Fast CDO computations in the affine Markov chain model

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Abstract

Abstract: It is shown that credit basket derivatives such as CDOs which depend on a large number $M$ of firms ($M \geq 100$ is typical in some contexts) can be modeled in a parsimonious and computationally efficient manner within the affine Markov chain (AMC) framework for multifirm credit migration introduced in a companion paper [Hurd and Kuznetsov(2004)]. The proposed method has a number of merits. First, since our AMC models extend the intensity based doubly stochastic framework for multifirm default to a credit migration setting, they can be flexibly fit to observed market bond data for the individual constituent firms, and they can in principle explain the dynamics of this data. Second, the method handles some of the variations of CDOs such as nonhomogeneous hazard rates and unequal notational amounts that industry practitioners need to use. Thirdly, the approximation schemes we use can be verified in special cases, and prove to perform to basis point accuracy with typical parameter choices. Finally, in our model, prices and sensitivities for such derivatives are reduced to low dimensional integrals which can often be computed on a desktop computer in fractions of seconds. In this paper we develop an illustrative version of the modeling framework and present a number of sample CDO computations which illustrate the power of the method.

Key words: CDO, multifirm credit migration, stochastic intensity, default correlation, credit spread
1 Introduction

The search for accurate, fast, flexible methods for pricing and hedging credit derivatives such as CDOs (collateralized debt obligations) and other large scale credit securities has become of primary importance to the finance industry in the past few years. Structured products such as these have become the paradigm of how to securitize risk in the market.

The affine Markov chain (AMC) framework introduced in [Hurd and Kuznetsov(2004)] is an approach to the joint modeling of multifirm credit migration and default which generalizes the intensity based methods initiated in [Jarrow and Turnbull(1995), Lando(1998), Duffie and Singleton(1999)] and extended to large scale problems in [Duffie and Garleanu(2001)]. We were also inspired by the affine credit default framework of [Chen and Filipović(2004)]. The new modeling premise which underlies the framework is that firms undergo credit rating migrations which are independent conditioned on an intensity which is simply the speed of a stochastic time change. We assume that the rating classes are “ideal” in that they represent the best estimate of a firm’s credit worthiness at any instant in time and therefore that firms with the same rating have identical default risk. In the basic AMC framework adopted here, default dependence is introduced by a single stochastic time change which represents the “speed” of the credit market. This stochastic time change, together with other market factors such as the interest rate, is efficiently modeled by a multivariate affine process. Various simple components of the time change are included which build in distinct default correlation mechanisms: a diffusive component represents the “normal market”, a component in which the speed jumps leads to jumps in hazard rates, while jumps in the time itself lead to the possibility of simultaneous defaults (i.e. a “contagion” effect). The concept of a stochastic market time has been used by numerous authors to explain various models of asset prices. For example, the well known VG model [Madan et al.(1999)] can be thought of as geometric Brownian motion with a gamma–distributed stochastic time change. Stochastic volatility models also have a similar interpretation. Our use of the time–change concept is in the same spirit. In [Hurd and Kuznetsov(2004)] it was shown how essential default computations for one, two or more firms could be reduced to explicit formulas or one dimensional integrals. The present paper addresses how in the AMC setting, the exact formulas for pricing CDOs on $M$ firms can be computed efficiently using one of several approximation schemes. These approximations short-circuit the “curse of dimensionality” and result in formulas which are again essentially one dimensional integrals which can be computed in fractions of seconds.

Our modeling approach can be contrasted with the standard industry approaches to CDOs known as copula or factor methods, of which the best known versions are those of [Laurent and Gregory(2003), Hull and White(2004), Andersen et al.(2003)]. In those models advantages of flexibility and ease of calibration are offset by the deficiency that hazard rates and default correlations are introduced without regard to the dynamics of the underlying companies (essentially by fitting functional forms such as a one factor normal copula to current market data). This means in par-
ticular that those methods provide no means to model correlations in credit spread changes. On the other hand, the AMC approach, being a consistent model of dynamic correlations and credit spreads, has no such deficiency. In view of their differences, a direct comparison of the AMC modeling approach to the copula approach is difficult: however, we show how the benchmark CDO computations found in [Hull and White(2004)] can be qualitatively reproduced by a very simple version of our method. In general, our main results for CDO pricing show how a far more realistic model can be computed leading to results which are qualitatively consistent with, but much more extensive than those coming from the copula/factor methods. The two frameworks lead to algorithms for CDOs which perform at comparable speeds.

