BESOV CAPACITY AND HAUSDORFF MEASURES IN METRIC MEASURE SPACES

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Abstract. This paper studies Besov p-capacities as well as their relationship to Hausdorff measures in Ahlfors regular metric spaces of dimension Q for $1 < Q < p < \infty$. Lower estimates of the Besov p-capacities are obtained in terms of the Hausdorff content associated with gauge functions h satisfying the decay condition $\int_0^1 h(t)^{1/(p-1)} \frac{dt}{t} < \infty$.

1. Introduction

In this paper (X, d, μ) is a proper (that is, closed bounded subsets of X are compact) and unbounded metric space. In addition, it is Ahlfors Q-regular for some $Q > 1$. That is, there exists a constant $C = c_\mu$ such that, for each $x \in X$ and all $r > 0$,

$$C^{-1} r^Q \leq \mu(B(x,r)) \leq C r^Q.$$

For $1 < Q, p < \infty$ we define

$$B_p(X) = \{u \in L^p(X) : ||u||_{B_p(X)} < \infty\},$$

where

$$||u||_{B_p(X)} = ||u||_{L^p(X)} + [u]_{B_p(X)} \tag{1}$$

with

$$[u]_{B_p(X)} = \left(\int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} \, d\mu(x) \, d\mu(y) \right)^{1/p}. \tag{2}$$

The expressions $||u||_{B_p(X)}$ and $[u]_{B_p(X)}$ from (1) and (2) are called the *Besov norm* and the *Besov seminorm* of u respectively. We have

$$[u]_{B_p(X)} = 0 \text{ if and only if } u \text{ is constant } \mu\text{-a.e.} \tag{3}$$

Besov spaces were studied extensively on Ahlfors Q-regular subsets of $\mathbf{R}^n$ by Jonsson and Wallin in [JW84]. In [JW84] the authors studied mainly trace and extension results. Embeddings of homogeneous Besov spaces have recently been studied by Xiao in [Xia06].

Recently Besov spaces have been used in the study of quasiconformal mappings in metric spaces and in geometric group theory. See [BP03] and [Bou07].

Capacities associated with Besov spaces were studied by Netrusov in [Net92] and [Net96], by Adams and Xiao in [AX03], and by Adams and Hurri-Syrjänen in [AHS03]. Bourdon in [Bou07] studied Besov B_p-capacity in the metric setting under the assumption that X is compact. He worked with functions from $A_p(X)$, the algebra of

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continuous functions that are in $B_p(X)$. The algebra $A_p(X)$ does not separate in general the points of X when $1 \le p \le Q < \infty$. See [Bou07], [BP03], and [Bou04].

In this paper we assume that $1 < Q < p < \infty$ unless stated otherwise. Under the assumption $1 < Q < p < \infty$ we develop a theory of Besov B_p-capacity on X and prove that this capacity is a Choquet set function. We also obtain lower bounds for the relative Besov capacity in terms of the Hausdorff content associated with gauge functions h satisfying a certain integrability condition under the additional hypothesis that X admits a weak $(1,\widetilde{p})$-Poincaré inequality with $1 \le \widetilde{p} < Q < p < \infty$. Some of the ideas used here follow [KM96], [KM00], [BP03], and [Bou07].

2. Preliminaries

In this section we present the standard notation to be used throughout this paper. Here and throughout this paper $B(x,r) = \{y \in X : d(x,y) < r\}$ is the open ball with center $x \in X$ and radius $r > 0$, $\overline{B}(x,r) = \{y \in X : d(x,y) \le r\}$ is the closed ball with center $x \in X$ and radius $r > 0$, while $S(x,r) = \{y \in X : d(x,y) = r\}$ is the closed sphere with center $x \in X$ and radius $r > 0$. For a positive number λ, $\lambda B(a,r) = B(a,\lambda r)$ and $\lambda \overline{B}(a,r) = \overline{B}(a,\lambda r)$.

Under the hypothesis of Ahlfors regularity, it is known that the metric measure space (X,d,μ) is proper if and only if it is complete. Throughout this paper (X,d,μ) is assumed to be a complete and unbounded Ahlfors Q-regular metric space for some $Q > 1$.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. $C(a,b,\dots)$ is a constant that depends only on the parameters $a,b,\dots$. Here Ω will denote a nonempty open subset of X. For $E \subset X$, the boundary, the closure, and the complement of E with respect to X will be denoted by ∂E, $\overline{E}$, and $X \setminus E$, respectively; diam E is the diameter of E with respect to the metric d and $E \subset\subset F$ means that $\overline{E}$ is a compact subset of F.

For two sets $A, B \subset X$, we define $\operatorname{dist}(A,B)$, the distance between A and B, by

$$\operatorname{dist}(A,B) = \inf_{a\in A, b\in B} d(a,b).$$

For $\Omega \subset X$, $C(\Omega)$ is the set of all continuous functions $u : \Omega \to \mathbf{R}$. Moreover, for a measurable $u : \Omega \to \mathbf{R}$, supp u is the smallest closed set such that u vanishes on the complement of supp u. We also use the spaces

$$\begin{aligned} C_0(\Omega) &= \{\varphi \in C(\Omega) : \operatorname{supp} \varphi \subset\subset \Omega\}, \\ \operatorname{Lip}(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz}\}, \\ \operatorname{Lip}_{loc}(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is locally Lipschitz}\}, \\ \operatorname{Lip}_0(\Omega) &= \operatorname{Lip}(\Omega) \cap C_0(\Omega). \end{aligned}$$

Let $f : \Omega \to \mathbf{R}$ be integrable. For $E \subset \Omega$ measurable with $0 < \mu(E) < \infty$, we define

$$f_E = \frac{1}{\mu(E)} \int_E f \, d\mu(x).$$

We say that a locally integrable function $u : X \to \mathbf{R}$ belongs to $\mathrm{BMO}(X)$, the space of functions of bounded mean oscillation, if

$$[u]_{\mathrm{BMO}(X)} = \sup_{a\in X} \sup_{r>0} \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |u - u_{B(a,r)}| dx < \infty.$$

3. Besov spaces

In this section we prove some basic properties of the Besov spaces $B_p(X)$ and their closed subspaces $B_p(\Omega)$ and $B_p^0(\Omega)$, where $\Omega \subset X$ is an open set. We also present standard lemmas needed for the proofs of our main results.

We know that in the Euclidean case $B_p(\mathbf{R}^n)$ is a reflexive Banach space and moreover, $\mathcal{S}$ is dense in $B_p(\mathbf{R}^n)$ where $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ is the Schwartz class. See [AH96, Theorem 4.1.3] and [Pee76, Chapter 3]. We would like to prove similar results about reflexivity and density when (X, d, μ) is an Ahlfors Q-regular metric space with $Q > 1$. It is easy to see that every Lipschitz function with compact support belongs to $B_p(X)$ whenever X is proper and $1 < Q < p < \infty$.

We have the following lemmas regarding the reflexivity of $B_p(X)$ and the embedding of $B_p(X)$ into $\mathrm{BMO}(X)$ whenever (X, d, μ) is an Ahlfors Q-regular metric space with $1 < Q, p < \infty$.

Lemma 3.1. *Suppose $1 < Q, p < \infty$ and that X is an Ahlfors Q-regular metric space. Then $B_p(X)$ is a reflexive space.*

Proof. Let ν be a measure on the product space $X \times X$ given by

$$d\nu(x, y) = d(x, y)^{-2Q} d\mu(x) d\mu(y).$$

We endow the product space $L^p(X, \mu) \times L^p(X \times X, \nu)$ with the product norm. Namely, for $(u, g) \in L^p(X, \mu) \times L^p(X \times X, \nu)$ we let

$$||(u, g)||_{L^p(X,\mu)\times L^p(X\times X,\nu)} = ||u||_{L^p(X,\mu)} + ||g||_{L^p(X\times X,\nu)}.$$

Clearly this product space is reflexive because it is a product of two reflexive spaces. Since $B_p(X)$ embeds isometrically into a closed subspace of this reflexive product space, we have that $B_p(X)$ is itself a reflexive space. This finishes the proof. □

Lemma 3.2. *Suppose $1 < Q, p < \infty$ and that X is an Ahlfors Q-regular metric space. There exists a constant $C = C(Q, p, c_\mu)$ such that $[u]_{\mathrm{BMO}(X)} \le C[u]_{B_p(X)}$ whenever $u \in L^1_{loc}(X)$.*

Proof. Indeed, let $u \in L^1_{loc}(X)$ be such that $[u]_{B_p(X)} < \infty$. Suppose that $B = B(a, R)$ is a ball in X. It is easy to see that there exists a constant $C = C(Q, c_\mu)$ such that

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |u(x) - u_B|^p d\mu(x) &\le \frac{1}{\mu(B)^2} \int_B \int_B |u(x) - u(y)|^p d\mu(x) d\mu(y) \\ &\le C \int_B \int_B \frac{|u(x) - u(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y). \end{aligned} \tag{4}$$

Therefore,

$$[u]_{\mathrm{BMO}(X)} \le C(Q, p, c_\mu)[u]_{B_p(X)} \tag{5}$$

and the claim follows. □

Remark 3.3. We notice that in the above two lemmas the claims hold for general Q and p in $(1, \infty)$.

From now on throughout the remainder of the paper it will be assumed that $1 < Q < p < \infty$.

For an open set $\Omega \subset X$ we define

$$B_p(\Omega) = \{u \in B_p(X) : u = 0\ \mu\text{-a.e. in } X \setminus \Omega\}.$$

For a function $u \in B_p(\Omega)$ we let $||u||_{B_p(\Omega)} = ||u||_{B_p(X)}$.

We notice that $B_p(\Omega)$ is a closed subspace of $B_p(X)$ with respect to the Besov norm, hence it is itself a reflexive space.

We define $B^0_p(\Omega)$ as the closure of $\mathrm{Lip}_0(\Omega)$ in $B_p(X)$. Since $\mathrm{Lip}_0(\Omega) \subset B_p(\Omega)$, it follows that $B^0_p(\Omega) \subset B_p(\Omega)$, so we can say that $B^0_p(\Omega)$ is the closure of $\mathrm{Lip}_0(\Omega)$ in $B_p(\Omega)$.

Lemma 3.4. *$B_p(\Omega)$ is closed under truncations. In particular, bounded functions in $B_p(\Omega)$ are dense in $B_p(\Omega)$.*

Proof. As in the proof of [Cos07, Lemma 2.1], it is not difficult to see that $v_{(\lambda)} = \min(v, \lambda) \in B_p(\Omega)$ for every $\lambda \geq 0$. Indeed, we have $||v_{(\lambda)}||_{L^p(X)} \leq ||v||_{L^p(X)}$ and $[v_{(\lambda)}]_{B_p(\Omega)} \leq [v]_{B_p(\Omega)}$.

To prove the second assertion, for positive integers k we define the function v_k by $v_k = \max(-k, \min(v, k))$. From the first assertion it follows that $v_k \in B_p(\Omega)$ with $||v_k||_{B_p(\Omega)} \leq ||v||_{B_p(\Omega)}$. Furthermore, we have $|v_k(x)| \leq |v(x)|$ for every $x \in X$ and from the Lebesgue Dominated Convergence Theorem it follows that $||v_k - v||_{L^p(\Omega)} \to 0$. We also notice that $|v_k(x) - v_k(y)| \leq |v(x) - v(y)|$ for every $x, y \in X$ and since $|(v_k(x) - v_k(y)) - (v(x) - v(y))| \to 0$ for almost every $(x, y) \in X \times X$, it follows from the Lebesgue Dominated Convergence Theorem that $[v_k - v]_{B_p(\Omega)} \to 0$ as $k \to \infty$. □

For a measurable function $u : \Omega \to \mathbf{R}$, we let $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$.

Lemma 3.5. *If $u_j \to u$ in $B_p(\Omega)$ and $v_j \to v$ in $B_p(\Omega)$, then $\min(u_j, v_j) \to \min(u, v)$ in $B_p(\Omega)$.*

Proof. The proof is similar to the proof of [Cos07, Lemma 2.2] and omitted. □

Next we show that the space $B^0_p(\Omega)$ is a lattice.

Lemma 3.6. *If $u, v \in B^0_p(\Omega)$, then $\min(u, v)$ and $\max(u, v)$ are in $B^0_p(\Omega)$. Moreover, if $u \in B^0_p(\Omega)$ is nonnegative, then there is a sequence of nonnegative functions $\varphi_j \in \mathrm{Lip}_0(\Omega)$ converging to u in $B_p(\Omega)$.*

Proof. It is enough to show, due to Lemma 3.5, that u^+ is in $B^0_p(\Omega)$ whenever u is in $\mathrm{Lip}_0(\Omega)$. But this is immediate, because $u^+ \in \mathrm{Lip}_0(\Omega)$ whenever $u \in \mathrm{Lip}_0(\Omega)$. This finishes the proof. □

Lemma 3.7. *Let φ be a Lipschitz function with compact support in X. If $u \in B_p(X)$, then $u\varphi \in B_p(X)$ with*

$$||u\varphi||_{B_p(X)} \leq C\, ||u||_{B_p(X)},$$

where C depends on Q, p, c_μ, the Lipschitz constant of φ, and the diameter of $\operatorname{supp} \varphi$.

