Houston Journal of Mathematics
© 2009 University of Houston
Volume 35, No. 4, 2009

STRONG A-INFINITY WEIGHTS AND SOBOLEV CAPACITIES IN METRIC MEASURE SPACES

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Communicated by Stephen W. Semmes

Abstract. This article studies strong A-infinity weights in Ahlfors Q-regular unbounded and geodesic metric measure spaces satisfying a weak $(1,s)$-Poincaré inequality for some s in $(1,Q]$. For a fixed s in $(Q-1,Q]$, it is shown that a function u yields a strong A-infinity weight of the form $w=exp(Qu)$ whenever the minimal s-weak upper gradient of u has sufficiently small Morrey s norm.

1. Introduction

In this paper (X,d,μ) is a complete and unbounded metric measure space. In addition, we assume that (X,d,μ) is Ahlfors Q-regular for some $Q>1$. That is, there exists a constant $C=c_\mu$ such that, for each $x\in X$ and all $r>0$,

$$C^{-1}r^Q \le \mu(B(x,r)) \le Cr^Q.$$

We will also assume that (X,d,μ) satisfies a weak $(1,s)$-Poincaré inequality for some $s\in(1,Q]$. That is, there exist constants $C>0$ and $\lambda\ge 1$ such that for all balls B with radius r, all measurable functions u on X and all upper gradients g of u we have

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu \le Cr\left(\frac{1}{\mu(\lambda B)}\int_{\lambda B} g^s\,d\mu\right)^{1/s}, \tag{1.1}$$

where λB represents the ball concentric with B with radius λ times the radius of B whenever $\lambda>0$, and u_E denotes the average of u on the measurable set

2000 *Mathematics Subject Classification.* Primary: 30C99.
Key words and phrases. Strong A-infinity weights, Newtonian spaces, Poincaré inequality, Sobolev capacity.
Work partially supported by NSERC, by the Emil Aaltonen foundation, by the Fields Institute, and by the NSF grant DMS 0244421.

$E \subset X$ with respect to the measure μ whenever $0 < \mu(E) < \infty$. For a discussion on spaces satisfying a Poincaré inequality, see for example [HeK]. We recall that a nonnegative Borel function g is an *upper gradient* for a measurable function $u : X \to [-\infty, \infty]$ if for all nonconstant rectifiable paths $\gamma : [0, l_\gamma] \to X$ we have

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g\,ds \tag{1.2}$$

whenever $u(\gamma(0))$ and $u(\gamma(l_\gamma))$ are both finite and $\int_\gamma g\,ds = \infty$ otherwise. Here and throughout the paper the rectifiable curve $\gamma : [0, l_\gamma] \to X$ is assumed to be parameterized by the arc length ds, where l_γ is the length of γ. For a discussion about rectifiable paths and path integration in metric spaces see [HeK, Section 2] and [Hei, Chapter 7].

Furthermore, X is assumed to be geodesic. That is, every pair of points can be joined by a curve whose length is the distance between the points. It is known that if μ is a Borel and Ahlfors Q-regular measure and X is geodesic and satisfies a weak $(1, s)$-Poincaré inequality for some $1 < s \leq Q < \infty$, then (X, d, μ) in fact admits a Poincaré inequality with $\lambda = 1$ in (1.1), but possibly with a different constant $C > 0$. (See [Hei, Theorem 9.5].)

We study sufficient conditions under which we get strong A_∞-weights in X. A nontrivial doubling measure ν on X is a Radon measure for which there exists a constant $C > 1$ such that

$$0 < \nu(2B) \leq C\nu(B)$$

for all balls B.

To every doubling measure ν on X we can associate a quasidistance on X defined by

$$\delta_\nu(x, y) = \nu(B_{xy})^{1/Q}, \tag{1.3}$$

where $B_{xy} := B(x, d(x, y)) \cup B(y, d(y, x))$. To say that $\delta_\nu(x, y)$ is a quasidistance means by definition that $\delta_\nu : X \times X \to [0, \infty)$ is symmetric, vanishes if and only if $x = y$, and satisfies

$$\delta_\nu(x, z) \leq C(\delta_\nu(x, y) + \delta_\nu(y, z)) \tag{1.4}$$

for some $C \geq 1$ and all $x, y, z \in X$. If (1.4) was satisfied with $C = 1$, then the quasidistance δ_ν would in fact be a distance function.

We call ν a *metric doubling measure* if the quasidistance δ_ν is comparable to a distance δ'_ν; that is, there exists a distance function δ'_ν on X and a constant $C > 0$ such that

$$C^{-1}\delta_\nu(x, y) \leq \delta'_\nu(x, y) \leq C\delta_\nu(x, y) \text{ for all } x, y \in X. \tag{1.5}$$

We say that a nonnegative function $w \in L^1_{loc}(X, \mu)$ is an A_p-*weight* with respect to the measure μ for some $1 < p < \infty$ and we write $w \in A_p(\mu)$ if there exists a constant $C \geq 1$ such that

$$\left(\frac{1}{\mu(B)}\int_B w(x)^{-1/(p-1)}\,d\mu(x)\right)^{p-1}\frac{1}{\mu(B)}\int_B w(x)\,d\mu(x) \leq C$$

for all balls $B \subset X$. We say that w is a A_∞-weight with respect to the measure μ and we write $w \in A_\infty(\mu)$ if w is an A_p-weight with respect to μ for some p in $(1, \infty)$. That is,

$$A_\infty(\mu) = \cup_{p>1} A_p(\mu).$$

We define w to be a strong A_∞-weight if it is the density of a metric doubling measure ν and moreover, it is an A_∞-weight with respect to μ. That is,

$$d\nu(x) = w(x)\,d\mu(x)$$

where $w \in A_\infty(\mu)$ and ν is a metric doubling measure.

