O}(n^{-2m}),$$

whereas, for $M^n = \mathbb{S}^n$:

$$\int_{\mathbb{R}} g(t)(\phi_m)_* \hat{d}\mu_n(t) = g(0) + \mathcal{O}(n^{-m}).$$

Remark: It follows readily from the above analysis that the estimates in Theorem 2 are valid for sequences $\phi_{m(n)}; n = 1, 2, ...$ with $m(n) \sim n^\alpha$, provided $0 \leq \alpha < 1$. Thus, in particular, when $m(n) = n^\alpha$, the error terms in Theorem 1 are $\mathcal{O}(n^{-n^\alpha + 2\alpha})$ and $\mathcal{O}(n^{-2n^\alpha + 2\alpha})$ respectively.

4 The general estimate

In the last section, we showed that for the complex and quaternionic projective spaces, the 1-Lipschitz zonal eigenfunctions of the Laplacian have stronger dimensional concentration than the LMG estimate predicts. Our objective here is to extend this refined concentration estimate to a wider class of 1-Lipschitz observables containing the zonal eigenfunctions.

Consider the complex vector space, V_{λ_1}, of eigenfunctions of the Laplacian, Δ, with eigenvalue, λ_1. There is a natural space of linearly independent, zonal eigenfunctions in V_{λ_1}, which we will denote by $\mathcal{Z}(M^n; \lambda_1)$. For instance, on the sphere, $\mathbb{S}^n$,

$$\mathcal{Z}(\mathbb{S}^n; \lambda_1) := \{x_1, ..., x_{n+1}\}, \tag{39}$$

where for $k = 1, ..., n+1$, x_k denote the restrictions to $\mathbb{S}^n$ of the standard coordinate functions on $\mathbb{R}^{n+1}$. On complex projective space,

$$\mathcal{Z}(\mathbb{CP}^n; \lambda_1) := \left\{ |z_k|^2 - \frac{1}{n+1} \right\}_{1 \leq k \leq n+1}, \tag{40}$$

where $[z_k]$ denote the standard homogeneous coordinates on $\mathbb{C}^{n+1}$ restricted to $\mathbb{S}^{2n+1}$. The space $\mathcal{Z}(\mathbb{HP}^n; \lambda_1)$ is identical to (40), provided we replace complex multiplication by quaternionic multiplication. Recall, we say that $f_n \in Lip^1(M^n)$, provided $f_n \in Lip(M^n)$ and there exists a constant, $\kappa > 0$, independent of the dimension, n, such that $\|f_n\|_{Lip} \leq \kappa$.

Definition: $\mathcal{I}^*(M^n)$ is defined to be the finitely-generated ideal over $Lip^1(M^n)$, generated by the elements of $\mathcal{Z}(M^n; \lambda_1)$.

In the following, we will, without loss of generality, always assume that $\int f_n = 0$. Moreover, given geodesic spherical coordinates (r, θ) centered at a point $p \in M$, we define the spherical mean $\mathcal{M}_{f_n}$ by:

$$\mathcal{M}_{f_n}(r) := \int_{\mathbb{S}^{2n-1}} f_n(r,\omega)\,\hat{d}\omega. \tag{41}$$

We can now state our result:

Theorem 2 *(i) Let $M^n = G^n/K^n$ be a rank-one symmetric space and $g \in C_0^\infty(\mathbb{R})$. Suppose $f_n \in \mathcal{I}(M^n)$ is generated by a zonal eigenfunction, $\phi(z,\overline{z}) = |z_k|^2 - (n+1)^{-1}$; that is $f_n(z,\overline{z}) = \phi_1(z,\overline{z}) \cdot h(z,\overline{z};n)$. Assume moreover, that there exist a constant $C > 0$ independent of n such that*

$$\|\mathcal{M}_{|h|^2}\|_\infty \leq C.$$

When $M^n = \mathbb{CP}^n, \mathbb{HP}^n$:

$$\int_{\mathbb{R}} g(t)(f_n)_*\hat{d}\mu_n(t) = g(0) + \mathcal{O}(n^{-2}).$$

as $n \to \infty$. On the other hand, when $M^n = \mathbb{S}^n$:

$$\int_{\mathbb{R}} g(t)(f_n)_*\hat{d}\mu_n(t) = g(0) + \mathcal{O}(n^{-1}).$$

Moreover, these estimates are sharp.
(ii) Let $f_n \in \mathcal{I}(\mathbb{CP}^n)$ with $f_n = \sum_{j=1}^{n+1} |z_j|^2 \cdot h_j(z,\overline{z})$. Suppose that, for $j = 1, ..., n+1$, there exists a constant $C > 0$ independent of the dimension, such that $h_j \in Lip^1(M^n)$ satisfies:

$$\sum_{j=1}^{n+1} \|h_j\|_\infty^2 \leq C \;\; \text{and} \;\; \sum_{j=1}^{n+1} \|\nabla h_j\|_\infty^2 \leq Cn.$$

Then, the estimates in part (i) are valid for these f_n's.