Proof. We can assume without loss of generality that $\varphi \not\equiv 0$. Let R be the diameter of $\operatorname{supp} \varphi$. We choose $x_0 \in \operatorname{supp} \varphi$ such that $\operatorname{supp} \varphi \subset \overline{B}$, where $B = B(x_0, R)$. Let $L > 0$ be a constant such that $|\varphi(x) - \varphi(y)| \leq L d(x, y)$ for every $x, y \in X$. Note that $||\varphi||_{L^\infty(X)} \leq 2LR$. We also notice that

$$||u\varphi||_{L^p(X)} \leq ||\varphi||_{L^\infty(X)}\, ||u||_{L^p(X)},$$

hence $u\varphi \in L^p(X)$. Observe that

$$\int_X \int_X \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y) = I_1 + 2I_2,$$

where

$$I_1 = \int_{2B} \int_{2B} \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y) \tag{6}$$

and

$$I_2 = \int_{2B} \int_{X \setminus 2B} \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y). \tag{7}$$

For every $x, y \in X$ we have

$$|u(x)\varphi(x) - u(y)\varphi(y)| \leq |u(x) - u(y)|\, |\varphi(x)| + |u(y)|\, |\varphi(x) - \varphi(y)|.$$

Therefore

$$I_1 \leq 2^p(||\varphi||^p_{L^\infty(X)} [u]^p_{B_p(X)} + I_{11}), \tag{8}$$

where

$$I_{11} = \int_{2B} \int_{2B} \frac{|u(y)|^p\, |\varphi(x) - \varphi(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y).$$

From the definition of I_{11} we have, since φ is Lipschitz with constant L,

$$\begin{aligned} I_{11} &\leq \int_{2B} \int_{2B} \frac{L^p\, |u(y)|^p}{d(x,y)^{2Q-p}} d\mu(x)\, d\mu(y) \\ &= L^p \int_{2B} |u(y)|^p \left(\int_{2B} d(x,y)^{p-2Q} d\mu(x) \right) d\mu(y). \end{aligned} \tag{9}$$

We have

$$\int_{2B} d(x,y)^{p-2Q} d\mu(x) \leq C(Q, p, c_\mu) R^{p-Q} \tag{10}$$

for every $y \in 2B$, where we recall that R is the radius of B. From (9) and (10) we get

$$\begin{aligned} I_{11} &\leq C(Q, p, c_\mu)\, L^p\, R^{p-Q} \int_{2B} |u(y)|^p d\mu(y) \\ &\leq C(Q, p, c_\mu)\, L^p\, R^{p-Q}\, ||u||^p_{L^p(X)}. \end{aligned} \tag{11}$$

Since φ is supported in B, it follows from the definition of I_2 that

$$I_2 = \int_B \int_{X \setminus 2B} \frac{|u(y)|^p\, |\varphi(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y).$$

Hence

$$I_2 \leq ||\varphi||^p_{L^\infty(X)} \int_B \int_{X \setminus 2B} \frac{|u(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

and since $d(x,y) \geq \frac{d(x,x_0)}{2}$ whenever $x \in X \setminus 2B$ and $y \in B$, we get

$$I_2 \leq 2^{2Q}\, ||\varphi||^p_{L^\infty(X)} \int_B |u(y)|^p\, d\mu(y) \int_{X \setminus 2B} \frac{1}{d(x,x_0)^{2Q}} d\mu(x).$$

Hence

$$I_2 \le C(Q,p,c_\mu)\,||\varphi||^p_{L^\infty(X)}\,R^{-Q}\int_B |u(y)|^p d\mu(y) \le C(Q,p,c_\mu)\,||\varphi||^p_{L^\infty(X)}\,R^{-Q}\,||u||^p_{L^p(X)}. \tag{12}$$

From (8), (11), (12), and the fact that $I = I_1 + 2I_2$, we get that $u\varphi \in B_p(X)$ with

$$||u\varphi||_{B_p(X)} \le C||u||_{B_p(X)}, \tag{13}$$

where the constant C is as required. This finishes the proof. □

Lemma 3.8. *Let φ be a Lipschitz function with compact support in X. Suppose u_k is a sequence in $B_p(X)$ converging to u in $B_p(X)$. Then $u_k\varphi$ converges to $u\varphi$ in $B_p(X)$.*

Proof. From Lemma 3.7, we have that $u_k\varphi \in B_p(X)$ for every $k \ge 1$ and $u\varphi \in B_p(X)$. Moreover, Lemma 3.7 implies

$$||u_k\varphi - u\varphi||_{B_p(X)} \le C||u_k - u||_{B_p(X)} \tag{14}$$

for every $k \ge 1$, and since $u_k \to u$ in $B_p(X)$, it follows that $u_k\varphi \to u\varphi$ in $B_p(X)$. This finishes the proof. □

Remark 3.9. Let $\Omega, \widetilde{\Omega}$ be bounded and open subsets of X with $\Omega \subset\subset \widetilde{\Omega}$. Suppose that φ is a function in $\mathrm{Lip}_0(\widetilde{\Omega})$ with Lipschitz constant $C(Q,c_\mu)/\mathrm{dist}(\Omega, X\setminus\widetilde{\Omega})$ such that

$$0 \le \varphi \le 1 \text{ and } \varphi = 1 \text{ in } \Omega. \tag{15}$$

By an argument similar to the one from Lemma 3.7, one can show that $u\varphi \in B_p(\widetilde{\Omega})$ whenever $u \in B_p(X)$ and $\varphi \in \mathrm{Lip}_0(\widetilde{\Omega})$ satisfies (15). Moreover, in this case

$$||u\varphi||_{B_p(\widetilde{\Omega})} \le C||u||_{B_p(X)}$$

for all $u \in B_p(X)$ and the constant $C > 0$ can be chosen to depend only on Q, p, c_μ, $\mathrm{dist}(\Omega, X\setminus\widetilde{\Omega})$, and the diameter of $\widetilde{\Omega}$.

Remark 3.10. It is easy to see that $u\varphi \in B_p(X)$ whenever u, φ are bounded functions in $B_p(X)$. Moreover,

$$||u\varphi||_{L^p(X)} \le \min(||u||_{L^\infty(X)}||\varphi||_{L^p(X)},\, ||\varphi||_{L^\infty(X)}||u||_{L^p(X)})$$

and

$$[u\varphi]_{B_p(X)} \le ||u||_{L^\infty(X)}[\varphi]_{B_p(X)} + ||\varphi||_{L^\infty(X)}[u]_{B_p(X)}.$$

Lemma 3.11. *Let $B = B(x_0,R) \subset X$ and η be a $C(c_\mu)/R$-Lipschitz function supported in $2B$ such that $0 \le \eta \le 1$. Then there exists a constant $C = C(Q,p,c_\mu)$ such that*

$$[\eta(v - v_B)]_{B_p(X)} \le C[v]_{B_p(X)}$$

whenever $v \in L^1_{loc}(X)$ with $[v]_{B_p(X)} < \infty$.

Proof. Let $v \in L^1_{loc}(X)$ such that $[v]_{B_p(X)} < \infty$. Then $v \in L^p_{loc}(X)$ and this implies, since $\eta \in \mathrm{Lip}_0(2B)$, that $\eta(v - v_B) \in L^p(X)$. We repeat to some extent the argument of Lemma 3.7 with $\varphi = \eta$ and $u = v - v_B$. We can choose $L = \frac{C(c_\mu)}{R}$ and we note that $||\eta||_{L^\infty(X)} \le 1$. Hence

$$\int_X\int_X \frac{|u(x)\eta(x) - u(y)\eta(y)|^p}{d(x,y)^{2Q}} d\mu(x)\,d\mu(y) = I_1 + 2\,I_2, \tag{16}$$

where

$$I_1 = \int_{4B}\int_{4B} \frac{|u(x)\eta(x) - u(y)\eta(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

and

$$I_2 = \int_{4B}\int_{X\setminus 4B} \frac{|\eta(x)u(x) - \eta(y)u(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

We notice that $I_1 \leq 2^p(I_{10} + I_{11})$, where

$$I_{10} = \int_{4B}\int_{4B} \frac{|\eta(y)(u(x) - u(y))|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

and

$$I_{11} = \int_{4B}\int_{4B} \frac{|u(x)(\eta(x) - \eta(y))|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y).$$

We have

$$I_{10} \leq \int_{4B}\int_{4B} \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y) \leq [v]^p_{B_p(X)} \tag{17}$$

since $||\eta||_{L^\infty(X)} \leq 1$. As in (11) we get with $L = \frac{C(c_\mu)}{R}$

$$I_{11} \leq C(Q,p,c_\mu)\, R^{-Q} \int_{4B} |v(y) - v_B|^p d\mu(y). \tag{18}$$

Because η is supported in $2B$, it follows from the definition of I_2 that in fact

$$I_2 = \int_{2B}\int_{X\setminus 4B} \frac{|v(y) - v_B|^p\, |\eta(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y).$$

As in Lemma 3.7 we get

$$I_2 \leq C(Q,p,c_\mu)\, R^{-Q} \int_{2B} |v(y) - v_B|^p d\mu(y). \tag{19}$$

From (16), (17), (18), (19), and the fact that $I_1 \leq 2^p(I_{10} + 2I_{11})$, we have that $\eta(v - v_B) \in B_p(X)$ with

$$\begin{aligned} [\eta(v - v_B)]^p_{B_p(X)} &\leq C(Q,p,c_\mu) \int_{4B}\int_{4B} \frac{|v(x) - v(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y) \\ &\leq C(Q,p,c_\mu)\, [v]^p_{B_p(X)}. \end{aligned}$$

This finishes the proof. □

We now show that every function in $B_p(X)$ can be approximated by locally Lipschitz functions in $B_p(X)$.

Proposition 3.12. $\mathrm{Lip}_{loc}(X) \cap B_p(X)$ *is dense in* $B_p(X)$. *More precisely, if* $u \in L^1_{loc}(X)$ *has finite Besov p-seminorm, then there exists a sequence* $u_\varepsilon, \varepsilon > 0$, *in* $\mathrm{Lip}_{loc}(X)$ *such that:*

(i) $[u_\varepsilon - u]_{B_p(X)} \to 0$ *as* $\varepsilon \to 0$,

(ii) $||u_\varepsilon - u||_{L^p(X)} \to 0$ *as* $\varepsilon \to 0$.

Proof. For every $\varepsilon > 0$ we construct a family of balls $B(x_i, \varepsilon)$ with bounded overlap that cover X and we form a c_1/ε-Lipschitz partition of unity associated with the cover $\{B(x_i, \varepsilon)\}$ as in [KL02]. Here $c_1 = c_1(c_\mu)$. More precisely, we choose a family of balls $B(x_i, \varepsilon), i = 1, 2, \dots$, such that

$$X \subset \bigcup_{i=1}^{\infty} B(x_i, \varepsilon)$$

and

$$\sum_{i=1}^{\infty} \chi_{6B(x_i,\varepsilon)} < c_0 = c_0(Q, c_\mu). \tag{20}$$

Now we choose a sequence of c_1/ε-Lipschitz functions $\varphi_i, i = 1, 2, \dots$, such that $0 \le \varphi_i \le 1$, $\varphi_i = 0$ on $X \setminus 6B(x_i, \varepsilon)$, $\varphi_i \ge 1/c_0$ on $3B(x_i, \varepsilon)$, where c_0 is the constant from (20) and such that

$$\sum_{i=1}^{\infty} \varphi_i = 1$$

on X. We define the approximation by setting

$$u_\varepsilon(x) = \sum_{i=1}^{\infty} \varphi_i(x) u_{3B(x_i,\varepsilon)}$$

for every $x \in X$. Then u_ε is a locally Lipschitz function.