Strong A_∞-weights in $\mathbf{R}^n$ were introduced in the early 90's by David and Semmes in [DS] and [Sem] when trying to identify the subclass of A_∞-weights that are comparable to the Jacobian determinants of quasiconformal mappings.

Question 1.1. In the Euclidean setting, metric doubling measures have densities that are A_∞-weights. (See [Sem].) An open question in the metric setting is whether or not metric doubling measures necessarily have A_∞-densities. This question has been answered affirmatively recently by Korte and Maasalo in [KorM] for Ahlfors Q-regular spaces that satisfy a weak $(1,1)$-Poincaré inequality.

In the last few years strong A_∞-weights were studied by Bonk, Heinonen, and Saksman in [BHS1] and [BHS2] and by the author in [Cos1].

In the Euclidean setting Bonk and Lang proved in [BL] that if ν_0 is a signed Radon measure on $\mathbf{R}^2$ such that $\nu_0^+(\mathbf{R}^2) < 2\pi$ and $\nu_0^-(\mathbf{R}^2) < \infty$, then $(\mathbf{R}^2, \widetilde{D}_{\nu_0})$ is bi-Lipschitz equivalent to $\mathbf{R}^2$ endowed with the Euclidean metric, where

$$\widetilde{D}_{\nu_0}(x,y) = \inf\left\{\int_\alpha e^u ds : \alpha \text{ analytic curve connecting } x, y\right\},$$

u is a solution of $-\Delta u = \nu_0$ with $|\nabla u| \in L^2(\mathbf{R}^2)$, and $\nu_0 = \nu_0^+ - \nu_0^-$ is the Jordan decomposition of ν_0. In particular, it is proved that $w = e^{2u}$ is comparable to the Jacobian of a quasiconformal mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$, which implies that w is a strong A_∞-weight.

Here we prove a result in (X, d, μ), related to [Cos1, Theorem 5.1] and to the result from [BL]. It states that A_∞-weights of the form $w = e^{Qu}$ are strong A_∞-weights if u is a locally integrable function that has an upper gradient g in the

Morrey space $\mathcal{L}^{s,Q-s}(X,\mu)$ with small Morrey s norm $||\cdot||_{\mathcal{L}^{s,Q-s}(X,\mu)}$ for some $s>1$ in $(Q-1,Q]$.

For $1 \le s \le Q$, the Morrey space $\mathcal{L}^{s,Q-s}(X,\mu)$ is defined to be the linear space of locally μ-integrable functions u on X such that $||u||_{\mathcal{L}^{s,Q-s}(X,\mu)} < \infty$, where

$$||u||_{\mathcal{L}^{s,Q-s}(X,\mu)} = \sup_{a\in X}\sup_{r>0}\left(r^{s-Q}\int_{B(a,r)}|u(x)|^s\,d\mu(x)\right)^{1/s}.$$

In particular $\mathcal{L}^{Q,0}(X,\mu) = L^Q(X,\mu)$. We refer to [Gia, p. 65] for more information about Morrey spaces in the Euclidean setting and their use in the theory of partial differential equations.

If (X,d,μ) is an Ahlfors Q-regular metric space with $Q>1$ satisfying a weak $(1,s)$-Poincaré inequality for some $s\in(1,Q]$, it follows from (1.1) that there exists a constant C depending only s and on data of X such that

$$[u]_{\mathrm{BMO}(X,\mu)} \le C||g||_{\mathcal{L}^{s,Q-s}(X,\mu)} \tag{1.6}$$

whenever g is an upper gradient of u. Here and throughout the paper $[u]_{\mathrm{BMO}(X,\mu)}$ is the *bounded mean oscillation* seminorm that measures the oscillation of u on balls in X, given by

$$[u]_{\mathrm{BMO}(X,\mu)} = \sup_{a\in X}\sup_{r>0}\frac{1}{\mu(B(a,r))}\int_{B(a,r)}|u(x)-u_{B(a,r)}|\,d\mu(x).$$

Acknowledgements. This article was written when the author was a visiting member of the Fields Institute in 2008. Part of the research was done when the author was a visiting postdoctoral researcher at the Helsinki University of Technology in 2007. The author wishes to thank Mario Bonk for useful conversations leading to Lemma 3.2 and Carlos Pérez for helpful discussions regarding the John-Nirenberg lemma.

2. Preliminaries

In this section we recall standard definitions and results. The open ball with center $x\in X$ and radius $r>0$ is denoted $B(x,r)=\{y\in X : d(x,y)<r\}$, the closed ball by $\overline{B}(x,r)=\{y\in X : d(x,y)\le r\}$, and the sphere by $S(x,r)=\{y\in X : d(x,y)=r\}$. Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. $C(a,b,\ldots)$ is a constant that depends only on the parameters $a,b,\ldots$. Here Ω will denote a nonempty open subset of X. For $E\subset X$, the closure and the complement of E with respect to X will be denoted by $\overline{E}$ and $X\setminus E$ respectively;

diam E is the diameter of E with respect to the metric d and $E \subset\subset F$ means that $\overline{E}$ is a compact subset of F.