The organization of the paper is as follows. Section 2 introduces the main modeling ingredients for multifirm credit migration in the AMC framework, and illustrates the method by computing one-firm transition and default probabilities in terms of basic building block functions called $G_1, G_2, G_3$. Section 3 describes the basic structure of a CDO tranche as a swap between a premium leg and an insurance leg. The main results of this section are Theorems 3.1 and 3.2 which give formulas for the two legs. Section 4 gives some approximation schemes for fast computation of CDOs. Section 5 presents numerical results for CDO spreads in the typical AMC setting introduced in [Hurd and Kuznetsov(2004)] while Section 6 compares the benchmark computations for the normal copula model from [Hull and White(2004)] to those of a very simple version of our model. Section 7 discusses sensitivity computations for CDOs. Finally, the formulas for the building block functions $G_1, G_2, G_3$ are given in an appendix.

In summary, the present paper introduces an efficient and flexible approach for evaluating prices and hedge ratios for large scale credit derivative securities such as CDOs. The results pass visual inspection to be a plausible description of real credit markets, but much more research will be needed to validate the quantitative aspects of the model.

## 2 The affine Markov chain model

The credit model of [Hurd and Kuznetsov(2004)] for $M$ firms is built from the following basic ingredients:

1. A vector of independent $K + 1$-state Markov chains $\tilde{Y}_t = (\tilde{Y}_t^1, \tilde{Y}_t^2, \ldots, \tilde{Y}_t^M)$ where $\tilde{Y}_t^k \in \{0, 1, \ldots, K\}$ and 0 is an absorbing state.

2. The “market time” defined to be a stochastic time change process $\tau_t$ (a “subordinator”);

3. The spot interest rate process $r_t$.

We assume that $\tilde{Y}$ is independent of $r_t, \tau_t$. For simplicity in this paper we assume a fixed recovery fraction $R_0$ at default: stochastic recovery was included in [Hurd and Kuznetsov(2004)].
The Markov chains have identical Markov generators $\mathcal{L}_Y$ (a $K+1 \times K+1$ matrix) and thus identical node-to-node transition probabilities $P(\bar{Y}_t = j|\bar{Y}_0 = i)$ given by the semigroup $e^{t\mathcal{L}_Y}$. The market time may have absolutely continuous and jump components. Then we define the credit migration process to be $Y_t = \bar{Y}_{\tau_t}$. The state of $Y^i_t \in \{0, 1, \ldots, K\}$ represents the credit rating of firm $i$ at time $t$, where the absorbing state 0 is the default state. For example, we may map these states to Standard and Poor’s rating classes:

$$\{0, 1, \ldots, 7\} \leftrightarrow \{‘\text{default}', \text{CCC}, \text{BB}, \text{BBB}, \text{A}, \text{AA}, \text{AAA}\}.$$

The time of default of the $i$th firm is the stopping time

$$t_i^* = \inf\{t|Y^i_t = 0\}. \tag{1}$$

Thus the picture which describes the basic AMC model is that firms of the same rating have identical migration and default probabilities. Conditioned on $r_t, \tau_t$, firms undergo independent credit migration with identical transition probabilities: eventually every firm defaults. The stochastic time change $\tau_t$ leads to correlations between firm defaults. When the stochastic clock is running fast, migration and hence defaults happen relatively frequently; if the stochastic clock jumps, then simultaneous defaults may occur. To create an interesting range of possibilities, we model the time change and interest rate in terms of three underlying independent factors: a two-dimensional affine process $Z_t = (Z^1_t, Z^2_t)$ and a Poisson process $Z^3_t$. Then:

- The interest rate $r_t = \langle M_r \cdot Z_t \rangle$
- The time change has absolutely continuous and jump components

$$\tau_t = \tau^{(ac)}_t + \tau^{(jump)}_t = \int_0^t \langle M_r \cdot Z_s \rangle ds + m_r Z^3_t. \tag{2}$$

Here $m_r \geq 0$ while $M_r$ and $M_{\tau}$ are coefficient vectors from $\mathbb{R}_+^2$.