(i) We note that

$$u_\varepsilon(x) - u(x) = \sum_{i=1}^{\infty} \varphi_i(x)(u_{3B(x_i,\varepsilon)} - u(x)) \tag{21}$$

for every $x \in X$. We notice from (20) that for every $x \in X$ at most c_0 nonzero terms appear in (21). Therefore

$$u_\varepsilon(x) - u(x) - (u_\varepsilon(y) - u(y)) = \sum_{i=1}^{\infty} \Big[\varphi_i(x)(u_{3B(x_i,\varepsilon)} - u(x)) - \varphi_i(y)(u_{3B(x_i,\varepsilon)} - u(y))\Big]$$

with at most $2c_0$ nonzero terms appearing in the infinite sum for every $(x, y) \in X \times X$. Consequently, we obtain

$$[u_\varepsilon - u]^p_{B_p(X)} \le (2c_0)^p \sum_{i=1}^{\infty} [\varphi_i(u_{3B(x_i,\varepsilon)} - u)]^p_{B_p(X)}, \tag{22}$$

where c_0 is the bounded overlap constant appearing in (20). However, from Lemma 3.11 there exists a constant $C = C(Q, p, c_\mu)$ such that

$$[\varphi_i(u_{3B(x_i,\varepsilon)} - u)]^p_{B_p(X)} \le C \int_{12B(x_i,\varepsilon)} \int_{12B(x_i,\varepsilon)} \frac{|u(x) - u(y)|^p}{d(x, y)^{2Q}} d\mu(x)\, d\mu(y)$$

for every $i = 1, 2, \dots$. From this and (22) we obtain

$$[u_\varepsilon - u]^p_{B_p(X)} \le C \sum_{i=1}^{\infty} \int_{12B(x_i,\varepsilon)} \int_{12B(x_i,\varepsilon)} \frac{|u(x) - u(y)|^p}{d(x, y)^{2Q}} d\mu(x)\, d\mu(y), \tag{23}$$

where $C = C(Q, p, c_\mu)$. If we let

$$A_\varepsilon = \{(x, y) \in X \times X : d(x, y) < 24\varepsilon\},$$

we have from (20) and (23) that

$$[u_\varepsilon - u]^p_{B_p(X)} \le C(Q,p,c_\mu) \int_X \int_X \frac{|u(x)-u(y)|^p}{d(x,y)^{2Q}} \chi_{A_\varepsilon}(x,y) d\mu(x)\, d\mu(y).$$

An application of Lebesgue Dominated Convergence Theorem yields $[u_\varepsilon - u]_{B_p(X)} \to 0$ as $\varepsilon \to 0$. Moreover, we also notice that $[u_\varepsilon]_{B_p(X)} \le C(Q,p,c_\mu)[u]_{B_p(X)}$ for every $\varepsilon > 0$.

(ii) By using (20) and the fact that φ_i forms a partition of unity we obtain, via an argument similar to the one from Lemma 3.2

$$\begin{aligned} ||u_\varepsilon - u||^p_{L^p(X)} &\le c_0^p \sum_{i=1}^{\infty} ||\varphi_i(u_{3B(x_i,\varepsilon)} - u)||^p_{L^p(X)} \\ &\le c_0^p \sum_{i=1}^{\infty} \int_{6B(x_i,\varepsilon)} |u(x) - u_{3B(x_i,\varepsilon)}|^p d\mu(x) \\ &\le C(Q,p,c_\mu)\, \varepsilon^Q \int_X \int_X \frac{|u(x)-u(y)|^p}{d(x,y)^{2Q}} d\mu(x) d\mu(y), \end{aligned} \tag{24}$$

where c_0 is the constant from (20). This implies immediately that $||u_\varepsilon - u||_{L^p(X)} \to 0$ as $\varepsilon \to 0$. This finishes the proof. □

Proposition 3.13. $\mathrm{Lip}_0(X)$ *is dense in* $B_p(X)$.

Proof. Let $u \in B_p(X)$. Without loss of generality we can assume that u is bounded and locally Lipschitz. We fix $x_0 \in X$. For every integer $k \ge 2$, we define $\varphi_k : X \to \mathbf{R}$ by

$$\varphi_k(x) = \begin{cases} 1 & \text{if } 0 \le d(x,x_0) \le k, \\ \left(\log \frac{k^2}{d(x,x_0)}\right) / \log k & \text{if } k < d(x,x_0) \le k^2, \\ 0 & \text{if } d(x,x_0) > k^2. \end{cases}$$

Then $\varphi_k \in B_p(X)$ and moreover, $[\varphi_k]^p_{B_p(X)} \le C(\log k)^{1-p}$. (See (25).)

Let $u_k = u\varphi_k$. Then $u_k \in \mathrm{Lip}_0(X)$ and

$$||u - u_k||_{L^p(X)} \le ||u \chi_{X \setminus B(x_0,k)}||_{L^p(X)} \to 0 \text{ as } k \to \infty.$$

We also have

$$\begin{aligned} [u - u_k]_{B_p(X)} &\le \left(\int_X \int_X \frac{(1-\varphi_k(y))^p |u(x)-u(y)|^p}{d(x,y)^{2Q}} \, d\mu(x)\, d\mu(y) \right)^{1/p} \\ &+ ||u||_{L^\infty(X)} [\varphi_k]_{B_p(X)} \to 0 \end{aligned}$$

as $k \to \infty$. This finishes the proof. □

Lemma 3.14. *Let* $v \in B_p(\Omega)$.

(i) *If* $\operatorname{supp} v \subset\subset \Omega$, then $v \in B_p^0(\Omega)$.

(ii) If $u \in B_p^0(\Omega)$ and $0 \le v \le u$ in X, then $v \in B_p^0(\Omega)$.

Proof. The proof is similar to the proof of [Cos07, Lemma 2.10]. We present it for the convenience of the reader.

For the proof of (i), let $\psi \in \mathrm{Lip}_0(\Omega)$ be such that $\psi = 1$ on the support of v. If a sequence $v_j \in \mathrm{Lip}_0(X)$ converges to v in $B_p(X)$, then from Lemma 3.8 we see that $\psi v_j \in \mathrm{Lip}_0(\Omega)$ converges to $\psi v = v$ in $B_p(X)$, therefore $v \in B_p^0(\Omega)$.

As to assertion (ii), let $\varphi_j \in \mathrm{Lip}_0(\Omega)$ be an approximating sequence for $u \in B_p^0(\Omega)$. From Lemma 3.6 we can assume that the functions φ_j are nonnegative. We can assume without loss of generality that $v = u = 0$ everywhere on $X \setminus \Omega$. Then $\min(v, \varphi_j)$ has as support a compact subset of Ω and hence belongs to $B_p^0(\Omega)$. Moreover, since $\min(v, \varphi_j)$ converges to $\min(u, v) = v$ in $B_p(\Omega)$ (see Lemma 3.5), we have $v \in B_p^0(\Omega)$. □

Lemma 3.15. *Suppose that $\Omega \subset\subset X$. Let $u \in B_p(\Omega)$ be such that $u = 0$ on $X \setminus \Omega$ and $\lim_{\Omega \ni x \to y} u(x) = 0$ for all $y \in \partial\Omega$. Then $u \in B_p^0(\Omega)$.*

Proof. Recalling that $u = u^+ - u^-$, we may assume as in [Cos07, Lemma 2.11] that u is nonnegative. The function $u_\varepsilon = \max(u - \varepsilon, 0)$ is in $B_p(\Omega)$ for $\varepsilon > 0$ and has compact support in Ω. Thus $u_\varepsilon \in B_p^0(\Omega)$, $||u_\varepsilon||_{B_p(\Omega)} \le ||u||_{B_p(\Omega)}$ for every $\varepsilon > 0$ and $u_\varepsilon \to u$ both in $L^p(X)$ and pointwise as $\varepsilon \to 0$. The convexity and reflexivity of $B_p^0(\Omega)$ together with Mazur's lemma [Yos80, p. 120] imply that $u \in B_p^0(\Omega)$.

□

4. Relative Besov capacity

In this section, we establish a general theory of the relative Besov capacity and study how this capacity is related to Hausdorff measures.

For $E \subset \Omega$ we define

$$BA(E, \Omega) = \{u \in B_p^0(\Omega) : u \ge 1 \text{ on a neighborhood of } E\}.$$

We call $BA(E, \Omega)$ the set of *admissible functions for the condenser* (E, Ω). The *relative Besov p-capacity* of the pair (E, Ω) is denoted by

$$\operatorname{cap}_{B_p}(E, \Omega) = \inf\{[u]^p_{B_p(\Omega)} : u \in BA(E, \Omega)\}.$$

If $BA(E, \Omega) = \emptyset$, we set $\operatorname{cap}_{B_p}(E, \Omega) = \infty$.

Since $B_p^0(\Omega)$ is closed under truncations from below by 0 and from above by 1 and since these truncations do not increase the Besov p-seminorm, we may restrict ourselves to those admissible functions u for which $0 \le u \le 1$.

Remark 4.1. If K is a compact subset of the bounded and open set $\Omega \subset X$, we get the same Besov B_p-capacity for (K, Ω) if we restrict ourselves to a smaller set of admissible functions, namely

$$BW(K, \Omega) = \{u \in \operatorname{Lip}_0(\Omega) : u = 1 \text{ in a neighborhood of } K\}.$$

Indeed, let $u \in BA(K, \Omega)$; we may clearly assume that $u = 1$ in a neighborhood $U \subset\subset \Omega$ of K. Then we choose a cut-off Lipschitz function η, $0 \le \eta \le 1$ such that $\eta = 1$ in $X \setminus U$ and $\eta = 0$ in a neighborhood $\widetilde{U}$ of K, $\widetilde{U} \subset\subset U$. Now, if $\varphi_j \in \operatorname{Lip}_0(\Omega)$ is a sequence converging to u in $B_p^0(\Omega)$, then $\psi_j = 1 - \eta(1 - \varphi_j)$ is a sequence belonging to $BW(K, \Omega)$ which converges to $1 - \eta(1 - u)$ in $B_p^0(\Omega)$. (See Lemma 3.8.) But $1 - \eta(1 - u) = u$. This establishes the assertion, since $BW(K, \Omega) \subset BA(K, \Omega)$. In fact, it is easy to see that if $K \subset \Omega$ is compact we get the same Besov B_p-capacity if we consider

$$B\widetilde{W}(K, \Omega) = \{u \in \operatorname{Lip}_0(\Omega) : u = 1 \text{ on } K\}.$$

It is also useful to observe that if $\psi \in B_p^0(\Omega)$ is such that $\varphi - \psi \in B_p^0(\Omega \setminus K)$ for some $\varphi \in B\widetilde{W}(K, \Omega)$, then

$$\operatorname{cap}_{B_p}(K, \Omega) \le [\psi]^p_{B_p(\Omega)}.$$

4.1. **Basic properties of the relative Besov capacity.** A capacity is a monotone, subadditive set function. The following theorem expresses, among other things, that this is true for the relative Besov p-capacity.

Theorem 4.2. *Suppose (X, d, μ) is a proper and unbounded Ahlfors Q-regular metric space with $1 < Q < p < \infty$. Let $\Omega \subset X$ be a bounded open set. The set function $E \mapsto \mathrm{cap}_{B_p}(E, \Omega)$, $E \subset \Omega$, enjoys the following properties:*

(i) *If $E_1 \subset E_2$, then $\mathrm{cap}_{B_p}(E_1, \Omega) \leq \mathrm{cap}_{B_p}(E_2, \Omega)$.*

(ii) *If $\Omega_1 \subset \Omega_2$ are open, bounded, and $E \subset \Omega_1$, then*

$$\mathrm{cap}_{B_p}(E, \Omega_2) \leq \mathrm{cap}_{B_p}(E, \Omega_1).$$

(iii) $\mathrm{cap}_{B_p}(E, \Omega) = \inf\{\mathrm{cap}_{B_p}(U, \Omega) : E \subset U \subset \Omega,\ U \text{ open}\}$.

(iv) *If K_i is a decreasing sequence of compact subsets of Ω with $K = \bigcap_{i=1}^{\infty} K_i$, then*

$$\mathrm{cap}_{B_p}(K, \Omega) = \lim_{i\to\infty} \mathrm{cap}_{B_p}(K_i, \Omega).$$

(v) *If $E_1 \subset E_2 \subset \ldots \subset E = \bigcup_{i=1}^{\infty} E_i \subset \Omega$, then*

$$\mathrm{cap}_{B_p}(E, \Omega) = \lim_{i\to\infty} \mathrm{cap}_{B_p}(E_i, \Omega).$$

(vi) *If $E = \bigcup_{i=1}^{\infty} E_i \subset \Omega$, then*

$$\mathrm{cap}_{B_p}(E, \Omega) \leq \sum_{i=1}^{\infty} \mathrm{cap}_{B_p}(E_i, \Omega).$$

Proof. The proof is very similar to the proof of [Cos07, Theorem 3.1] and is therefore omitted. □

A set function that satisfies properties (i), (iv) and (v) is called a *Choquet capacity* (relative to Ω). We may thus invoke an important capacitability theorem of Choquet and state the following result. See [Doo84, Appendix II].

Theorem 4.3. *Suppose (X, d, μ) is a metric measure space as in Theorem 4.2. Suppose that Ω is a bounded open set in X. The set function $E \mapsto \mathrm{cap}_{B_p}(E, \Omega)$, $E \subset \Omega$, is a Choquet capacity. In particular, all Borel subsets (in fact, all analytic) subsets E of Ω are capacitable, i.e.,*

$$\mathrm{cap}_{B_p}(E, \Omega) = \sup\{\mathrm{cap}_{B_p}(K, \Omega) : K \subset E \text{ compact}\}$$

whenever $E \subset \Omega$ is Borel (or analytic).