For a measurable $u : \Omega \to \mathbf{R}$, supp u is the smallest closed set such that u vanishes on the complement of supp u. We also use the spaces

$$\begin{aligned} Lip(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz}\}, \\ Lip_0(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz and supp } \varphi \subset\subset \Omega\}. \end{aligned}$$

2.1. **Newtonian spaces.** We introduce now some definitions and known results about Newtonian spaces to be used in this paper. Let $1 \leq s < \infty$. The s-modulus of a family of nonconstant rectifiable paths Γ in X is the number

$$\inf_{\rho} \int_X \rho^s \, d\mu,$$

where the infimum is taken over all non-negative Borel measurable functions ρ such that for all nonconstant rectifiable paths γ which belong to Γ we have

$$\int_\gamma \rho \, ds \geq 1.$$

It is known that the s-modulus is an outer measure on the collection of all nonconstant rectifiable paths in X.

A property is said to hold for s-*almost all* paths, if the set of nonconstant rectifiable paths for which the property fails is of zero s-modulus. If (1.2) holds for s-almost all nonconstant rectifiable paths γ, then g is said to be a s-weak upper gradient of u. We could state the definition of the weak $(1, s)$-Poincaré inequality by requiring the inequality (1.1) to hold for all s-weak upper gradients of u. (See [KosM].) Similarly we can define weak (q, s)-Poincaré inequalities for $q > 1$.

From now on, throughout the rest of the paper, we assume that $1 < s < \infty$. We define the space $\widetilde{N}^{1,s}(X)$ to be the collection of all the functions u that are s-integrable and have a s-integrable s-weak upper gradient g. This space is equipped with the norm

$$||u||_{\widetilde{N}^{1,s}(X)} = \Big(||u||^s_{L^s(X)} + \inf ||g||^s_{L^s(X)}\Big)^{1/s},$$

where the infimum is taken over all s-weak upper gradients of u. The *Newtonian space* on X is the quotient space $N^{1,s}(X) = \widetilde{N}^{1,s}(X)/\sim$ with the norm $||u||_{N^{1,s}(X)} = ||u||_{\widetilde{N}^{1,s}(X)}$, where $u \sim v$ if and only if $||u-v||_{\widetilde{N}^{1,s}(X)} = 0$. For basic properties of the Newtonian spaces we refer to [Sha1]. Cheeger in [Che] gives an alternative definition which leads to the same space when $1 < s < \infty$. For future reference we recall some known facts (see [KiM] and [Sha2]):

(i) The functions in $\widetilde{N}^{1,s}(X)$ are defined outside a path family of s-modulus zero. This implies that the functions in $\widetilde{N}^{1,s}(X)$ cannot be changed arbitrarily on sets of measure zero.

(ii) If $1 < s < \infty$, every function u that has a s-integrable s-weak upper gradient has in fact a minimal s-integrable s-weak upper gradient in X, denoted by g_u, in the sense that if g is another s-weak upper gradient of u, then $g_u \leq g$ μ-a.e. in X.

(iii) For every $c \in \mathbf{R}$ the minimal s-weak upper gradient satisfies $g_u = 0$ μ-a.e. on the set $\{x \in X : u(x) = c\}$.

(iv) If $u \in N^{1,s}(X)$ and v is a bounded Lipschitz continuous function, then $uv \in N^{1,s}(X)$ and $g_{uv} \leq |u|g_v + |v|g_u$ μ-a.e.

We emphasize that these properties hold without any additional assumptions on the measure μ and on the space X.

The s-capacity of a set $E \subset X$ is defined by (see [BBS])

$$C_s(E) = \inf_u ||u||^s_{N^{1,s}(X)},$$

where the infimum is taken over all functions $u \in N^{1,s}(X)$ whose restriction on E is bounded below by 1. A property is said to hold *s-quasieverywhere* (or s-q.e.), if it holds everywhere except on a set of s-capacity zero. A function is *s-quasicontinuous*, if there is an open set of arbitrarily small s-capacity such that the function is continuous when restricted to the complement of the set. Every function in $\widetilde{N}^{1,s}(X)$ is defined s-quasieverywhere. Moreover, if $u, v \in N^{1,s}(X)$ and $u = v$ μ-a.e., then $u = v$ s-quasieverywhere. In particular, this implies that u and v belong to the same equivalence class in $N^{1,s}(X)$.

We introduce the notion of a local Newtonian space as follows.

Definition 1. We say that u belongs to the *local Newtonian space* $N^{1,s}_{loc}(X)$ if $u \in N^{1,s}(\Omega)$ for every open set $\Omega \subset\subset X$. If $u \in N^{1,s}_{loc}(X)$ with $1 < s < \infty$, then u has a minimal s-weak upper gradient g_u in X in the following sense: if $\Omega \subset\subset X$ is an open set and g is the minimal upper gradient of u in Ω, then $g_u = g$ μ-a.e. in Ω.

From now on throughout the rest of the paper we assume that the measure μ is Borel and Ahlfors Q-regular for some $Q > 1$. Furthermore we assume that the space supports a weak $(1, s)$-Poincaré inequality for some $1 < s \leq Q$. We recall a few useful properties of Newtonian spaces that hold under these additional assumptions (see [BBS] and [KiM]):

(i) The space X is proper (that is, closed and bounded sets are compact).

(ii) Lipschitz functions are dense in $N^{1,s}(X)$ and Lipschitz functions which vanish in the complement of an open set Ω are dense in $N_0^{1,s}(\Omega)$, where

$$N_0^{1,s}(\Omega) = \{u \in N^{1,s}(X) : u = 0 \text{ } s\text{-q.e. in } X \setminus \Omega\}.$$

(iii) Every function in $N^{1,s}(X)$ is s-quasicontinuous.