To be specific, let $Z^1$ be a CIR process with Markov generator

$$\mathcal{L}_{Z^1} f(x) = a(1-x)f'(x) + cxf''(x), \tag{3}$$

and $Z^2$ be an affine process with jumps defined by its Markov generator

$$\mathcal{L}_{Z^2} f(x) = \lambda_2 (f(x + h_2) - f(x)) - h_2 \lambda_2 x f'(x). \tag{4}$$

Finally, $Z^3$ (a jump part of the time change) is taken to be a Poisson process with fixed jump size $h_3$ and intensity $\lambda_3 = h_3^{-1}$:

$$Z^3_t = h_3 \Pi(h_3^{-1}t). \tag{5}$$

Note that $Z^2$ undergoes jumps of size $h_2 > 0$ with intensity $\lambda_2$ and then decays exponentially (with the speed of decay given by $h_2 \lambda_2$). Also $Z^1_t, Z^2_t, Z^3_t$ are normalized to have long term means of 1, 1, $t$ respectively so $M^1_r + M^2_r + m_r$ is equal to the average speed of the time change.
Remark 2.1. Note that the $M$ firms are exchangeable, since the components of $\tilde{Y}$ are independent and firms of the same ratings class are identically distributed.

The main computational building blocks are the functions $G_1$ and $G_2$ defined by

$$G_1(t; z; u, v) = E^0_x \left[ e^{-\int_0^t (u \cdot Z_s) ds} e^{-\langle v \cdot Z_t \rangle} \right]$$
(6)

and

$$G_2(t; z; u, v, w) = E^0_x \left[ e^{-\int_0^t (u \cdot Z_s) ds} \langle w \cdot Z_t \rangle e^{-\langle v \cdot Z_t \rangle} \right].$$
(7)

Here $z, u, v$ and $w$ are vectors in $\mathbb{R}^2$. Due to our choice of underlying factors these functions can be computed explicitly (see the appendix).

We will also need the explicit expression for the Laplace transform of $Z^3_t$

$$G_3(t; v) = E \left[ e^{-vZ^3_t} \right] = \exp \left( h_3^{-1} t (e^{-vh_3} - 1) \right).$$
(8)

2.1 Transition probabilities for the process $Y_t$

As we showed in [Hurd and Kuznetsov(2004)], this setup is computationally efficient: the affine structure of the stochastic time change works beautifully with the Markov chains. To illustrate, we show how one can compute node-to-node transition probabilities for each firm.

For simplicity, we assume that $L_Y$ can be diagonalized (recall each component of $Y$ has the same generator):

$$L_Y = QDQ^{-1},$$
(9)

where $D = \text{diag}\{\alpha_0, \alpha_1, \ldots, \alpha_K\}$ is a diagonal matrix and $Q = (q_{ij})_{i,j=0\ldots K}$ is a matrix whose columns are the corresponding eigenvectors of $L_Y$. Let the elements of $Q^{-1}$ be denoted as $Q^{-1} = (\tilde{q}_{ij})_{i,j=0\ldots K}$.