Proof of (i): To begin, we will assume that $M^n = \mathbb{CP}^n$ and without loss of generality, it suffices to assume that:

$$f_n(z,\overline{z}) = |z_{n+1}|^2 \cdot h(z,\overline{z};n). \tag{42}$$

Given $g \in C_0^\infty(\mathbb{R})$, we must give an asymptotic expansion for the push-forward integral:

$$I_g(f_n) := \int_{\mathbb{R}} g(t)(f_n)_*\hat{d}\mu_n(t) = \int_0^{\pi/2}\int_{\mathbb{S}^{2n-1}} g(f_n(r,\theta))(\cos r)(\sin r)^{2n-1}\hat{d}\theta\hat{d}r.$$

By a second-order Taylor expansion, we have for each $r \in [0,\pi/2]$,

$$\int_{\mathbb{S}^{2n-1}} g(f_n(r,\theta))\hat{d}\theta = g\left(\int_{\mathbb{S}^{2n-1}} f_n(r,\omega)\hat{d}\omega\right) + \mathcal{O}_g(1)\int_{\mathbb{S}^{2n-1}} \left|f_n(r,\theta) - \int_{\mathbb{S}^{2n-1}} f_n(r,\omega)\,\hat{d}\omega\right|^2 \hat{d}\theta.$$

Denoting by $E(r;n)$ the last term on the RHS of the above equation, we have by the Cauchy-Schwartz inequality,

$$E(r;n) \leq \cdot 2(\cos r)^4 \int_{\mathbb{S}^{2n-1}} |h(r,\theta;n)|^2 \, \hat{d}\theta \leq 2C(\cos r)^4. \tag{43}$$

Note that by the estimate in (43),

$$\frac{\int_0^{\pi/2} E(r;n)(\cos r)(\sin r)^{2n-1} dr}{\int_0^{\pi/2} (\cos r)(\sin r)^{2n-1} dr} \leq 2C \left(\frac{\int_0^{\pi/2} (\cos r)^5 (\sin r)^{2n-3} dr}{\int_0^{\pi/2} (\cos r)(\sin r)^{2n-1} dr} \right) = \mathcal{O}(n^{-2}).$$

Recapping, we have so far shown that:

$$I_g(f_n) = \frac{\int_0^{\pi/2} g(\mathcal{M}_{f_n}(r)) \, (\cos r)(\sin r)^{2n-1} dr}{\int_0^{\pi/2} (\cos r)(\sin r)^{2n-1} dr} + \mathcal{O}(n^{-2}). \tag{44}$$

Next, we integrate over the radial variable, r. To wit, consider the second-order Taylor expansion:

$$g(\mathcal{M}_{f_n}(r)) = g(0) + g'(0) \, \mathcal{M}_{f_n}(r) + \mathcal{O}_g(1) \, \mathcal{M}^2_{f_n}(r).$$

Since by assumption, $\int f_n = 0$, it follows that:

$$I_g = g(0) + \mathcal{O}(1) \cdot \left(\frac{\int_0^{\pi/2} \mathcal{M}^2_{f_n}(r) \, (\cos r)(\sin r)^{2n-1} dr}{\int_0^{\pi/2} (\cos r)(\sin r)^{2n-1} dr} \right) + \mathcal{O}(n^{-2}). \tag{45}$$

Again, by the Cauchy-Schwartz inequality,

$$|\mathcal{M}_{f_n}(r)|^2 \leq (\cos r)^4 \, |\mathcal{M}_{|h|^2}(r)| \leq C(\cos r)^4. \tag{46}$$

So, it follows that:

$$\frac{\int_0^{\pi/2} \mathcal{M}_{f_n}(r)^2 \, (\cos r)(\sin r)^{2n-1} dr}{\int_0^{\pi/2} (\cos r)(\sin r)^{2n-1} dr} \leq C \left(\frac{\int_0^{\pi/2} (\cos r)^5 (\sin r)^{2n-1} dr}{\int_0^{\pi/2} (\cos r)(\sin r)^{2n-1} dr} \right) = \mathcal{O}(n^{-2}).$$

Therefore,

$$I_g(f_n) = g(0) + \mathcal{O}(n^{-2}). \tag{47}$$

The analogous result for $\mathbb{HP}^n$ with $\mathcal{O}(n^{-2})$ error term is proved in the same way. On the other hand, the corresponding result for $\mathbb{S}^n$ has an $\mathcal{O}(n^{-1})$ error. Moreover, it follows by our earlier computations for the first, non-trivial, 1-Lipschitz eigenfunctions that all these estimates are sharp.