Now we introduce the relative Sobolev s-capacity as in [Cos2]. See also [Bjo].

Definition 2. Let $1 < s, Q < \infty$. Suppose (X, d, μ) is a proper and unbounded Ahlfors Q-regular metric space that satisfies a weak $(1, s)$-Poincaré inequality. Let $\Omega \subset\subset X$ be open. For $E \subset \Omega$ we let

$$A(E, \Omega) = \{u \in N_0^{1,s}(\Omega) : u \geq 1 \text{ in a neighborhood of } E\}.$$

We call $A(E, \Omega)$ the set of *admissible functions for the condenser* (E, Ω). We define the relative s-capacity of the pair (E, Ω) by

$$\operatorname{cap}_s(E, \Omega) = \inf \left\{ \int_\Omega g_u^s \, d\mu : u \in A(E, \Omega) \right\}.$$

3. Main results

In this section we present the results about strong A_∞-weights. We prove the following theorem.

Theorem 3.1. *Let $1 < s \leq Q < \infty$ be fixed. We assume that $s > Q - 1$. Suppose (X, d, μ) is an Ahlfors Q-regular and geodesic unbounded metric measure space satisfying a weak $(1, s)$-Poincaré inequality. Let $u \in N_{loc}^{1,s}(X)$ be such that it has a minimal s-weak upper gradient g_u in the Morrey space $\mathcal{L}^{s,Q-s}(X, \mu)$. There exists a constant $\varepsilon > 0$ depending only on s and on the data of X such that if*

$$||g_u||_{\mathcal{L}^{s,Q-s}(X,\mu)} < \varepsilon,$$

then $w = e^{Qu}$ is a strong A_∞-weight with data depending only on s and on the data associated with X.

For $r \in (0, \infty)$ we define the Hausdorff r-content of a set $E \subset X$ by

$$\Lambda_r^\infty(E) = \inf\{\sum_i \operatorname{diam}(G_i)^r : E \subset \bigcup_i G_i\},$$

where the infimum is taken over all coverings of E by open sets G_i.

The following lemma is a generalization of [BHS1, Lemma 3.11]. We again thank Mario Bonk for his contribution to this result.

Lemma 3.2. *Suppose (X, d, μ) is a proper and unbounded geodesic Ahlfors Q-regular metric space admitting a weak $(1, Q)$-Poincaré inequality for some $1 < Q < \infty$. Suppose $0 < r \leq 1$. Let $x, y \in X$ and let $E \subset X$ be a bounded Borel set. Suppose that $B_1, \ldots, B_k$ are open balls such that $x \in B_1, y \in B_k$ and $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-1$. Then there exists a constant $c_1 = c_1(r)$ depending on the data of X with the following property: if*

$$\Lambda_r^\infty(E) \leq c_1\, d(x, y)^r, \tag{3.1}$$

then

$$\sum_{i \in \mathcal{G}_0} \operatorname{diam}(B_i)^r > \frac{1}{20^r} d(x, y)^r, \tag{3.2}$$

where

$$\mathcal{G}_0 = \left\{ i = 1, \ldots, k : \mu(E \cap B_i) \leq \frac{1}{2} \mu(B_i) \right\}. \tag{3.3}$$

Proof. We choose a family $\mathcal{I} \subset \{1, \ldots, k\}$ such that

$$B_i \cap B_j = \emptyset \text{ whenever } i \neq j \in \mathcal{I} \text{ and } \bigcup_{i=1}^{k} B_i \subset \bigcup_{i \in \mathcal{I}} 5B_i.$$

(See [Hei, Theorem 1.2].) For every $i = 1, \ldots, k-1$ let $x_i \in B_i \cap B_{i+1}$. We let $x_0 = x$ and $x_k = y$. Since X is geodesic, we have that for every $i = 1, \ldots, k$ there exists a rectifiable path γ_i in B_i connecting x_{i-1} and x_i. This yields a rectifiable path γ in $\cup_{i=1}^k B_i$ connecting x and y and therefore

$$\Lambda_r^\infty\Big(\bigcup_{i \in \mathcal{I}} 5B_i\Big) \geq \Lambda_r^\infty\Big(\bigcup_{i=1}^{k} B_i\Big) \geq d(x, y)^r. \tag{3.4}$$

We can assume without loss of generality that $\Lambda_r^\infty(E) > 0$. Let $(D_j)_{j \in \mathcal{J}}$ be a countable covering by open balls for E such that

$$\frac{1}{5} D_i \cap \frac{1}{5} D_j = \emptyset$$

whenever $i \neq j \in \mathcal{J}$ (see [Hei, Theorem 1.16]) and such that

$$\sum_{j \in \mathcal{J}} \operatorname{diam}(D_j)^r < 2^{r+1} \Lambda_r^\infty(E).$$

For every $i \in \mathcal{I}$ we define

$$\mathcal{F}_i = \{ j \in \mathcal{J} : D_j \cap B_i \neq \emptyset \}.$$

We denote

$$\mathcal{G} = \{i \in \mathcal{I} : \operatorname{diam}(D_j) < \operatorname{diam}(B_i) \text{ for all } j \in \mathcal{F}_i\} \text{ and } \mathcal{B} = \mathcal{I} \setminus \mathcal{G}.$$

Suppose $i \in \mathcal{B}$. Then there exists $j = j_i \in \mathcal{F}_i$ such that

$$D_{j_i} \cap B_i \neq \emptyset \text{ and } \operatorname{diam}(D_{j_i}) \geq \operatorname{diam}(B_i).$$