4.2. **Upper estimates for the relative Besov capacity.** Next we derive some upper estimates for the relative Besov capacity. Similar estimates have been obtained earlier by Bourdon in [Bou07]. We follow his methods.

Theorem 4.4. *Let (X, d, μ) be a metric measure space as in Theorem 4.2. There exists a constant $C = C(Q, p, c_\mu) > 0$ depending only on Q, p and c_μ such that*

$$\mathrm{cap}_{B_p}(B(x_0, r), B(x_0, R)) \leq C \left(\log \frac{R}{r}\right)^{1-p} \tag{25}$$

for every $0 < r < \frac{R}{2}$ and every $x_0 \in X$.

Proof. We use the function $u : X \to \mathbf{R}$,

$$u(x) = \begin{cases} 1 & \text{if } 0 \le d(x,x_0) \le r, \\ \left(\log \frac{d(x,x_0)}{R}\right) / \left(\log \frac{r}{R}\right) & \text{if } r < d(x,x_0) < R, \\ 0 & \text{if } d(x,x_0) \ge R. \end{cases}$$

Then $u \in B_p(X)$ because it is Lipschitz with compact support. Since u is continuous on X and 0 outside $B(x_0,R)$, we have in fact from Lemma 3.15 that $u \in B_p^0(B(x_0,R))$. In fact $u \in BA(B(x_0,r), B(x_0,R))$ since $u = 1$ on $B(x_0,r)$. Let $v(x) = u(x) \log \frac{R}{r}$. We will get an upper bound for $[v]_{B_p(B(x_0,R))}$. Let $k \ge 3$ be the smallest integer such that $2^{k-1}r \ge R$. For $i = 1, \dots, k$ we define $B_i = B(x_0, 2^i r) \setminus \overline{B}(x_0, 2^{i-1}r)$. We also define $B_0 = B(x_0,r)$ and $B_{k+1} = X \setminus B(x_0, 2^k r)$. We have

$$[v]^p_{B_p(B(x_0,R))} = \sum_{0 \le i,j \le k+1} I_{i,j} = \sum_{0 \le i,j \le k+1} \int_{B_i} \int_{B_j} \frac{|v(x) - v(y)|^p}{d(x,y)^{2Q}} \, d\mu(x) \, d\mu(y).$$

Obviously we have $I_{i,j} = I_{j,i}$. We majorize $I_{i,j}$ by distinguishing a few cases. For $j \le k$ and $0 \le i \le j-2$ we have from the definition of v that $|v(x) - v(y)| \le j - i + 1$ whenever $x \in B_i$ and $y \in B_j$, hence

$$I_{i,j} \le C_0 (j-i+1)^p \, (2^j r)^{-2Q} \, (2^i r)^Q \, (2^j r)^Q,$$

that is $I_{i,j} \le C_1 (j-i)^p 2^{(i-j)Q}$. For $0 \le i \le j \le k$ we notice, since v is $\frac{1}{2^{i-1}r}$-Lipschitz on $\bigcup_{j \ge i} B_j$ that

$$I_{i,j} \le (2^{i-1}r)^{-p} \int_{B_i} \int_{B_j} \frac{1}{d(x,y)^{2Q-p}} \, d\mu(x) \, d\mu(y).$$

Moreover, we have

$$\int_{B_j} \frac{1}{d(x,y)^{2Q-p}} \, d\mu(x) \le C_2 (\text{diam } B_j)^{p-Q}$$

for every $y \in B(x_0, 2^i r)$, where C_2 depends only on p, Q and c_μ. Hence for $0 \le i \le j \le k$ we have

$$I_{i,j} \le C_3 (2^{i-1}r)^{-p} (2^i r)^Q (2^j r)^{p-Q} \le C_4 2^{(j-i)(p-Q)}.$$

In particular, for $j - 1 \le i \le j \le k$, the integral $I_{i,j}$ is bounded by a constant that depends only on p, Q and c_μ. Now we have to bound $I_{i,j}$ when $j = k+1$. Since v is constant on $B_k \cup B_{k+1}$, we have $I_{i,k+1} = 0$ for $i \in \{k, k+1\}$. For $0 \le i \le k-1$ we have

$$I_{i,k+1} \le (k-i+1)^p \int_{B_i} \int_{B_{k+1}} \frac{1}{d(x,y)^{2Q}} \, d\mu(x) \, d\mu(y).$$

But there exists $C_5 > 0$ such that

$$\int_{B_{k+1}} \frac{1}{d(x,y)^{2Q}} \, d\mu(x) \le C_5 (2^{k+1} r)^{-Q}$$

for every $y \in X$ with $d(y, x_0) \le 2^{k-1} r$. Hence $I_{i,k+1} \le C_6 (k-i+1)^p 2^{(i-k-1)Q}$. Finally we have

$$[v]^p_{B_p(B(x_0,R))} \le C_7 k + C_8 \sum_{0 \le i \le j \le k+1} (j-i)^p 2^{(i-j)Q}.$$

The last sum is equal to

$$\sum_{l=1}^{k+1} (k+2-l) l^p 2^{-lQ}.$$

But $k+2-l \leq k+1$ and there exists $a>1$ such that $l^p 2^{-lQ} \leq C_9 a^{-l}$ for $l \geq 1$. Hence

$$[v]^p_{B_p(B(x_0,R))} \leq C_{10} \log \frac{R}{r}$$

and

$$[u]^p_{B_p(B(x_0,R))} \leq C_{10} \left(\log \frac{R}{r}\right)^{1-p}.$$

The claim follows with $C = C_{10}$. □

For a fixed $r>0$ we construct the dyadic partition of X as in [Chr90, Theorem11]. That is, a family of open sets $\mathcal{D}_r = \{K^\alpha_{m,r} : m \in \mathbf{Z}, \alpha \in I_m\}$ such that

(i) $\mu(X \setminus \bigcup_\alpha K^\alpha_{m,r}) = 0, \forall m$.

(ii) If $l \geq m$ then either $K^\beta_{l,r} \subset K^\alpha_{m,r}$ or $K^\beta_{l,r} \cap K^\alpha_{m,r} = \emptyset$.

(iii) For each (m,α) and each $l<m$ there is a unique β such that $K^\alpha_{m,r} \subset K^\beta_{l,r}$.

(iv) For every (m,α) there exists a ball $B^\alpha_{m,r} = B(x^\alpha_{m,r}, 10^{-m}r)$ such that

$$\frac{1}{10} B^\alpha_{m,r} \subset K^\alpha_{m,r} \subset 3B^\alpha_{m,r}.$$

We call these open sets "dyadic cubes".

Two distinct dyadic cubes K, K' in $\mathcal{D}_r$ are *adjacent* if there exists an integer k such that either

(i) K, K' are in generation k and $\overline{K} \cap \overline{K'} \neq \emptyset$, or

(ii) one of the cubes K, K' is in generation k, the other one is in generation $k+1$ the one in generation k contains the other one.

Similarly, if $K_0 \subset X$ is a dyadic cube in $\mathcal{D}_r$, we denote by $\mathcal{D}_r(K_0)$ the dyadic subcubes of K_0.

For two adjacent cubes $K, K' \in \mathcal{D}_r$ we have

$$\begin{aligned} |f_{\overline{K}} - f_{\overline{K'}}|^p &= \left| \frac{1}{\mu(\overline{K})} \int_{\overline{K}} f(x)\,d\mu(x) - \frac{1}{\mu(\overline{K'})} \int_{\overline{K'}} f(y)\,d\mu(y) \right|^p \\ &= \left| \frac{1}{\mu(\overline{K})} \frac{1}{\mu(\overline{K'})} \int_{\overline{K}} \int_{\overline{K'}} (f(x)-f(y))\,d\mu(x)\,d\mu(y) \right|^p \\ &\leq \frac{1}{\mu(\overline{K})} \frac{1}{\mu(\overline{K'})} \int_{\overline{K}} \int_{\overline{K'}} |f(x)-f(y)|^p\,d\mu(x)\,d\mu(y) \\ &\leq C \int_{\overline{K}} \int_{\overline{K'}} \frac{|f(x)-f(y)|^p}{d(x,y)^{2Q}}\,d\mu(x)\,d\mu(y), \end{aligned}$$

where C is a constant that depends only on the Ahlfors regularity of X.

For the following lemma see [BP03, Lemma 3.5].

Lemma 4.5. *There exists a constant C depending only on the Ahlfors regularity of X such that*

$$C^{-1} d(\eta,\zeta)^{-2Q} \leq \sum_{K,K' \in \mathcal{D}_r \ \text{adjacent}} \frac{\chi_{\overline{K}}(\eta)\chi_{\overline{K'}}(\zeta)}{\mu(\overline{K})\mu(\overline{K'})} \leq C d(\eta,\zeta)^{-2Q}$$

for μ-a.e. $\eta, \zeta \in X$.

We also have (see [BP03, Theorem 3.4]):

Lemma 4.6. *There exists a constant C depending only on p and on the Ahlfors regularity of X such that*

$$\begin{aligned} C^{-1}[f]^p_{B_p(X)} &\leq \sum_{K,K'\in\mathcal{D}_r \text{ adjacent}} \frac{1}{\mu(\overline{K})}\frac{1}{\mu(\overline{K'})}\int_{\overline{K}}\int_{\overline{K'}} |f(x)-f(y)|^p\,d\mu(x)\,d\mu(y) \\ &\leq C[f]^p_{B_p(X)} \end{aligned}$$

for every $f \in B_p(X)$.

This implies (see [BP03, Lemma 3.5]):

Lemma 4.7. *There exists a constant C depending only on p and on the Ahlfors regularity of X such that*

$$\sum_{K,K'\in\mathcal{D}_r \text{ adjacent}} |f_{\overline{K}} - f_{\overline{K'}}|^p \leq C[f]^p_{B_p(X)} \tag{26}$$

for every $f \in B_p(X)$.

4.3. **Hausdorff measure and relative Besov capacity.** Now we examine the relationship between Hausdorff measures and the B_p-capacity. Let h be a real-valued and increasing function on $[0,\infty)$ such that $\lim_{t\to 0} h(t) = h(0) = 0$ and $\lim_{t\to\infty} h(t) = \infty$. Such a function h is called a *measure function.* Let $0 < \delta \leq \infty$. Suppose $\Omega \subset X$ is open. For $E \subset \overline{\Omega}$ we define

$$\Lambda^\delta_{h,\overline{\Omega}}(E) = \inf \sum_i h(r_i),$$

where the infimum is taken over all coverings of E by open sets G_i in $\overline{\Omega}$ with diameter r_i not exceeding δ. The set function $\Lambda^\infty_{h,\overline{\Omega}}$ is called the *h-Hausdorff content relative to* $\overline{\Omega}$. Clearly $\Lambda^\delta_{h,\overline{\Omega}}$ is an outer measure for every $\delta \in (0,\infty]$ and every open set $\Omega \subset X$. We write $\Lambda^\delta_h(E)$ for $\Lambda^\delta_{h,X}(E)$.

Moreover, for every $E \subset \overline{\Omega}$, there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E} \subset \overline{\Omega}$ and $\Lambda^\delta_{h,\overline{\Omega}}(E) = \Lambda^\delta_{h,\overline{\Omega}}(\widetilde{E})$. Clearly $\Lambda^\delta_{h,\overline{\Omega}}(E)$ is a decreasing function of δ. It is easy to see that $\Lambda^\delta_{h,\overline{\Omega_2}}(E) \leq \Lambda^\delta_{h,\overline{\Omega_1}}(E)$ for every $\delta \in (0,\infty]$ whenever Ω_1 and Ω_2 are open sets in X such that $E \subset \overline{\Omega_1} \subset \overline{\Omega_2}$. This allows us to define the *h-Hausdorff measure relative to* $\overline{\Omega}$ of $E \subset \overline{\Omega}$ by

$$\Lambda_{h,\overline{\Omega}}(E) = \sup_{\delta>0} \Lambda^\delta_{h,\overline{\Omega}}(E) = \lim_{\delta\to 0} \Lambda^\delta_{h,\overline{\Omega}}(E).$$

The measure $\Lambda_{h,\overline{\Omega}}$ is Borel regular; that is, it is an additive measure on Borel sets of $\overline{\Omega}$ and for each $E \subset \overline{\Omega}$ there is a Borel set G such that $E \subset G \subset \overline{\Omega}$ and $\Lambda_{h,\overline{\Omega}}(E) = \Lambda_{h,\overline{\Omega}}(G)$. (See [Fed69, p. 170] and [Mat95, Chapter 4].) If $h(t) = t^s$, we write Λ_s for $\Lambda_{t^s,X}$. It is immediate from the definition that $\Lambda_s(E) < \infty$ implies $\Lambda_u(E) = 0$ for all $u > s$. The smallest $s \geq 0$ that satisfies $\Lambda_u(E) = 0$ for all $u > s$ is called the *Hausdorff dimension of E.*

There are many excellent books on Hausdorff measures, including Federer [Fed69], Mattila [Mat95], and Rogers [Rog70].