Solving the Kolmogorov equation shows that the probability semigroup for the process $Y_t$ is given (in matrix form)

$$P_y(t) = e^{tL_Y} = Q e^{tD} Q^{-1},$$
(10)

and the node-to-node transition probabilities for the process $\tilde{Y}_t^i$ are given by

$$p_{yj}(t) = P_{0,y} (\tilde{Y}_t^i = j) = \sum_{k=0}^K q_{yk} \tilde{q}_{kj} e^{\alpha_k t}.$$  
(11)

Here is the result for the real time credit migration process $Y_t$:

Lemma 2.2. Node to node rating transition probabilities for the $i$th firm are given by

$$P_{0,z,y}(Y_t^i = j) = \sum_{k=0}^K q_{yk} \tilde{q}_{kj} G_1(t, z; -\alpha_i M, 0) G_3(t, -m_r \alpha_k).$$
(12)
Proof:

\[
P_{0,z,y}(Y^i_t = j) = E_{0,z,y}[I\{\tilde{Y}_{\tau_t} = j\}] = E_{0,z,y}[E_{0,z,y}[I\{\tilde{Y}_{\tau_t} = j\} | \tau]]
\]

\[
= E_{0,z} \left[ \sum_{k=0}^{K} q_{yj} q_{kj} e^{\alpha_k \tau_t} \right] = \sum_{k=0}^{K} q_{yj} q_{kj} E_{0,z} [e^{\alpha_k \tau_t}] .
\]

The result follows since

\[
E_{0,z} [e^{\alpha_i \tau_t}] = E_{0,z} \left[ e^{\alpha_i \int_0^t (M \cdot Z_s) ds + \alpha_i Z_t^i} \right] = G_1(t, z; -\alpha_i M \tau, 0) G_3(t, -m \tau \alpha_i).
\]

### 3 Pricing CDOs

Collateralized debt obligations (CDOs) are basket credit derivatives involving a large number of companies. The underlying security is a portfolio of coupon paying corporate bonds on \( M \) firms: the \( i \)th bond is taken to have face value \( N_i \) (called the “notional”). A “synthetic CDO tranche” can be regarded as a credit swap between two parties, the insured and the insurer. The two components of the swap, the “premium leg” and the “insurance leg”, are the basic credit contingent claims. We now show how each can be priced separately by risk neutral expectation. Henceforth it is assumed that all probabilities are computed in the risk neutral measure.

The components of a CDO tranche are derivatives on the total loss of the bond portfolio due to default of the constituent names. The loss at time \( t \) as a fraction of the total notional is given by

\[
L_t = \sum_{i=1}^{M} \left( 1 - R_{t_i}^i \right) \frac{N_i}{N} I\{t_i^* < t\}
\]

where

- \( M \) is the number of firms;
- \( N_i \) is the notional of firm \( i \);
- \( N = \sum_{i=1}^{M} N_i \) is the total notional of the portfolio;
- \( t_i^* \) is the default time of the firm \( i \).

The total loss process can be rewritten as

\[
L_t = \tilde{L}_{\tau_t},
\]

where

\[
\tilde{L}_t = \sum_{i=1}^{M} (1 - R_0) \frac{N_i}{N} I\{\tilde{Y}_t^i = 0\}.
\]
The main results on CDO prices will be expressed in terms of the integrated cumulative distribution function of $\tilde{L}_t$:

$$\tilde{C}(x, t) = E[(x - \tilde{L}_t)^+] = \int_0^x \int_0^z dp(y, t) \, dz$$

### 3.1 Credit premium

A generalized premium leg for a CDO can be regarded as a contingent claim which is paid by the insured to the insurer as a fee for default insurance. It is assumed that the insured party pays continuously in time over the period $[0, T]$ at a stochastic rate $U_t$, and that $U_t$ depends only on the loss process $L_t$; $U_t = U(L_t)$ for some function $U(x)$. A typical example is the premium leg for a CDO tranche for fractional losses in a range $[\bar{x}, \bar{x}] \subset [0, 1]$

$$U(x) = \frac{1}{\bar{x} - x} [(\bar{x} - x)^+ - (\bar{x} - x)^+]$$

Special cases are the senior tranche $\bar{x} = 1, x > 0$, and the equity tranche $x = 0, \bar{x} < 1$. In section 5 we consider the following standard tranches:

$$[0, 0.3], [0.03, 0.07], [0.07, 0.10], [0.10, 0.15], [0.15, 0.30], [0.30, 1.0].$$

The assumption of continuous payments in time is for simplicity of exposition only. A more pragmatic choice such as quarterly payments can be easily accommodated.