We notice that

$$5B_i \subset 14D_{j_i} \text{ and } \operatorname{diam}(5B_i) < 14 \operatorname{diam}(D_{j_i}).$$

Therefore

$$\begin{aligned} \Lambda_r^\infty\Big(\bigcup_{i\in\mathcal{B}} 5B_i\Big) &\leq \sum_{j\in\mathcal{J}} \operatorname{diam}(14D_j)^r \\ &\leq 28^r \sum_{j\in\mathcal{J}} \operatorname{diam}(D_j)^r < 28^r\, 2^{r+1}\Lambda_r^\infty(E). \end{aligned} \tag{3.5}$$

We let

$$\mathcal{G}_1 = \{i \in \mathcal{G} : \sum_{j\in\mathcal{F}_i} \operatorname{diam}(D_j)^r < c_0 \operatorname{diam}(B_i)^r\} \text{ and } \mathcal{G}_2 = \mathcal{G} \setminus \mathcal{G}_1$$

for some c_0 to be chosen later. We want to evaluate

$$\sum_{i\in\mathcal{G}_2} \operatorname{diam}(5B_i)^r.$$

Before we do that, we notice that there exists a number M depending only on the data of X such that every ball D_j intersects at most M pairwise disjoint balls B_i of bigger diameter. Therefore

$$\begin{aligned} \Lambda_r^\infty\Big(\bigcup_{i\in\mathcal{G}_2} 5B_i\Big) &\leq \sum_{i\in\mathcal{G}_2} \operatorname{diam}(5B_i)^r \leq 10^r \sum_{i\in\mathcal{G}_2} \operatorname{diam}(B_i)^r \\ &\leq c_0^{-1}\, 10^r \sum_{i\in\mathcal{G}_2} \left(\sum_{j\in\mathcal{F}_i} \operatorname{diam}(D_j)^r \right) \\ &\leq c_0^{-1}\, 10^r \sum_{i\in\mathcal{G}} \left(\sum_{j\in\mathcal{F}_i} \operatorname{diam}(D_j)^r \right) \\ &\leq c_0^{-1}\, M\, 10^r \sum_{j\in\mathcal{J}} \operatorname{diam}(D_j)^r \\ &< c_0^{-1}\, M\, 10^r\, 2^{r+1}\Lambda_r^\infty(E). \end{aligned} \tag{3.6}$$

We show now that if c_0 is taken small enough, then

$$\mu(E\cap B_i) \le \frac{1}{2}\mu(B_i) \text{ for every } i\in\mathcal{G}_1.$$

Indeed, for all $i\in\mathcal{G}_1$ we have

$$\begin{aligned}\mu(E\cap B_i) &\le \mu(\bigcup_{j\in\mathcal{F}_i} D_j\cap B_i) \le \sum_{j\in\mathcal{F}_i}\mu(D_j) \le c_\mu \sum_{j\in\mathcal{F}_i}\mathrm{diam}(D_j)^Q \\ &\le c_\mu \mathrm{diam}(B_i)^{Q-r}\sum_{j\in\mathcal{F}_i}\mathrm{diam}(D_j)^r \le c_0c_\mu\mathrm{diam}(B_i)^Q.\end{aligned}$$

So, if we let $c_0 = \frac{1}{2}c_\mu^{-2}2^{-Q}$, we get

$$\mu(E\cap B_i) \le \frac{1}{2}\mu(B_i) \text{ for every } i\in\mathcal{G}_1.$$

¿From (3.4), (3.5), (3.6), the subadditivity of Λ_r^∞, and the fact that $\mathcal{I}=\mathcal{G}_1\cup\mathcal{G}_2\cup\mathcal{B}$, it follows that

$$d(x,y)^r \le \Lambda_r^\infty(\bigcup_{i\in\mathcal{I}}5B_i) < \sum_{i\in\mathcal{G}_1}\mathrm{diam}(5B_i)^r + (28^r + c_0^{-1}\,M\,10^r)\,2^{r+1}\Lambda_r^\infty(E).$$

If we choose c_1 such that $2^{r+1}(28^r+c_0^{-1}\,M\,10^r)\,c_1 = 1-2^{-r}$, then we notice that

$$10^r\sum_{i\in\mathcal{G}_1}\mathrm{diam}(B_i)^r \ge \sum_{i\in\mathcal{G}_1}\mathrm{diam}(5B_i)^r > 2^{-r}d(x,y)^r$$

whenever $\Lambda_r^\infty(E)\le c_1 d(x,y)^r$. Since $\mathcal{G}_1\subset\mathcal{G}_0$, this finishes the proof of Lemma 3.2. □

Lemma 3.3. *Suppose* $1 < s, Q < \infty$. *Suppose that* (X,d,μ) *is a complete and unbounded Ahlfors Q-regular metric measure space that satisfies a weak* $(1,s)$-*Poincaré inequality. Let* $\Omega\subset\subset X$ *be open and let* $E\subset\Omega$. *Suppose* $u\in N_0^{1,s}(\Omega)$ *is compactly supported in* Ω. *If* $u\ge 1$ *on* E, *then*

$$\mathrm{cap}_s(E,\Omega) \le \int_\Omega g_u^s\,d\mu.$$

Proof. Since $u\in N_0^{1,s}(\Omega)$ is compactly supported in Ω, there exists a sequence $\varphi_j\in Lip_0(\Omega)$ converging to u in $N^{1,s}(X)$. Without loss of generality we can assume that all the functions φ_j are supported in an open set $U\subset\subset\Omega$ and that the sequence φ_j converges to u pointwise μ-a.e. Since φ_j is a Cauchy sequence in $N_0^{1,s}(\Omega)$, there is a subsequence, denoted again by φ_j, such that

$$||\varphi_j-\varphi_{j+1}||_{N^{1,s}(X)} < 2^{-2j} \text{ for every } j\ge 1.$$