For $\Omega \subset X$ open and $\delta > 0$ the set function $\Lambda^\delta_{h,\overline{\Omega}}$ has the following property:

(i) If K_i is a decreasing sequence of compact sets in $\overline{\Omega}$, then

$$\Lambda^{\delta}_{h,\overline{\Omega}}(\bigcap_{i=1}^{\infty} K_i) = \lim_{i\to\infty} \Lambda^{\delta}_{h,\overline{\Omega}}(K_i).$$

Moreover, if $\Omega \subset\subset X$ and h is a continuous measure function, then $\Lambda^{\delta}_{h,\overline{\Omega}}$ satisfies the following additional properties:

(ii) If E_i is an increasing sequence of arbitrary sets in $\overline{\Omega}$, then

$$\Lambda^{\delta}_{h,\overline{\Omega}}(\bigcup_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \Lambda^{\delta}_{h,\overline{\Omega}}(E_i).$$

(iii) $\Lambda^{\delta}_{h,\overline{\Omega}}(E) = \sup\{\Lambda^{\delta}_{h,\overline{\Omega}}(K) : K \subset E \text{ compact}\}$ whenever $E \subset \overline{\Omega}$ is a Borel set. (See [Rog70, Chapter 2:6].)

We have the following proposition:

Proposition 4.8. *Suppose (X, d, μ) is an Ahlfors Q-regular metric space with $Q > 1$. Let $h : [0,\infty) \to [0,\infty)$ be a measure function.*

(a) If $\liminf_{t\to 0} h(t)t^{-Q} = 0$, then $\Lambda^{\delta}_h(X) = 0$.

(b) If $\liminf_{t\to 0} h(t)t^{-Q} > 0$, then there is an increasing function $h^* : [0,\infty) \to [0,\infty)$ such that $h^*(0) = 0$, h^* is continuous, $t \mapsto h(t)\,t^{-Q}$, $0 < t < \infty$, is decreasing and there exists a constant $C = C(Q, c_\mu)$ such that for all $E \subset X$ and all $\delta > 0$

$$C^{-1}\Lambda^{\delta}_h(E) \le \Lambda^{\delta}_{h^*}(E) \le C\Lambda^{\delta}_h(E).$$

Proof. The proof is similar to the proof of [AH96, Proposition 5.1.8] and omitted. □

If $h : [0,\infty) \to [0,\infty)$ is a continuous increasing measure function such that $t \mapsto h(t)\,t^{-Q}$, $0 < t < \infty$, is decreasing, we know that $\Lambda_h(E) = 0$ if and only if $\Lambda^{\infty}_h(E) = 0$. (See [AH96, Proposition 5.1.5].) If $h(t) = t^s$, $0 < s < \infty$, we write Λ^{∞}_s for $\Lambda^{\infty}_{t^s,X}$.

Theorem 4.9. *Suppose $1 \le \widetilde{p} < Q < p < \infty$. Let (X, d, μ) be a complete and unbounded Ahlfors Q-regular metric space that supports a weak $(1,\widetilde{p})$-Poincaré inequality. Suppose $h : [0,\infty) \to [0,\infty)$ is a continuous increasing measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. Let $K_{0,r} \in \mathcal{D}_r$ be a dyadic cube of generation 0 and let $x_0 \in X$ be such that $B(x_0, r/10) \subset K_{0,r}$. There exists a positive constant $C_1' = C_1'(Q, p, c_\mu)$ such that*

$$\frac{\Lambda^{\infty}_h(E \cap \overline{K}_{k,r})}{\left(\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \le C_1' k^{p-1}\mathrm{cap}_{B_p}(E \cap \overline{K}_{k,r}, B(x_0, r/10)) \tag{27}$$

for every $E \subset X$, every $k > 1, r > 0$, and for every $K_{k,r} \in \mathcal{D}_r(K_{0,r})$ cube of generation k such that $B(x_0, 10^{-k}r) \cap \overline{K}_{k,r} \ne \emptyset$.

Proof. We fix $r > 0$ and $k > 1$. Suppose $K_{k,r} \in \mathcal{D}_r(K_{0,r})$ is a dyadic subcube of $K_{0,r}$ of generation k such that $\overline{K}_{k,r} \cap B(x_0, 10^{-k}r) \ne \emptyset$.

Let $E \subset X$. From the fact that there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E} \subset X$ and $\mathrm{cap}_{B_p}(E \cap \overline{K}_{k,r}, B(x_0, r/10)) = \mathrm{cap}_{B_p}(\widetilde{E} \cap \overline{K}_{k,r}, B(x_0, r/10))$, we can assume that E is a Borel set. All the sets $E \cap \overline{K}_{k,r}$ considered here lie in $7B(x_0, 10^{-k}r) \subset\subset B(x_0, r/10)$ whenever $k \ge 2$. Indeed, $K_{k,r}$, which is contained in a ball of radius $10^{-k}3r$, was chosen such that $\overline{K}_{k,r} \cap B(x_0, 10^{-k}r) \ne \emptyset$. From this observation, the discussion before Proposition 4.8, and the fact that $\mathrm{cap}_{B_p}(\cdot, B(x_0, r/10))$ is a Choquet capacity, it follows that it is enough to consider only compact sets E in order to prove the theorem.

There is nothing to prove if either $\Lambda^{\infty}_{h,\overline{K}_{0,r}}(E \cap \overline{K}_{k,r}) = 0$ or if $\int_0^{10^{-k}r} h(t)^{p'-1} \frac{dt}{t} = \infty$. So we can assume without loss of generality that $\alpha = \Lambda^{\infty}_{h,\overline{K}_{0,r}}(E \cap \overline{K}_{k,r}) > 0$ and that $\int_0^{10^{-k}r} h^{p'-1}(t) \frac{dt}{t} < \infty$.

For every $\zeta \in S(x_0, r/10)$ there exists an increasing sequence $(K_{s,\zeta})_{s \le 0}$ of dyadic subcubes of $K_{0,r}$ such that $K_{s,\zeta}$ is a cube of generation s for every integer $s \le 0$ and

$$\bigcap_{s \le 0} \overline{K}_{s,\zeta} = \{\zeta\}.$$

We denote by s^0_ζ the sequence $(\overline{K}_{s,\zeta})_{s \le 0}$.

For every $\eta \in \overline{K}_{k,r}$ there exists a decreasing sequence $(K_{s+k,\eta})_{s \ge 0}$ of dyadic subcubes of $K_{k,r}$ such that $K_{s+k,\eta}$ is of generation $s+k$ for every $s \ge 0$ and

$$\bigcap_{s \ge 0} \overline{K}_{s+k,\eta} = \{\eta\}.$$

We denote by s^1_η the sequence $(\overline{K}_{s+k,\eta})_{s \ge 0}$. Let $I = \{K_{0,r}, \dots, K_{k,r}\}$ be a shortest decreasing sequence of cubes connecting $K_{0,r}$ and $K_{k,r}$.

For $(\zeta, \eta) \in S(x_0, r/10) \times \overline{K}_{k,r}$ we define $\gamma_{\zeta,\eta} = (\overline{K}_{s,\zeta,\eta})_{s \in \mathbf{Z}}$, where

$$K_{s,\zeta,\eta} = \begin{cases} K_{s,\zeta} & \text{if } s \le 0 \\ K_{s,r} & \text{if } 0 \le s \le k \\ K_{s,\eta} & \text{if } s \ge k. \end{cases}$$

For $K, K' \in \mathcal{D}_r$ we define

$$\mathcal{C}(K, K') = \{(\zeta, \eta) \in S(x_0, \frac{r}{10}) \times \overline{K}_{k,r} : K = K_{s,\zeta,\eta},\ K' = K_{s+1,\zeta,\eta} \text{ for some } s \in \mathbf{Z}\}.$$

We notice that $\mathcal{C}(K, K') = \emptyset$ if K, K' are not adjacent or if they are adjacent but of the same generation.

Since X is an Ahlfors Q-regular complete metric space that satisfies a weak $(1, \widetilde{p})$-Poincaré inequality with $1 \le \widetilde{p} < Q$, there exists (see [Kor07, Theorem 4.2]) a constant C depending only on $\widetilde{p}$ and on the data of X such that

$$C^{-1} t^{Q-\widetilde{p}} \le \Lambda^{\infty}_{Q-\widetilde{p}}(S(x,t)) \le C t^{Q-\widetilde{p}}$$

for all closed spheres $S(x,t)$ of radius t in X. We also have $\alpha = \Lambda^{\infty}_h(E \cap \overline{K}_{k,r}) > 0$. Therefore, by applying Frostman's lemma (see [Mat95, Theorem 8.8]), there exists a constant $C > 0$ and probability measures ν_0 on $S(x_0, r/10)$ and ν_1 on $E \cap \overline{K}_{k,r}$ such that for every ball $B(x,t)$ of radius t in X we have

$$\nu_0(B(x,t)) \le C \left(\frac{t}{r}\right)^{Q-\widetilde{p}} \text{ and } \nu_1(B(x,t)) \le C \frac{h(t)}{\alpha}. \tag{28}$$

For $K, K' \in \mathcal{D}_r$ we define

$$m(\overline{K}, \overline{K'}) = \nu_0 \times \nu_1(\mathcal{C}(K, K')).$$

We notice that $m(\overline{K}, \overline{K'}) m(\overline{K'}, \overline{K}) = 0$ for every pair of cubes $K, K' \in \mathcal{D}_r$. Moreover, if $m(\overline{K}, \overline{K'}) \ne 0$, then this implies that K and K' are adjacent but of different generations.

Let f be in $BW(E, B(x_0, r/10))$. Then, since f is continuous, we have that

$$f_{\overline{K}_v} \to f(y)$$

for every $y \in X$ for every nested sequence $\overline{K}_v$ of r-dyadic cubes containing y and converging to y. It follows that

$$1 = f(\eta) - f(\zeta) \leq \sum_{s\in\mathbf{Z}} (f_{\overline{K}_{s+1,\zeta,\eta}} - f_{\overline{K}_{s,\zeta,\eta}})$$

whenever $\eta \in E \cap \overline{K}_{k,r}$ and $\zeta \in S(x_0, r/10)$.

We obtain with the definition of $m(\overline{K}, \overline{K'})$ and by Hölder's inequality, that

$$\begin{aligned}
1 &\leq \int_{S(x_0,r/10)} \int_{E\cap\overline{K}_{k,r}} \sum_{s\in\mathbf{Z}} (f_{\overline{K}_{s+1,\zeta,\eta}} - f_{\overline{K}_{s,\zeta,\eta}}) d\nu_1(\eta)\, d\nu_0(\zeta) \\
&\leq \int_{S(x_0,r/10)} \int_{\overline{K}_{k,r}} \sum_{s\in\mathbf{Z}} |f_{\overline{K}_{s+1,\zeta,\eta}} - f_{\overline{K}_{s,\zeta,\eta}}| d\nu_1(\eta)\, d\nu_0(\zeta) \\
&= \sum_{K,K'\in\mathcal{D}_r \text{ adjacent}} |f_{\overline{K}} - f_{\overline{K'}}| m(\overline{K}, \overline{K'}) \\
&\leq \left(\sum_{K,K'\in\mathcal{D}_r \text{ adjacent}} |f_{\overline{K}} - f_{\overline{K'}}|^p \right)^{1/p} \left(\sum_{K,K'\in\mathcal{D}_r \text{ adjacent}} m(\overline{K}, \overline{K'})^{p'} \right)^{1/p'} \\
&\leq C[f]_{B_p(X)} \left(\sum_{K,K'\in\mathcal{D}_r \text{ adjacent}} m(\overline{K}, \overline{K'})^{p'} \right)^{1/p'},
\end{aligned}$$

where we used (26) for the last inequality. Here the constant C depends only on p and on the Ahlfors regularity of X. For a nonnegative integer s we let

$$E_{0,s} = \{(K, K') \in \mathcal{D}_r \times \mathcal{D}_r : K = K_{-s-1,\zeta},\ K' = K_{-s,\zeta} \text{ for some } \zeta \in S(x_0, r/10)\}$$

and similarly

$$E_{1,s} = \{(K, K') \in \mathcal{D}_r \times \mathcal{D}_r : K = K_{s+k,\eta},\ K' = K_{s+k+1,\eta} \text{ for some } \eta \in \overline{K}_{k,r}\}.$$

We notice that we can break $\sum = \sum_{K,K'\in\mathcal{D}_r} m(\overline{K}, \overline{K'})^{p'}$ into 3 parts, namely

$$\sum = \sum_{s=0}^{\infty} \sum_{(K,K')\in E_{0,s}} m(\overline{K}, \overline{K'})^{p'} + \sum_{K,K'\in I} m(\overline{K}, \overline{K'})^{p'} + \sum_{s=0}^{\infty} \sum_{(K,K')\in E_{1,s}} m(\overline{K}, \overline{K'})^{p'}.$$

We recall that $I = \{K_{0,r}, \dots, K_{k,r}\}$ is a shortest decreasing sequence of cubes in $\mathcal{D}_r$ connecting $K_{0,r}$ and $K_{k,r}$. Thus, the sum in the middle is exactly k. We get upper bounds for the first and the third term in the sum. We notice that for every $s \geq 0$ we have

$$\sum_{(K,K')\in E_{0,s}} m(\overline{K}, \overline{K'}) = 1$$

since $\nu_0 \times \nu_1$ is a probability measure. On the other hand, there exists a constant C' depending only on p and on the Hausdorff dimension of X such that

$$m(\overline{K}, \overline{K'}) \leq C' \frac{h(10^{-s-k}r)}{\alpha} \text{ for every } (K, K') \in E_{1,s}$$

for every integer $s \geq 0$ and

$$m(\overline{K}, \overline{K'}) \leq C' 10^{(\widetilde{p}-Q)s} \text{ for every } (K, K') \in E_{0,s}$$

for every integer $s \geq 0$.