The price of the premium leg is given by

$$V^U = E_{0, z} \left[ \int_0^T e^{-\int_0^t r_s ds} U(L_t) \, dt \right],$$

and our first main result is the following formula:

**Theorem 3.1.** If $U$ is a bounded measurable function then

$$V^U = \int_0^\infty H^U(\tau) F^P(\tau; z) \, d\tau,$$

where the function $H^U(\tau)$ is given by

$$H^U(\tau) = E[U(\tilde{L}_\tau)] = \frac{1}{\bar{x} - x} [\tilde{C}(\bar{x}, \tau) - \tilde{C}(x, \tau)]$$

and thus depends only on the parameters of the loss process $\tilde{L}_t$ and the payoff function $U$. The function $F^P(\tau; z)$ is given by

$$F^P(\tau; z) = \int_0^T E_{0, z} \left[ e^{-\int_0^t r_s ds} \delta(\tau_t - \tau) \right] \, dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega \tau} \left[ \int_0^T G_1(t, z; M_r + i\omega M_\tau, 0) G_3(t, i\omega M_\tau) \, dt \right] \, d\omega,$$
and thus depends only on the parameters of the interest rate and time change processes.

**Proof:** First note

\[
V^U = \int_0^T E_0 \left[ H^U(\tau_t) e^{-\int_0^\tau_s r_s ds} \right] dt
\]  

where \( H^U(\tau) \) is bounded and measurable. It is now sufficient to prove (17) for the (dense) set of complex exponential functions of the form \( H^U(\tau) = e^{-iw\tau}, w \in \mathbb{R} \) with the result extending by linearity to general bounded measurable \( H \) supported on \( \mathbb{R}_+ \). By the Fourier inversion theorem and the basic building block formulas, we have

\[
\int_0^T E_0 \left[ e^{-iw\tau_t} e^{-\int_0^\tau_s r_s ds} \right] dt = \int_0^T G_1(t, z; M_r + iwM_r, 0) G_3(t, iwm_\tau) dt
\]

\[
= \int_{-\infty}^\infty e^{-iw\tau} F^P(\tau; z) d\tau
\]

which completes the proof.

**Remark 3.2.**

1. To compute the function \( F^P(\tau; z) \) given by equation (19) it is sufficient to evaluate the integral in \( t \) numerically and then perform the inverse Fourier transform in \( w \).

2. An important advantage of our formula is that it separates the effects of the stochastic time change (hidden in \( F^P(\tau; z) \)) from all information about the Markov chains \( Y, \) the loss process \( L_t \) and the payoff function \( U \) (hidden in \( H^U(\tau) \)).

3. Thus, in particular, computing the function \( F^P \) for \( n \) choices of parameters and \( H^U \) for \( m \) parameter choices leads immediately to \( n \times m \) CDO prices.

### 3.2 Credit insurance

The insurer pays the insured a tranche of the losses by default of firms in the basket portfolio. A general claim of this type is defined by

\[
W^S = E_0 \left[ \int_0^T e^{-\int_0^\tau_s r_s ds} dS_t \right]
\]  

where \( S_t = S(L_t) \) for some deterministic function \( S(L) \) with \( S(0) = 0 \). A typical example is the default leg of a CDO tranche with range \([x, \bar{x}]\) where:

\[
S(x) = \frac{1}{\bar{x} - x} \left[ (x - \bar{x})^+ - (x - x)^+ \right] = 1 - U(x)
\]
We can simplify the expression for \( W^S \) by integrating by parts (\( S(t) \) is of finite variation)

\[
W^S = E_{0,z} \left[ e^{-\int_0^T r_s ds} S_T + \int_0^T r_t e^{-\int_0^t r_s ds} S_t dt \right].
\] (23)

The analog of theorem 3.1 for the price of the insurance leg is:

**Theorem 3.3.** If \( S \) is bounded and measurable with \( S(0) = 0 \) then

\[
W^S = \int_0^\infty H^S(\tau) F^I(\tau; z) d\tau,
\] (24)

where \( H^S(\tau) \) is given by \( H^S(\tau) = E[S(\bar{L}_\tau)] = 1 - H_U(\tau) \). The function \( F^I(\tau; z) \) depends only on the parameters of the interest rate and time change processes and is given by

\[
F^I(\tau; z) = \frac{1}{2\pi} \int_\mathbb{R} e^{i\omega \tau} \left\{ \int_0^T G_2(t, z; M_r + i\omega M_r, 0, M_r) G_3(t, iwm_r) dt \right\} dw.
\] (25)

Figure 2 shows a typical shape of the function \( F^I \) and six tranches of the function \( H^S \) when \( M = 100 \). The value of the insurance leg is obtained by integrating \( F^I \) against the appropriate \( H^S \).

### 3.3 CDO prices and tranche spreads

A simple CDO tranche \([\underline{x}, \bar{x}] \subset [0, 1]\) is a swap (a contract with zero value at time 0) of a multiple \( p \) (called the CDO spread) of the premium leg \( V^U \) with \( U \) given by (15) for a default leg \( W^S \) with \( S \) given by (22). Thus \( p = W^S/V^U \) is selected to balance the two legs. At future times the CDO price \( W^S - pV^U \) evolves stochastically.

### 4 Approximating the \( H \) functions

The process \( \tilde{L}_t \) is the sum of \( M \) independent random variables which is itself independent of the affine processes \( Z \). Of course for large \( M \) the distribution of \( \tilde{L}_t \) is in general hard to compute explicitly. We propose here some methods for high speed approximations.
4.1 A normal approximation

Since all the random variables \( I\{\tilde{Y}_i = 1\} \) are independent, one approach is to use the central limit theorem to approximate the distribution of \( \tilde{L}_t \) as

\[
\tilde{L}_t \overset{d}{\approx} L(t, \xi) = m(t) + \xi \sigma(t),
\]

where \( \xi \) is Gaussian \( N(0, 1) \). The mean \( m(t) \) and variance \( \sigma^2(t) \) of \( \tilde{L}_t \) can easily be computed

\[
m(t) = \sum_{k=1}^{K} \alpha_k p_{k1}(t), \quad \alpha_k = \sum_{i=1}^{M} (1 - R_0) I\{\tilde{Y}_0^i = k\} \frac{N_i}{N} \]
\[
\sigma^2(t) = \sum_{k=2}^{K} \beta_k p_{k1}(t)(1 - p_{k1}(t)), \quad \beta_k = \sum_{i=1}^{M} (1 - R_0)^2 I\{\tilde{Y}_0^i = k\} \frac{N_i^2}{N^2}
\]

Provided the sequence of notionals \( N_i \) is bounded above and below uniformly in \( i \), we note that \( m(t) = O(1), \sigma(t) = O(M^{-\frac{1}{2}}) \) and the approximation is accurate to \( O(M^{-1}) \).

The distribution of real time loss process thus is approximated by

\[
L_t = \tilde{L}_{\tau_t} \overset{d}{\approx} L(\tau_t, \xi) = m(\tau_t) + \xi \sigma(\tau_t).
\]

The paper [Hurd et al.(2004)] analyzes the error connected with this type of approximation.

The explicit formula for \( H^U, H^S \) in the normal approximation is

\[
H^U(\tau) = 1 - H^S(\tau) = \frac{\sigma(\tau)}{\bar{x} - x} \left[ \tilde{\Phi} \left( \frac{\bar{x} - m(\tau)}{\sigma(\tau)} \right) - \Phi \left( \frac{\bar{x} - m(\tau)}{\sigma(\tau)} \right) \right]
\]

where

\[
\tilde{\Phi}(x) = \int_{-\infty}^{x} \Phi(y) dy = x \Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},
\]

and \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \) is the cumulative distribution function of \( N(0, 1) \).