For the open set

$$E_j = \{x \in X : |\varphi_j(x) - \varphi_{j+1}(x)| > 2^{-j}\}$$

we have

$$C_s(E_j) \le 2^{js} ||\varphi_j - \varphi_{j+1}||^s_{N^{1,s}(X)} < 2^{-js}.$$

If we put

$$G_j = \bigcup_{k=j}^{\infty} E_k,$$

we have from the subadditivity of the s-capacity that

$$C_s(G_j)^{1/s} \le \sum_{k=j}^{\infty} C_s(E_k)^{1/s} \le \sum_{k=j}^{\infty} 2^{-k} = 2^{1-j}.$$

Thus the sequence φ_j converges uniformly outside open sets of arbitrarily small s-capacity to a quasicontinuous function v and we can assume without loss of generality that $v = 0$ on $X \setminus U$. Moreover, $v \in N_0^{1,s}(\Omega)$ because $N_0^{1,s}(\Omega)$ is a Banach space. On the other hand, φ_j converges to u μ-a.e. in X. Thus u and v are two functions in $N^{1,s}(X)$ that agree μ-a.e., hence they agree s-q.e. on X. We let

$$E_0 = \{x \in X : u(x) \neq v(x)\} \text{ and } E_1 = E \setminus E_0.$$

We fix $\varepsilon \in (0,1)$. We choose an open set $G \subset U$ such that $C_s(G) < \varepsilon$ and $\varphi_j \to v$ uniformly on $X \setminus G$. We let

$$\widetilde{G}_j = \{x \in X : \varphi_j(x) > 1 - \varepsilon\}.$$

Then $\widetilde{G}_j$ is open and

$$E_1 \setminus G \subset \widetilde{G}_j \text{ for } j \ge j_\varepsilon.$$

Consequently, for $j \ge j_\varepsilon$ we have via the subadditivity of the relative s-capacity (see [Cos2, Theorem 3.2 (vi)])

$$\mathrm{cap}_s(E, \Omega) = \mathrm{cap}_s(E_1, \Omega) \le \mathrm{cap}_s(\widetilde{G}_j, \Omega) + \mathrm{cap}_s(G, \Omega).$$

Since $\varphi_j > 1 - \varepsilon$ on $\widetilde{G}_j$, we have

$$\mathrm{cap}_s(\widetilde{G}_j, \Omega) \le (1-\varepsilon)^{-s} \int_\Omega g^s_{\varphi_j} \, d\mu,$$

and hence by letting $j \to \infty$, we obtain

$$\mathrm{cap}_s(E, \Omega) \le (1-\varepsilon)^{-s} \int_\Omega g^s_u \, d\mu + \varepsilon.$$

By letting $\varepsilon \to 0$ we obtain the desired conclusion. □

We prove Theorem 3.1 now.

Proof. Since μ is a Borel and Ahlfors Q-regular measure and X is a geodesic metric space that satisfies a weak $(1,s)$-Poincaré inequality for some s in $(1,Q]$, it follows from [HaK, Theorem 5.1], [KZ, Theorem 1.0.1] and [Hei, Theorem 4.18] that (X,d,μ) satisfies in fact a (s,s)-Poincaré inequality with $\lambda = 1$.

We have that $u \in N^{1,s}_{loc}(X)$ has a minimal s-weak upper gradient g_u in the Morrey space $\mathcal{L}^{s,Q-s}(X,\mu)$, hence we can assume without loss of generality that u is a Borel s-quasicontinuous function.

Since g_u has small $\mathcal{L}^{s,Q-s}(X,\mu)$ norm, it follows from (1.6) that u has small $\mathrm{BMO}(X,\mu)$ seminorm. Therefore, from John-Nirenberg lemma, it follows that $w(x) = e^{Qu(x)}$ is an A_∞-density with respect to μ for some doubling measure ν with data depending on X. That is, (see [MP, Theorem 1.4], [MMNO, Theorem A], and [Buc, Theorem 2.2]), there exists a constant C depending on s and on data of X such that

$$\frac{1}{\mu(B)}\int_B e^{Q(u(x)-u_B)}\,d\mu(x) < C \text{ and } \int_{2B} w(x)\,d\mu(x) \le C\int_B w(x)\,d\mu(x) \tag{3.7}$$

for every ball $B \subset X$. We write $d\nu(x) = w(x)\,d\mu(x)$. We recall the definition of δ_ν from (1.4). We shall show that there exists a constant $C \in (0,1]$ such that

$$d_\nu(x_1,x_2) := \inf \sum_{i=1}^{k} \nu(B_i)^{1/Q} \ge C\delta_\nu(x_1,x_2) \tag{3.8}$$

for all $x_1, x_2 \in X$, where the infimum is taken over finite chains of open balls connecting x_1 and x_2 satisfying

$$x_1 \in B_1, x_2 \in B_k \text{ and } B_i \cap B_{i+1} \neq \emptyset \text{ for all } i = 1,\ldots,k-1. \tag{3.9}$$

Indeed, (3.8) implies both that d_ν is a distance and that is comparable to δ_ν as required in (1.5). Towards this end, fix $x_1, x_2 \in X$, $x_1 \neq x_2$. Let γ be a geodesic segment connecting x_1 and x_2, and let a be the midpoint of γ. We denote $R = d(x_1,x_2)$ and $B = B(a,R)$.