Therefore

$$\begin{aligned}\sum_{s=0}^{\infty}\sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'})^{p'} &= \sum_{s=0}^{\infty}\sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'})^{p'-1}m(\overline{K},\overline{K'})\\ &\leq C\alpha^{1-p'}\sum_{s\geq 0}h(10^{-s-k}r)^{p'-1}\left(\sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'})\right).\end{aligned}$$

But there exists a constant $C_0 = C_0(Q,p) > 1$ such that

$$\frac{1}{C_0}\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t} \leq \sum_{s\geq 0} h(10^{-k-s}r)^{p'-1} \leq C_0\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}$$

for every $r > 0$, every integer $k > 1$ and every continuous increasing measure function $h : [0,\infty) \to [0,\infty)$ such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. Hence

$$\sum_{s=0}^{\infty}\sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'})^{p'} \leq C\,\alpha^{1-p'}\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}.$$

From a similar computation we get

$$\begin{aligned}\sum_{s=0}^{\infty}\sum_{(K,K')\in E_{0,s}} m(\overline{K},\overline{K'})^{p'} &= \sum_{s=0}^{\infty}\sum_{(K,K')\in E_{0,s}} m(\overline{K},\overline{K'})^{p'-1}m(\overline{K},\overline{K'})\\ &\leq C\sum_{s\geq 0}10^{-(p'-1)(Q-\widetilde{p})s}\left(\sum_{(K,K')\in E_{0,s}} m(\overline{K},\overline{K'})\right) = C.\end{aligned}$$

So we get

$$\sum \leq C\left(\alpha^{1-p'}\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t} + k + 1\right).$$

It is easy to see that there exists a constant C depending only on p and on the Hausdorff dimension of X such that

$$\frac{\Lambda_h^{\infty}(\overline{K}_{k,r})}{\left(\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \leq C.$$

for every $r > 0$, every integer $k > 1$ and every continuous increasing measure function $h : [0,\infty) \to [0,\infty)$ such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. Hence

$$\sum \leq Ck\,\alpha^{1-p'}\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}.$$

Therefore we obtain

$$1 \leq C[f]_{B_p(B(x_0,r/10))}\left(k\,\alpha^{1-p'}\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{1/p'}$$

for every integer $k > 1$ and for every $f \in BW(E\cap\overline{K}_{k,r}, B(x_0,r/10))$. This implies that there exists a constant C_1' depending only on p and on the Hausdorff dimension of X such that

$$\frac{\Lambda^{\infty}_{h,\overline{K}_{0,r}}(E\cap\overline{K}_{k,r})}{\left(\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}}k^{1-p} \leq C_1'\mathrm{cap}_{B_p}(E\cap\overline{K}_{k,r}, B(x_0,r/10)).$$

This finishes the proof since $\Lambda_h^{\infty}(E\cap\overline{K}_{k,r}) \leq \Lambda^{\infty}_{h,\overline{K}_{0,r}}(E\cap\overline{K}_{k,r})$. □

As a consequence of Theorem 4.9, we obtain the following theorem.

Theorem 4.10. *Suppose (X, d, μ) is a metric measure space as in Theorem 4.9. Suppose $h : [0, \infty) \to [0, \infty)$ is a continuous increasing measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. There exists a positive constant $C_1 = C_1(Q, p, c_\mu)$ such that*

$$\frac{\Lambda_h^\infty(E \cap B(x, r))}{\left(\int_0^r h(t)^{p'-1} \frac{dt}{t}\right)^{p-1}} \le C_1 \left(\log \frac{R}{r}\right)^{p-1} \mathrm{cap}_{B_p}(E \cap B(x, r), B(x, R))$$

for every $E \subset X$, every $x \in X$, and every pair of positive numbers r, R such that $r < \frac{R}{2}$.

Proof. Fix $x \in X$ and r, R such that $0 < r < \frac{R}{2}$. Without loss of generality we can assume that $B(x, 100R) \subset K_{0,1000R}$. We choose $k \ge 3$ integer such that $10^{2-k}R \le r < 10^{3-k}R$. From the construction of the dyadic cubes and the fact that X is a Q-Ahlfors regular space with $Q > 1$, it follows that there exists a constant $C = C(Q, c_\mu)$ independent of k such that every ball of radius $10^{2-k}R$ intersects with at most C dyadic subcubes of $K_{0,1000R}$ from the kth generation. We leave the rest of the details to the reader. □

It follows easily that if X is a complete and unbounded Ahlfors Q-regular metric space as in Theorem 4.9, then there exists a constant $C = C(Q, p, \widetilde{p}, c_\mu)$ such that

$$\frac{\Lambda_1^\infty(E \cap B(a, R))}{R} \le C\mathrm{cap}_{B_p}(E \cap B(a, R), B(a, 2R)) \tag{29}$$

whenever $E \subset X$, $R > 0$, and $a \in X$.

As a corollary we have the following.

Corollary 4.11. *Suppose (X, d, μ) is a metric measure space as in Theorem 4.9. There exists a positive constant $C_2 = C_2(Q, p, \widetilde{p}, c_\mu)$ such that*

$$C_2 \left(\log \frac{R}{r}\right)^{1-p} \le \mathrm{cap}_{B_p}(B(x, r), B(x, R)) \tag{30}$$

for every $x \in X$ and every pair of positive numbers r, R such that $r < \frac{R}{2}$.

Proof. We apply Theorem 4.10 for $h(t) = t^{Q-\widetilde{p}}$. We notice (see [Kor07, Theorem 4.2]) that there exists a constant $C_2' = C_2'(Q, p, \widetilde{p}, c_\mu)$ such that

$$\frac{1}{C_2'} \le \frac{\Lambda_{Q-\widetilde{p}}^\infty(B(x, r))}{\left(\int_0^r t^{(p'-1)(Q-\widetilde{p})} \frac{dt}{t}\right)^{p-1}} \le C_2' \tag{31}$$

for every $x \in X$ and every $r > 0$. The rest is routine.

□

Theorem 4.4 and Corollary 4.11 easily yield the following theorem, (cf. [Bou07]).

Theorem 4.12. *Suppose (X, d, μ) is a metric measure space as in Theorem 4.9. There exists $C_0 = C_0(Q, p, c_\mu) > 0$ such that*

$$\frac{1}{C_0} \left(\log \frac{R}{r}\right)^{1-p} \le \mathrm{cap}_{B_p}(B(x, r), B(x, R)) \le C_0 \left(\log \frac{R}{r}\right)^{1-p} \tag{32}$$

for every $x \in X$ and every pair of positive numbers r, R such that $r < \frac{R}{2}$.

A set $E \subset X$ is said to be *of Besov B_p-capacity zero* if $\mathrm{cap}_{B_p}(E \cap \Omega, \Omega) = 0$ for all open and bounded $\Omega \subset X$. In this case we write $\mathrm{cap}_{B_p}(E) = 0$. The following lemma is obvious.

Lemma 4.13. *A countable union of sets of Besov B_p-capacity zero has Besov B_p-capacity zero.*

The next lemma shows that, if E is bounded, one needs to test only a single bounded open set Ω containing E in showing that E has zero Besov B_p-capacity.

Lemma 4.14. *Suppose that E is bounded and that there is a bounded neighborhood Ω of E with $\mathrm{cap}_{B_p}(E, \Omega) = 0$. Then $\mathrm{cap}_{B_p}(E) = 0$.*

Proof. The proof is similar to the proof of [Cos07, Lemma 3.13] and omitted. □

Corollary 4.15. *Suppose (X, d, μ) is a metric measure space as in Theorem 4.9. Let $E \subset X$ be such that $\mathrm{cap}_{B_p}(E) = 0$. Then $\Lambda_h(E) = 0$ for every measure function $h : [0, \infty) \to [0, \infty)$ such that*

$$\int_0^1 h(t)^{p'-1} \frac{dt}{t} < \infty. \tag{33}$$

In particular, the Hausdorff dimension of E is zero and $X \setminus E$ is connected.

Note that for every $\varepsilon > 0$ we can take $h = h_\varepsilon : [0, \infty) \to [0, \infty)$ in Corollary 4.15, where $h_\varepsilon(t) = |\log t|^{1-p-\varepsilon}$ for every $t \in (0, 1/2)$.

Proof. To prove the first claim, it is enough to assume, without loss of generality, that $h : [0, \infty) \to [0, \infty)$ is a continuous measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. (See Proposition 4.8.) If $\mathrm{cap}_{B_p}(E) = 0$, then there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E}$ and $\mathrm{cap}_{B_p}(\widetilde{E}) = 0$, hence we can assume without loss of generality that E is itself Borel. Since Λ_h^∞ is a countably subadditive set function and $\Lambda_h(E) = 0$ if and only if $\Lambda_h^\infty(E) = 0$ whenever h is a continuous measure function, it is enough to assume that E is bounded. Moreover, the discussion before Proposition 4.8 shows that it is enough to assume that E is in fact compact. For E compact the claim follows obviously from Theorem 4.10.

The second claim is a consequence of the first claim because for every $s \in (0, Q)$, the function $h_s : [0, \infty) \to [0, \infty)$ defined by $h_s(t) = t^s$ has the property (33). The third claim is a consequence of the Poincaré inequality. □

We also get upper bounds of the relative Besov p-capacity in terms of a certain Hausdorff measure.

Proposition 4.16. *Let $h : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism such that $h(t) = (\log \frac{1}{t})^{1-p}$ for all $t \in (0, \frac{1}{2})$. Suppose (X, d, μ) is a proper and unbounded Ahlfors Q-regular metric space. Let E be a compact subset of X. There exists a constant C depending only on p and on the Ahlfors regularity of X such that $\mathrm{cap}_{B_p}(E, \Omega) \leq C\Lambda_h(E)$ for every bounded and open set Ω containing E.*

Proof. The proof is similar to the proof of [Cos07, Proposition 3.17]. We present it for the convenience of the reader. We can assume without loss of generality that $\Lambda_h(E) < \infty$. Let Ω be a bounded open set containing E. We denote by δ the distance from E to the complement of Ω. Without loss of generality we can assume that $0 < \delta < 1$. We fix $0 < \varepsilon < 1$ such that $0 < \varepsilon < \frac{\delta^2}{4}$. Then $r < \varepsilon$ implies $\log\left(\frac{\delta}{2r}\right) \geq \frac{1}{2}\log\left(\frac{1}{r}\right)$. We cover E

by finitely many open balls $B(x_i, r_i)$ such that $r_i < \frac{\varepsilon}{2}$. Since we may assume that the balls $B(x_i, r_i)$ intersect E, we have $B(x_i, \frac{\delta}{2}) \subset \Omega$. As in [HKM93, p. 48] we obtain

$$\begin{aligned} \operatorname{cap}_{B_p}(E,\Omega) &\leq \sum_i \operatorname{cap}_{B_p}(B(x_i,r_i),\Omega) \leq \sum_i \operatorname{cap}_{B_p}(B(x_i,r_i),B(x_i,\frac{\delta}{2})) \\ &\leq C(n,p)\sum_i \left(\log\frac{1}{r_i}\right)^{1-p}. \end{aligned}$$

In the last step we also used formula (32) for the Besov B_p-capacity of spherical condensers together with our choice of ε. Taking the infimum over all such coverings and letting $\varepsilon \to 0$, we conclude $\operatorname{cap}_{B_p}(E, \Omega) \leq C\Lambda_h(E)$. This finishes the proof of the proposition. □

Proposition 4.16 gives another sufficient condition to obtain sets of Besov p-capacity zero.