4.2 A no default correction

The above continuous approximation to the conditional loss distribution of \( \tilde{L}_\tau \) makes one systematic error which becomes most important for small times and small default probability, namely they neglect the point mass corresponding to no default. To make a simple correction, we write \( \tilde{L}_\tau = \tilde{L}_{\tau, > 0} X(\tau) \) where

\[
\tilde{L}_{\tau, > 0} := \tilde{L}_{\tau} |_{L(\tau) > 0}
\]
\[
X_I = 1(\tilde{L}_\tau = 0).
\]
Then, rather than approximating $\tilde{L}$, we instead approximate $\tilde{L}_{\tau,>0}$ by a normal distribution with matched moments. Moments of $\tilde{L}_{\tau,>0}$ are easily computed using the relation between moment generating functions

$$\Phi_{\tilde{L}}(u) = p + (1 - p)\Phi_{\tilde{L}_{>0}}(u)$$

where $p = p(\tau) = P(\tilde{L}_{\tau} = 0))$.

### 4.3 The equal notional case

The special case when all obligors have equal notional amounts $N_i = N/M$ can be computed exactly relatively efficiently and provides us with a range of cases to test these approximation schemes. Because the loss given default of any single firm is $\Delta L = (1 - R_0)N/M$, the loss distribution $P_{i,k} = P[L_i^{\leq i} = k\Delta L]$ is explicitly determined by the following recursion:

$$P_{ik} = P_{i-1,k}p_i + P_{i-1,k-1}(1 - p_i)$$

where $p_i = P[Y_t^i = 1]$ and $L_i^{\leq i} = \sum_{j=1}^i 1(Y_t^j = 1)\Delta L$.

### 5 CDOs in the credit migration model

In this section we apply the model with $K = 7$ rating classes (these can be interpreted as the Moody’s or Standard and Poor’s system) to price some representative CDO tranches. In this section we consider only CDOs with equal notionals so that the accuracy of our approximation methods can be compared to the exact answers. When notionals are not equal, but the nonhomogeneity is not too wild, we expect the approximate methods to perform with similar accuracy. The first step is to specify the Markov generator $L_Y$. The relation between historical and risk neutral transition probabilities is discussed in [Jarrow et al.(1997)], but the question still remains to be completely understood. For our present expository purposes, we simply take an approximate logarithm of the historical generator from [Jarrow et al.(1997)]:

$$L_Y = \begin{pmatrix}
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.2856 & -0.4318 & 0.0928 & 0.0250 & 0.0142 & 0.0142 & 0.0000 \\
0.0753 & 0.0479 & -0.1928 & 0.0568 & 0.0073 & 0.0034 & 0.0021 \\
0.0273 & 0.0144 & 0.1181 & -0.2530 & 0.0813 & 0.0089 & 0.0025 \\
0.0049 & 0.0020 & 0.0174 & 0.0701 & -0.1711 & 0.0713 & 0.0047 \\
0.0010 & 0.0000 & 0.0048 & 0.0107 & 0.0688 & -0.1172 & 0.0309 \\
0.0000 & 0.0000 & 0.0030 & 0.0030 & 0.0105 & 0.0787 & -0.1043 \\
0.0000 & 0.0000 & 0.0000 & 0.0031 & 0.0020 & 0.0083 & 0.1019 \\
\end{pmatrix}$$

We use the two sets of parameters chosen in [Hurd and Kuznetsov(2004)]. In that paper, the parameters were chosen to yield credit spread curves and correlation curves which were quantitatively similar to the benchmark problems studied.
by [Hull and White(2004), Laurent and Gregory(2003)]. The “$\rho = 0.1$ model” is defined to have the parameters

$$a = c = 0.1, \quad h_2 = 3, \lambda_2 = 0.3, \quad h_3 = 3,$$

and is quantitatively similar to the one–factor Gaussian copula model with correlation $\rho = 0.1$. The “$\rho = 0.3$ model” is defined to have the parameters

$$a = c = 0.1, \quad h_2 = 15, \lambda_2 = 0.05, \quad h_3 = 5,$$

and is quantitatively similar to the one–factor Gaussian copula model with correlation $\rho = 0.3$. In both cases the marginal default time distribution is the same as in our model. In all models we take the fractional recovery to have the fixed value $R_0 = 0.4$, parameters for the interest rate are fixed to be $M_r = (0.05, 0)$ and the parameters for the time change are $M_r = (0.6, 1.2)$ and $m_r = 0.2$. The $Z$ process is initialized with $(Z_0^1, Z_0^2) = (1, 1)$.