Let $\eta \in Lip_0(6B)$ be a nonnegative $1/R$-Lipschitz function such that $\eta = 1$ on $3B$. Since u is s-quasicontinuous and Borel, it follows that $v(x) = \eta(x)\,|u(x)-u_{3B}|$ is a Borel s-quasicontinuous function in $N^{1,s}_0(6B)$ compactly supported in $6B$.

Let $E = \{x \in 3B : |u(x) - u_{3B}| > 1\}$. We have that E is a Borel set since u is a Borel function. Since v is an s-quasicontinuous function in $N^{1,s}_0(6B)$ compactly supported in $6B$, we have from the (s,s)-Poincaré inequality with $\lambda = 1$ and

Lemma 3.3 that

$$\begin{aligned} \operatorname{cap}_s(E, 6B) &\leq \int_{6B} g_v^s \, d\mu \leq \int_{6B} (\eta \, g_u + |u - u_{3B}| \, g_\eta)^s \, d\mu \\ &\leq C \int_{6B} g_u^s \, d\mu \leq C \, (6R)^{Q-s} ||g_u||^s_{\mathcal{L}^{s,Q-s}(X,\mu)}. \end{aligned}$$

This implies that

$$\frac{\operatorname{cap}_s(E, 6B)}{(6R)^{Q-s}} \leq C \, ||g_u||^s_{\mathcal{L}^{s,Q-s}(X,\mu)}, \tag{3.10}$$

which together with [Cos2, Theorem 4.4] yields

$$\frac{\Lambda_1^\infty(E)}{R} \leq C \, \frac{\operatorname{cap}_s(E, 6B)}{(6R)^{Q-s}} \leq C_0 ||g_u||^s_{\mathcal{L}^{s,Q-s}(X,\mu)}. \tag{3.11}$$

We choose $\varepsilon > 0$ such that $C_0 \, \varepsilon^s < c_1$ where c_1 is the constant from (3.1) and C_0 is the constant from the last inequality in (3.11).

Now let $B_1, \ldots, B_k$ be an arbitrary chain of balls connecting x_1 and x_2 as in (3.9). We assume first that $B_i \subset 3B$ for all $i = 1, \ldots, k$. Let $\mathcal{G}_0$ be defined like in (3.3). We have

$$\begin{aligned} \sum_{i=1}^k \nu(B_i)^{1/Q} &\geq \sum_{i\in\mathcal{G}_0} \nu(B_i)^{1/Q} \geq \sum_{i\in\mathcal{G}_0} \nu(B_i \setminus E)^{1/Q} \\ &= \sum_{i\in\mathcal{G}_0} \left(\int_{B_i\setminus E} e^{Qu(x)} \, d\mu(x) \right)^{1/Q} \geq \sum_{i\in\mathcal{G}_0} \left(\int_{B_i\setminus E} e^{Q(u_{3B}-1)} \, d\mu(x) \right)^{1/Q} \\ &= e^{u_{3B}-1} \left(\sum_{i\in\mathcal{G}_0} \mu(B_i \setminus E)^{1/Q} \right) \geq e^{u_{3B}-1} \sum_{i\in\mathcal{G}_0} \left(\frac{1}{2} \mu(B_i) \right)^{1/Q} \\ &\geq C e^{u_{3B}} \mu(3B)^{1/Q}. \end{aligned} \tag{3.12}$$

From (3.7), (3.12), and the definition of δ_ν there exists C such that

$$\sum_{i=1}^k \nu(B_i)^{1/Q} \geq C \left(\int_{3B} e^{Qu(x)} \, d\mu(x) \right)^{1/Q} \geq C \delta_\nu(x_1, x_2).$$

Next, if the chain (B_i) does not lie entirely in $3B$, then there exists a smallest number k' with $1 \leq k' \leq k$ such that $B_{k'} \cap S(a, 2R) \neq \emptyset$. Let x_0 be a point in $B_{k'} \cap S(a, 2R)$. Then $B_1, \ldots, B_{k'}$ is a chain of balls connecting x_1 and x_0 and $d(x_1, x_2) \leq d(x_1, x_0)$. If $B_{k'} \subset 3B$, then from the fact that $x_1 \in B_1 \cap 2B$ and from the definition of k' it follows that the subchain $B_1, \ldots, B_{k'}$ is contained in $3B$. Therefore we can apply the preceding argument to the chain $B_1, \ldots, B_{k'}$

connecting the points x_1 and x_0 to conclude that (3.8) holds; in the opposite case, diam $B_{k'} \geq R$. The doubling condition for ν then implies $\nu(B) \leq C\nu(B_{k'})$. Thus, (3.8) is true in all cases. This finishes the proof of Theorem 3.1. □

Theorem 3.1 yields the following consequence:

Theorem 3.4. *Let $1 < s \leq Q < \infty$ be fixed. We assume that $s > Q-1$. Suppose (X, d, μ) is a metric measure space as in Theorem 3.1. Let u be a Borel function in $N^{1,s}_{loc}(X)$ such that it has a minimal s-weak upper gradient g_u in the Morrey space $\mathcal{L}^{s,Q-s}(X,\mu)$. There exists a constant $\varepsilon > 0$ depending only on s and on the data of X such that if*

$$||g_u||_{\mathcal{L}^{s,Q-s}(X,\mu)} < \varepsilon,$$

then

$$\delta_\nu(x_1, x_2) \leq CD_\nu(x_1, x_2) \text{ for all } x_1, x_2 \text{ in } X, \tag{3.13}$$

where $d\nu(x) = e^{Qu(x)}\, d\mu(x)$, $C > 0$ is a constant depending only on s and on the data associated with X and

$$D_\nu(x,y) = \inf\left\{ \int_\gamma e^u ds : \ \gamma \text{ a rectifiable curve connecting } x, y \right\}. \tag{3.14}$$

One should compare the metrics D_ν in Theorem 3.4 to those studied in [BL], [Res] and [Cos1].