Theorem 4.17. *Let $h : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism such that $h(t) = (\log \frac{1}{t})^{1-p}$ for all $t \in (0, \frac{1}{2})$. Then $\Lambda_h(E) < \infty$ implies $\operatorname{cap}_{B_p}(E) = 0$ for every $E \subset X$.*

Proof. The proof is similar to the proof of [Cos07, Theorem 3.16]. Since Λ_h is a Borel regular measure, we may assume that E is a Borel set and furthermore, in light of the Choquet capacitability theorem, we may assume that E is compact. We let $M = C\Lambda_h(E)$, where C is the constant from Proposition 4.16. Since $\Lambda_h(E) < \infty$, we have that $\mu(E) = 0$, while Proposition 4.16 implies that $\operatorname{cap}_{B_p}(E, \Omega) \leq M$ for every bounded and open set Ω containing E. Let $\Omega \subset X$ be a bounded open set containing E. From Lemma 4.14 it is enough to show that $\operatorname{cap}_{B_p}(E, \Omega) = 0$. We choose a descending sequence of bounded open sets

$$\Omega = \Omega_1 \supset\supset \Omega_2 \supset\supset \ldots \supset\supset \cap_i \Omega_i = E$$

and a sequence $\varphi_i \in BW(E, \Omega_i)$ with $[\varphi_i]^p_{B_p(\Omega_i)} < M+1$. Then φ_i is a bounded sequence in $B_p(\Omega)$. Because φ_i converges pointwise to a function ψ which is 0 in $X \setminus E$ and 1 on E, we have from Mazur's lemma and the reflexivity of $B^0_p(\Omega)$ that $\psi \in B^0_p(\Omega)$. That is, there exists a subsequence denoted again by φ_i such that $\varphi_i \to \psi$ weakly in $B^0_p(\Omega)$ and a sequence $\widetilde{\varphi}_i$ of convex combinations of φ_j,

$$\widetilde{\varphi}_i = \sum_{j=i}^{j_i} \lambda_{i,j}\varphi_j, \quad \lambda_{i,j} \geq 0, \text{ and } \sum_{j=i}^{j_i} \lambda_{i,j} = 1,$$

such that $\widetilde{\varphi}_i \to \psi$ in $B^0_p(\Omega)$. Without loss of generality we can assume that $\widetilde{\varphi}_i \to \psi$ pointwise in X as $i \to \infty$. The convexity of the Besov seminorm and the choice of the sequence φ_i imply, together with the closedness of $BW(E, \Omega_i)$ under finite convex combinations, that $\widetilde{\varphi}_i \in BW(E, \Omega_i)$ for every integer $i \geq 1$. Since $\mu(E) = 0$, $\psi = 0$ in $X \setminus E$, and $\widetilde{\varphi}_i \to \psi$ in $B^0_p(\Omega)$, it follows that $||\psi||_{B_p(\Omega)} = 0$. This implies $||\widetilde{\varphi}_i||_{B_p(\Omega)} \to 0$ as $i \to \infty$, hence

$$0 \leq \operatorname{cap}_{B_p}(E, \Omega) \leq \lim_{i\to\infty} [\widetilde{\varphi}_i]^p_{B_p(\Omega)} = 0$$

This finishes the proof. □

5. Besov capacity and quasicontinuous functions

In this section we study a global Besov capacity and quasicontinuous functions in Besov spaces.

5.1. **Besov Capacity.**

Definition 5.1. For a set $E \subset X$ define

$$\mathrm{Cap}_{B_p}(E) = \inf\{||u||^p_{L^p(X)} + [u]^p_{B_p(X)} : u \in S(E)\},$$

where u runs through the set

$$S(E) = \{u \in B_p(X) : \; u = 1 \text{ in a neighborhood of } E\}.$$

Since $B_p(X)$ is closed under truncations from below by 0 and from above by 1 and since these truncations do not increase the Besov p-norm, we may restrict ourselves to those functions $u \in S(E)$ for which $0 \le u \le 1$. We get the same capacity if we consider the apparently larger set of admissible functions, namely

$$\widetilde{S}(E) = \{u \in B_p(X) : \; u \ge 1 \; \mu\text{-a.e. in a neighborhood of } E\}.$$

Moreover, we have the following lemma:

Lemma 5.2. *If K is compact, then*

$$\mathrm{Cap}_{B_p}(K) = \inf\{||u||^p_{L^p(X)} + [u]^p_{B_p(X)} : u \in S_0(K)\}$$

where $S_0(K) = S(K) \cap \mathrm{Lip}_0(X)$.

Proof. Let $u \in S(K)$. Since $B_p(X) = B^0_p(X)$, we may choose a sequence of functions $\varphi_j \in \mathrm{Lip}_0(X)$ converging to u in $B_p(X)$. Let U be a bounded and open neighborhood of K such that $u = 1$ in U. Let $\psi \in \mathrm{Lip}(X)$, $0 \le \psi \le 1$ be such that $\psi = 1$ in $X \setminus U$ and $\psi = 0$ in $\widetilde{U} \subset\subset U$, an open neighborhood of K. From Lemma 3.8 we see that the functions $\psi_j = 1 - (1-\varphi_j)\psi$ converge to $1 - (1-u)\psi$ in $B_p(X)$. This establishes the assertion since $1 - (1-u)\psi = u$.

□

We have a result similar to Theorem 4.2, namely:

Theorem 5.3. *The set function $E \mapsto \mathrm{Cap}_{B_p}(E), E \subset X$ is a Choquet capacity. In particular*

(i) *If $E_1 \subset E_2$, then $\mathrm{Cap}_{B_p}(E_1) \le \mathrm{Cap}_{B_p}(E_2)$.*

(ii) *If $E = \bigcup_i E_i$, then*

$$\mathrm{Cap}_{B_p}(E) \le \sum_i \mathrm{Cap}_{B_p}(E_i).$$

We have introduced two different capacities, and it is next shown that they have the same zero sets.

Let $\Omega, \widetilde{\Omega}$ be bounded and open subsets of X such that $\Omega \subset\subset \widetilde{\Omega}$. Let $\eta \in \mathrm{Lip}_0(\widetilde{\Omega})$ be a cut-off function as in Remark 3.9. Suppose K is a compact subset of Ω. Then, if $u \in S_0(K)$, we have that $u\eta$ is admissible for the condenser $(K, \widetilde{\Omega})$. Therefore

$$\mathrm{cap}_{B_p}(K, \widetilde{\Omega}) \le [u\eta]^p_{B_p(\widetilde{\Omega})} \le ||u\eta||^p_{B_p(\widetilde{\Omega})} \le C\,||u||^p_{B_p(X)} \tag{34}$$

where C depends only on Q, p, c_μ, diam $\widetilde{\Omega}$ and $\operatorname{dist}(\Omega, X \setminus \widetilde{\Omega})$. (See Remark 3.9.) Since $||u||_{B_p(X)} = ||u||_{L^p(X)} + [u]_{B_p(X)}$, we have

$$||u||^p_{B_p(X)} \leq 2^p(||u||^p_{L^p(X)} + [u]^p_{B_p(X)}). \tag{35}$$

From (34) and (35) we get, by taking the infimum over all $u \in S_0(K)$, that

$$\operatorname{cap}_{B_p}(K, \widetilde{\Omega}) \leq 2^p\, C \operatorname{Cap}_{B_p}(K), \tag{36}$$

where C is the constant from (34).

Since both $\operatorname{cap}_{B_p}(\cdot, \widetilde{\Omega})$ and $\operatorname{Cap}_{B_p}(\cdot)$ are Choquet capacities, we obtain:

Theorem 5.4. *There exists $C > 0$ depending only on Q, p, c_μ, $\operatorname{dist}(\Omega, X \setminus \widetilde{\Omega})$ and* diam $\widetilde{\Omega}$ *such that*

$$\operatorname{cap}_{B_p}(E, \widetilde{\Omega}) \leq C \operatorname{Cap}_{B_p}(E) \tag{37}$$

for every $E \subset \Omega$.

Corollary 5.5. *If* $\operatorname{Cap}_{B_p}(E) = 0$, *then* $\operatorname{cap}_{B_p}(E) = 0$.

We also have a converse result, namely:

Theorem 5.6. *If* $\operatorname{cap}_{B_p}(E) = 0$, *then* $\operatorname{Cap}_{B_p}(E) = 0$.

Proof. The proof is similar to the proof of [Cos07, Theorem 4.6] and omitted. □

Remark 5.7. For $E \subset X$ compact we see from the proof of Lemma 4.14 and Theorem 5.6 that it is enough to have $\operatorname{cap}_{B_p}(E, \Omega) = 0$ for one bounded open set $\Omega \subset X$ with $E \subset \Omega$ in order to have $\operatorname{Cap}_{B_p}(E) = 0$.

It is desirable to know when a set is negligible for a Besov space. If there is an isometric isomorphism between two normed spaces X and Y we write $X = Y$. In particular, if E is relatively closed subset of Ω, then by

$$B^0_p(\Omega \setminus E) = B^0_p(\Omega)$$

we mean that each function $u \in B^0_p(\Omega)$ can be approximated in B_p-norm by functions from $\operatorname{Lip}_0(\Omega \setminus E)$.

Theorem 5.8. *Suppose that E is a relatively closed subset of Ω. Then*

$$B^0_p(\Omega \setminus E) = B^0_p(\Omega)$$

if and only $\operatorname{Cap}_{B_p}(E) = 0$.

Proof. Suppose that $\operatorname{Cap}_{B_p}(E) = 0$. Let $\varphi \in \operatorname{Lip}_0(\Omega)$ and choose a sequence u_j of functions in $B_p(X)$ such that $0 \leq u_j \leq 1$, $u_j = 1$ in a neighborhood of E and $u_j \to 0$ in $B_p(X)$. For every $j \geq 1$ we define $w_j = (1 - u_j)\varphi$. Then from Remark 3.10 and the properties of the functions φ and u_j, it follows that w_j is a bounded sequence of functions in $B_p(X)$, compactly supported in $\Omega \setminus E$. Lemma 3.14 implies that w_j is a sequence in $B^0_p(\Omega \setminus E)$. Moreover, Lemma 3.8 implies, since $\varphi - w_j = u_j\varphi$ for every $j \geq 1$ and since $||u_j||_{B_p(X)} \to 0$, that w_j converges to φ in $B_p(X)$. Since w_j is a sequence in $B^0_p(\Omega \setminus E)$, it follows that $\varphi \in B^0_p(\Omega \setminus E)$. Hence

$$B^0_p(\Omega) \subset B^0_p(\Omega \setminus E)$$

and since the reverse inclusion is trivial, the sufficiency is established.

For the only if part, let $K \subset E$ be compact. It suffices to show that $\mathrm{Cap}_{B_p}(K) = 0$. Choose $\varphi \in \mathrm{Lip}_0(\Omega)$ with $\varphi = 1$ in a neighborhood of K. Since $B_p^0(\Omega \setminus E) = B_p^0(\Omega)$, we may choose a sequence of functions $\varphi_j \in \mathrm{Lip}_0(\Omega \setminus K)$ such that $\varphi_j \to \varphi$ in $B_p(\Omega)$. Consequently

$$\mathrm{Cap}_{B_p}(K) \leq \left(\lim_{j\to\infty} ||\varphi_j - \varphi||^p_{L^p(X)} + [\varphi_j - \varphi]^p_{B_p(X)} \right) = 0,$$

and the theorem follows. □

5.2. **Quasicontinuous functions.** We show that for each $u \in B_p(X)$ there is a function v such that $u = v$ μ-a.e. and that v is B_p-quasicontinuous, i.e. v is continuous when restricted to a set whose complement has arbitrarily small Besov B_p-capacity. Moreover, this quasicontinuous representative is unique up to a set of Besov B_p-capacity zero.

Definition 5.9. A function $u : X \to \mathbf{R}$ is B_p-*quasicontinuous* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\mathrm{Cap}_{B_p}(G) < \varepsilon$ and the restriction of u to $X \setminus G$ is continuous.

A sequence of functions $\psi_j : X \to \mathbf{R}$ converges B_p-*quasiuniformly* in X to a function ψ if for every $\varepsilon > 0$ there is an open set G such that $\mathrm{Cap}_{B_p}(G) < \varepsilon$ and $\psi_j \to \psi$ uniformly in $X \setminus G$.

We say that a property holds B_p-*quasieverywhere*, or simply q.e., if it holds except on a set of Besov B_p-capacity zero.

Theorem 5.10. *Let $\varphi_j \in C(X) \cap B_p(X)$ be a Cauchy sequence in $B_p(X)$. Then there is a subsequence φ_k which converges B_p-quasiuniformly in X to a function $u \in B_p(X)$. In particular, u is B_p-quasicontinuous and $\varphi_k \to u$ B_p-quasieverywhere in X.*

Proof. The proof is similar to the proof of [HKM93, Theorem 4.3] and omitted. □

Theorem 5.10 implies the following corollary.