Proof of (ii): By the Taylor expansion in Lemma 1:

$$\int_{\mathbb{R}} g(t)(f_n)_* \hat{d}\mu_n(t) = g(0) + E(n)$$

where,

$$|E(n)| \leq \frac{1}{4(n+1)} \int_{\mathbb{CP}^n} |\nabla f_n|^2 \hat{d}\mu_n.$$

Clearly,

$$|\nabla f_n|^2 \leq 2\sum_{j=1}^{n+1}\left[|z_j|^4\,|\nabla h_j|^2 + 2|h_j|^2(1-|z_j|^2)|z_j|^2\right].$$

So, it follows that:

$$|E(n)| \leq \frac{1}{2(n+1)}\cdot\sum_{j=1}^{n+1}\left(\|\nabla h_j\|_\infty^2\int_{\mathbb{CP}^n}|z_j|^4\hat{d}\mu_n + 2\|h_j\|_\infty^2\int_{\mathbb{CP}^n}(1-|z_j|^2)|z_j|^2\hat{d}\mu_n\right).$$

Here,

$$\int_{\mathbb{CP}^n}|z_j|^4\,\hat{d}\mu_n = \int_{\mathbb{R}}\left(\phi_1 - \frac{1}{n+1}\right)^2\hat{d}\mu_1 = \mathcal{O}(n^{-2}) \tag{48}$$

by our computations in section 2. On the other hand,

$$\int_{\mathbb{CP}^n}(1-|z_j|^2)|z_j|^2\,\hat{d}\mu_n \leq \int_{\mathbb{CP}^n}|z_j|^2\,\hat{d}\mu_n = \int_{\mathbb{R}}\left(\phi_1 - \frac{1}{n+1}\right)\hat{d}\mu_1 = \mathcal{O}(n^{-1}). \tag{49}$$

Part (ii) of Theorem 2 follows. Q.E.D.

Remarks:(1) Note that part (i) of Theorem 2 shows that, $h(z,\overline{z};n)$ need not even be uniformly bounded as $n\to\infty$ for the $\mathcal{O}(n^{-2})$ error to hold. All we need is that the second spherical mean $\mathcal{M}_{|h|^2}$ be uniformly bounded.
(2) Note that part (ii) of Theorem 1 extends in a straightforward fashion to both $\mathbb{HP}^n$ and $\mathbb{S}^n$. We simply replace $|z_j|^2$ by x_j^2 in the case of the sphere.

Suppose we now fix $k\in\mathbb{Z}^+$ independent of the dimension, n, and consider the usual embedding $\iota:\mathbb{CP}^k\to\mathbb{CP}^n$ given by:

$$\iota([z_1,...,z_{k+1}]) = [z_1,...,z_{k+1},0,...,0].$$

Let $\psi_m\in V_{\lambda_m}(\mathbb{CP}^k)$ be a 1-Lipschitz Laplace eigenfunction on $\mathbb{CP}^k$ in the m-th eigenspace. We can naturally think of ψ_m as a 1-Lipschitz Laplace eigenfunction on $\mathbb{CP}^n$ via the embedding, ι. The following is an easy consequence of Theorem 2 (ii):

Corollary 1 *Suppose $m\in\mathbb{Z}^+$ and $k\in\mathbb{Z}^+$ are fixed arbitrarily large, but independent of the dimensional parameter, n. Then, for arbitrary $g\in C_0^\infty(\mathbb{R})$ and n sufficiently large,*

$$\int_{\mathbb{R}} g(t)(\psi_m)_*\hat{d}\mu_n(t) = g(0) + \mathcal{O}(n^{-2m}).$$

Proof: We will suppose that $\psi_1\in V_{\lambda_1}(\mathbb{CP}^k)\subset V_{\lambda_1}(\mathbb{CP}^n)$. Then, we can write:

$$\psi_1(z,\overline{z}) = \sum_{i,j=1}^{n+1} c_{ij}z_i\overline{z_j}$$

where the c_{ij}'s are complex numbers satisfying $\sum_{i=1}^{n+1} c_{ii} = 0$ and $[z] \in \mathbb{S}^{2n+1}$. The error term in the computation of the push-forward integral, $I_{\psi_1}(g)$, is just the second moment, $\int \psi_1^2 \hat{d}\nu$. This consists of linear cominations of $(m+1)^2$ terms of the form:

$$I_{ijkl} = \int_{\mathbb{CP}^n} z_i \overline{z_j} z_k \overline{z_l} \, \hat{d}\mu_n. \tag{50}$$

Note that $I_{ijkl} = 0$ unless, either $i = j$ and $k = l$, or $i = l$ and $j = k$. In both cases, we are reduced to assuming that $c_{ij} = 0$ for $i \neq j$ and so, the estimate follows from Theorem 2 (ii), since the coefficients c_{ii} are just constants which are independent of the dimension, n. The argument for the higher eigenfunctions, ψ_m, is very similar and is left to the reader. Q.E.D.

Remark: There is a straightforward analogue of Corollary 1 for both $\mathbb{HP}^n$ and $\mathbb{S}^n$.

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