Question 3.5. Another open question is whether or not the inequality (3.13) can be reversed in general. The answer is yes in the Euclidean setting when $n \geq 2$. (See [Cos1, Theorem 5.4].)

Now we prove Theorem 3.4.

Proof. It is easy to see that D_ν is indeed symmetric, nonnegative and satisfies the triangle inequality. From (3.13) it would follow immediately that D_ν is a distance function dominating δ_ν. So fix x_1, x_2 in X. We can assume without loss of generality that $x_1 \neq x_2$. Like before, let a be a point such that $d(x_1, a) = d(a, x_2) = R/2$, where $R = d(x_1, x_2)$. We denote $B = B(a, R)$. Like in the proof of Theorem 3.1, let

$$v = \eta|u - u_{3B}| \text{ and } E = \{x \in 3B : |u(x) - u_{3B}| > 1\},$$

where $\eta \in Lip_0(6B)$ is a nonnegative $1/R$-Lipschitz function such that $\eta = 1$ on $3B$. We notice that E is a Borel set and v is a Borel and s-quasicontinuous function in $N^{1,s}_0(6B)$ compactly supported in $6B$.

Let γ be a rectifiable curve connecting x_1 and x_2 and let $|\gamma|$ be its image. We assume first that $|\gamma| \subset 3B$. We obviously have

$$\int_\gamma e^u ds \geq \int_{\gamma\cap(3B\setminus E)} e^u ds. \tag{3.15}$$

As in the proof of Theorem 3.1, we have

$$\frac{\Lambda_1^\infty(E)}{R} \leq C\,\frac{\mathrm{cap}_s(E,6B)}{(6R)^{Q-s}} \leq C_0 ||g_u||^s_{\mathcal{L}^{s,Q-s}(X,\mu)} < C_0\,\varepsilon^s,$$

hence

$$\begin{aligned}\Lambda_1^\infty(|\gamma|\cap(3B\setminus E)) &\geq \Lambda_1^\infty(|\gamma|\cap 3B) - \Lambda_1^\infty(|\gamma|\cap E)\\ &\geq R - \Lambda_1^\infty(E) \geq (1-c_1)R\end{aligned}$$

if $\varepsilon > 0$ is small enough, where c_1 is the constant from (3.1).

Thus we obtain

$$\begin{aligned}\int_\gamma e^u ds &\geq \int_{\gamma\cap(3B\setminus E)} e^u ds \geq \int_{\gamma\cap(3B\setminus E)} e^{u_{3B}-1} ds\\ &\geq \Lambda_1^\infty(|\gamma|\cap(3B\setminus E))e^{u_{3B}-1} \geq C\,R\,e^{u_{3B}}\\ &\geq C\left(\int_{3B} e^{Qu_{3B}}\,d\mu(x)\right)^{1/Q}\\ &\geq C\left(\int_{3B} e^{Qu(x)}\,d\mu(x)\right)^{1/Q},\end{aligned} \tag{3.16}$$

where the last inequality follows from (3.7). Therefore

$$\int_\gamma e^u ds \geq C\left(\int_{3B} e^{Qu(z)}\,d\mu(z)\right)^{1/Q} \geq C\delta_\nu(x_1,x_2) \text{ whenever } |\gamma|\subset 3B. \tag{3.17}$$

Now we assume that $|\gamma|\setminus 3B \neq \emptyset$. Suppose that γ is parameterized by its arc length parametrization. Let $t_0 = \inf\{t\in[0,l_\gamma] : \gamma(t)\notin B(a,2R)\}$. Then, since γ is a path with $\gamma(0)$, $\gamma(l_\gamma)\in 2B$, it follows that $0 < t_0 < l_\gamma$ and $\gamma([0,t_0]) \subset \overline{B}(a,2R)$. Let $x_0 = \gamma(t_0)$ and let $\widetilde{\gamma}$ be the restriction of γ to $[0,t_0]$. Then $x_0 \in S(a,2R)$ and $d(x_1,x_2) \leq d(x_1,x_0)$. Therefore

$$\begin{aligned}\Lambda_1^\infty(|\gamma|\cap(3B\setminus E)) &\geq \Lambda_1^\infty(|\widetilde{\gamma}|\cap(3B\setminus E)) \geq \Lambda_1^\infty(|\widetilde{\gamma}|) - \Lambda_1^\infty(|\widetilde{\gamma}|\cap E)\\ &\geq d(x_1,x_0) - \Lambda_1^\infty(E) \geq (1-c_1)R\end{aligned}$$

if $\varepsilon > 0$ is small enough, where c_1 is the constant from (3.1). By repeating the argument from (3.16) with $\widetilde{\gamma}$ instead of γ, we obtain

$$\int_{\widetilde{\gamma}} e^u ds \geq C \left(\int_{3B} e^{Qu(z)}\, d\mu(z) \right)^{1/Q}.$$

This finishes the proof of Theorem 3.4. □

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Received September 15, 2008

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