Corollary 5.11. *Suppose that $u \in B_p(X)$. Then there exists a B_p-quasicontinuous Borel function $v \in B_p(X)$ such that $u = v$ μ-a.e.*

Proof. Since $u \in B_p(X)$, from Theorem 3.13 there exists a sequence of functions φ_j in $\mathrm{Lip}_0(X)$ converging to u in $B_p(X)$. Passing to subsequences if necessary, we can assume that $\varphi_j \to u$ pointwise μ-a.e. in X and that

$$2^{jp} \left(||\varphi_{j+1} - \varphi_j||^p_{L^p(X)} + [\varphi_{j+1} - \varphi_j]^p_{B_p(X)} \right) < 2^{-j}$$

for every $j = 1, 2, \dots$. Defining $E_j = \{x \in X : |\varphi_{j+1} - \varphi_j| > 2^{-j}\}$ and letting $E = \cap_{k=1}^{\infty} \cup_{j=k} E_j$, the proof of Theorem 5.10 yields the existence of a function $v \in B_p(X)$, such that $\varphi_j \to v$ in $B_p(X)$ and pointwise in $X \setminus E$. Since E is a Borel set of Besov B_p-capacity zero and the functions φ_j are continuous, this finishes the proof. □

Theorem 5.12. *Let $u \in B_p(X)$. Then $u \in B_p^0(\Omega)$ if and only if there exists a B_p-quasicontinuous function v in X such that $u = v$ μ-a.e. in Ω and $v = 0$ q.e. in $X \setminus \Omega$.*

Proof. Fix $u \in B_p^0(\Omega)$ and let $\varphi_j \in \mathrm{Lip}_0(\Omega)$ be a sequence converging to u in $B_p(\Omega)$. By Theorem 5.10 there is a subsequence of φ_j which converges B_p-quasieverywhere in X to a B_p-quasicontinuous function v in X such that $u = v$ μ-a.e. in Ω and $v = 0$ q.e. in $X \setminus \Omega$. Hence v is the desired function.

To prove the converse, we assume first that Ω is bounded. Because the truncations of v converge to v in $B_p(\Omega)$, we can assume that v is bounded. Without loss of generality, since v is B_p-quasicontinuous and $v = 0$ q.e. outside Ω we can assume that in fact $v = 0$ everywhere in $X \setminus \Omega$. Choose open sets G_j such that v is continuous on $X \setminus G_j$ and $\mathrm{Cap}_{B_p}(G_j) \to 0$. By passing to a subsequence, we may pick a sequence φ_j in $B_p(X)$ such that $0 \le \varphi_j \le 1$, $\varphi_j = 1$ everywhere in G_j, $\varphi_j \to 0$ μ-a.e. in X, and

$$||\varphi_j||^p_{L^p(X)} + [\varphi_j]^p_{B_p(X)} \to 0.$$

Then from Remark 3.10 we have that $w_j = (1 - \varphi_j)v$ is a bounded sequence in $B_p(\Omega)$. Moreover, for every $j \ge 1$, we have $\lim_{x \to y, x \in \Omega} w_j(x) = 0$ for all $y \in \partial\Omega$. Thus, from Lemma 3.15, we have that w_j is a sequence in $B^0_p(\Omega)$. Clearly $w_j \to v$ in $L^p(X)$ and pointwise μ-a.e. in X. This, together with the boundedness of the sequence w_j in $B^0_p(\Omega)$, implies via Mazur's lemma that $v \in B^0_p(\Omega)$. The proof is complete in case Ω is bounded.

Assume that Ω is unbounded. We can assume again, without loss of generality, that v is bounded and that $v = 0$ everywhere in $X \setminus \Omega$. We fix $x_0 \in X$. For every $k \ge 2$ let $\varphi_k \in \mathrm{Lip}_0(B(x_0, k^2))$ be such that $0 \le \varphi_k \le 1$, $\varphi_k = 1$ on $B(x_0, k)$ and $[\varphi_k]_{B_p(X)} \le C(\ln k)^{1-p}$. (See (25).) Then $v_k = v\varphi_k \in B^0_p(\Omega \cap B(x_0, k^2)) \subset B^0_p(\Omega)$ for every $k \ge 2$ and like in Theorem 3.13, we get

$$||v - v_k||_{B_p(X)} \to 0,$$

which implies that $v \in B^0_p(\Omega)$. This finishes the proof. □

We denote by

$$Q^{B_p} = Q^{B_p}(X)$$

the set of all functions $u \in B_p(X)$ such that there exists a sequence $\varphi_j \in C(X) \cap B_p(X)$ converging to u both in $B_p(X)$ and B_p-quasiuniformly. It follows immediately from Theorem 5.10 that the functions in Q^{B_p} are B_p-quasicontinuous and for each $v \in B_p(X)$ there is $u \in Q^{B_p}$ such that $u = v$ μ-a.e. We soon show that, conversely, each B_p-quasicontinuous function v of $B_p(X)$ belongs to Q^{B_p}.

Theorem 5.13. *Let $u \in Q^{B_p}$. If $u \ge 1$ B_p-quasieverywhere on E, then*

$$\mathrm{Cap}_{B_p}(E) \le ||u||^p_{L^p(X)} + [u]^p_{B_p(X)}.$$

Proof. The proof is similar to the proof of [HKM93, Lemma 4.7] and omitted. □

This result has the following corollary.

Corollary 5.14. *Suppose that Ω is open and bounded and let $E \subset\subset \Omega$. Let $u \in Q^{B_p}$. Suppose that $u \ge 1$ quasieverywhere on E and that u has compact support in Ω. Then*

$$\mathrm{cap}_{B_p}(E, \Omega) \le [u]^p_{B_p(\Omega)}.$$

We know that Cap_{B_p} is an outer capacity. It satisfies the following compatibility condition (see [Kil98]):

Theorem 5.15. *Suppose that G is open and $\mu(E) = 0$. Then*

$$\mathrm{Cap}_{B_p}(G) = \mathrm{Cap}_{B_p}(G \setminus E). \tag{38}$$

Proof. The proof is similar to the proof of [Cos07, Theorem 4.15]. We present it for the convenience of the reader. Obviously we have $\mathrm{Cap}_{B_p}(G \setminus E) \le \mathrm{Cap}_{B_p}(G)$. Conversely, we can assume without loss of generality that $\mathrm{Cap}_{B_p}(G \setminus E) < \infty$. We fix $\varepsilon > 0$. There exists a function $u_\varepsilon \in B_p(X)$ and an open neighborhood W of $G \setminus E$ such that $u_\varepsilon = 1$ on W and

$$||u_\varepsilon||^p_{L^p(X)} + [u_\varepsilon]^p_{B_p(X)} < \mathrm{Cap}_{B_p}(G \setminus E) + \varepsilon.$$

Since $\mu(E) = 0$, we can assume without loss of generality that in fact $u_\varepsilon = 1$ on E. But then $u_\varepsilon = 1$ on $W \cup G$ which is an open neighborhood of G, hence

$$\mathrm{Cap}_{B_p}(G) \le ||u_\varepsilon||^p_{L^p(X)} + [u_\varepsilon]^p_{B_p(X)} < \mathrm{Cap}_{B_p}(G \setminus E) + \varepsilon.$$

The desired conclusion follows by letting $\varepsilon \to 0$.

□

We state now the uniqueness of a B_p-quasicontinuous representative.

Theorem 5.16. *Let f and g be B_p-quasicontinuous functions on X such that*

$$\mu(\{x : f(x) \neq g(x)\}) = 0.$$

Then $f = g$ B_p-quasieverywhere on X.

Proof. The proof is verbatim the proof from [Kil98, p. 262]. □

Combining Theorem 5.13 and Theorem 5.16 we obtain the following corollary.

Corollary 5.17. *Suppose that $E \subset X$. Then*

$$\mathrm{Cap}_{B_p}(E) = \inf\{||u||^p_{L^p(X)} + [u]^p_{B_p(X)}\},$$

where the infimum is taken over all B_p-quasicontinuous $u \in B_p(X)$ such that $u = 1$ B_p-quasieverywhere on E.

Corollary 5.11 and Theorem 5.16 imply that each $u \in B_p(X)$ has a "unique" quasi-continuous version.

Corollary 5.18. *Suppose that $u \in B_p(X)$. Then there exists a B_p-quasicontinuous function v such that $u = v$ μ-a.e. Moreover, if $\widetilde{v}$ is another B_p-quasicontinuous function such that $u = \widetilde{v}$ μ-a.e., then $v = \widetilde{v}$ B_p-quasieverywhere.*

We have a result similar to Corollary 5.18 for locally integrable functions with finite B_p-seminorm.

Corollary 5.19. *Suppose that $u \in L^1_{loc}(X)$ such that $[u]_{B_p(X)} < \infty$. Then there exists a B_p-quasicontinuous Borel function v such that $u = v$ μ-a.e. Moreover, if $\widetilde{v}$ is another B_p-quasicontinuous Borel function such that $u = \widetilde{v}$ μ-a.e., then $v = \widetilde{v}$ B_p-quasieverywhere.*

Proof. We prove the "uniqueness" first. Suppose $v, \widetilde{v}$ are two B_p-quasicontinuous Borel functions such that $v = u$ μ-a.e. and $\widetilde{v} = u$ μ-a.e. Let $w = v - \widetilde{v}$. We notice that w is B_p-quasicontinuous and belongs to $B_p(X)$ because $w = 0$ μ-a.e. in X. Hence from Corollary 5.18 we have that $w = 0$ B_p-quasieverywhere. The "uniqueness" is proved.

We prove now the existence. Fix $x_0 \in X$. For every integer $k \ge 1$ we choose a 2^{1-k}-Lipschitz function η_k supported in $B(x_0, 2^{k+1})$ such that $\eta_k = 1$ on $B(x_0, 2^k)$. We have

$$\eta_{k+1}\eta_k = \eta_k \tag{39}$$

for every integer $k \ge 1$. For a fixed integer $k \ge 1$, we define $u_k = \eta_k u$. Then $u_k \in L^p(X)$ because $u \in L^p_{loc}(X)$ and $\eta_k \in \mathrm{Lip}_0(B(x_0, 2^{k+1}))$. Moreover, from Lemma 3.11, it follows that $[\eta_k u - \eta_k u_{B(x_0,2^k)}]_{B_p(X)} < \infty$. From this and the fact that $\eta_k \in B_p(X)$ it follows that $u_k \in B_p(X)$. Therefore, from Corollary 5.11 it follows that there exists $\widetilde{u}_k \in B_p(X)$ a B_p-quasicontinuous Borel function such that $\widetilde{u}_k = u_k$ μ-a.e. in X. In particular, since $\eta_k = 1$ in $B(x_0, 2^k)$, this implies that $\widetilde{u}_k = u$ μ-a.e. in $B(x_0, 2^k)$. So, for every integer $k \geq 1$ we have that $\widetilde{u}_{k+1}$ is a B_p-quasicontinous Borel representative of $\eta_{k+1}u$, hence $\eta_k \widetilde{u}_{k+1}$ is a B_p-quasicontinuous Borel representative of $\eta_k \eta_{k+1} u = u_k$, where the equality follows from the definition of u_k and (39). This implies that both $\eta_k \widetilde{u}_{k+1}$ and $\widetilde{u}_k$ are two B_p-quasicontinuous Borel representatives of $u_k \in B_p(X)$, hence from Corollary 5.18 we can assume that $\widetilde{u}_k = \eta_k \widetilde{u}_{k+1}$ in $B(x_0, 2^k)$. Since $\eta_k = 1$ on $B(x_0, 2^k)$, this means in particular that we can assume that $\widetilde{u}_k(x) = \widetilde{u}_{k+1}(x)$ for every x in $B(x_0, 2^k)$.

So, we constructed a sequence of B_p-quasicontinuous Borel functions $\widetilde{u}_k$ in $B_p(X)$ satisfying the following properties:

$$\begin{aligned} \widetilde{u}_k(x) &= u(x) && \text{for } \mu\text{-a.e. } x \text{ in } B(x_0, 2^k) \\ \widetilde{u}_l(x) &= \widetilde{u}_k(x) && \text{for every } x \text{ in } B(x_0, 2^k) \text{ and } l \geq k \geq 1. \end{aligned}$$

We define $\widetilde{u} : X \to \overline{\mathbf{R}}$ by

$$\widetilde{u}(x) = \lim_{k \to \infty} \widetilde{u}_k(x).$$

Thus, $\widetilde{u}$ is a B_p-quasicontinuous Borel function and $u = \widetilde{u}$ μ-a.e. This proves the existence of a B_p-quasicontinuous Borel representative of u. The claim follows.

□

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