In each of the two sets of model parameters, we consider the six standard tranches for three typical 5 year CDOs consisting of $M = 20, 100$ and $400$ ($M = 100$ is the default value) similar firms, divided equally into the four investment grade ratings classes (BBB, A, AA, AAA).

When combined with the matrix $L_Y$ above, the model for individual firms leads to the following table of CDO tranche spreads (in basis points) for the two contracts described above, computed in the normal approximation and exactly:

<table>
<thead>
<tr>
<th>tranche</th>
<th>$\rho = 0.1$, Normal</th>
<th>$\rho = 0.1$, Exact</th>
<th>$\rho = 0.3$, Normal</th>
<th>$\rho = 0.3$, Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>N=20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0,0.03]</td>
<td>1703.5</td>
<td>1926.2</td>
<td>1268.5</td>
<td>1384.8</td>
</tr>
<tr>
<td>[0.03,0.07]</td>
<td>819.5</td>
<td>677.1</td>
<td>652.8</td>
<td>578.0</td>
</tr>
<tr>
<td>[0.07,0.10]</td>
<td>288.2</td>
<td>287.7</td>
<td>380.2</td>
<td>384.7</td>
</tr>
<tr>
<td>[0.10,0.15]</td>
<td>87.0</td>
<td>100.3</td>
<td>259.5</td>
<td>263.8</td>
</tr>
<tr>
<td>[0.15,0.30]</td>
<td>6.31</td>
<td>7.62</td>
<td>80.1</td>
<td>79.7</td>
</tr>
<tr>
<td>[0.30,1.00]</td>
<td>0.0</td>
<td>0.0</td>
<td>1.4</td>
<td>1.4</td>
</tr>
<tr>
<td>N=100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0,0.03]</td>
<td>2487.7</td>
<td>2483.5</td>
<td>1633.3</td>
<td>1625.9</td>
</tr>
<tr>
<td>[0.03,0.07]</td>
<td>666.0</td>
<td>661.9</td>
<td>533.7</td>
<td>534.3</td>
</tr>
<tr>
<td>[0.07,0.10]</td>
<td>180.0</td>
<td>181.1</td>
<td>386.1</td>
<td>386.7</td>
</tr>
<tr>
<td>[0.10,0.15]</td>
<td>38.4</td>
<td>39.1</td>
<td>276.1</td>
<td>276.2</td>
</tr>
<tr>
<td>[0.15,0.30]</td>
<td>1.28</td>
<td>1.31</td>
<td>61.7</td>
<td>61.7</td>
</tr>
<tr>
<td>[0.30,1.00]</td>
<td>0.0</td>
<td>0.0</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>N=400</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0,0.03]</td>
<td>2642.5</td>
<td>2641.6</td>
<td>1683.4</td>
<td>1682.0</td>
</tr>
<tr>
<td>[0.03,0.07]</td>
<td>645.8</td>
<td>645.5</td>
<td>517.0</td>
<td>517.0</td>
</tr>
<tr>
<td>[0.07,0.10]</td>
<td>155.3</td>
<td>155.4</td>
<td>388.9</td>
<td>388.9</td>
</tr>
<tr>
<td>[0.10,0.15]</td>
<td>28.4</td>
<td>28.4</td>
<td>287.5</td>
<td>287.5</td>
</tr>
<tr>
<td>[0.15,0.30]</td>
<td>0.68</td>
<td>0.69</td>
<td>56.4</td>
<td>56.4</td>
</tr>
<tr>
<td>[0.30,1.00]</td>
<td>0.0</td>
<td>0.0</td>
<td>0.6</td>
<td>0.6</td>
</tr>
</tbody>
</table>

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The computations on this table took 8.4 seconds when implemented in MATLAB on a 2.4GHz laptop.

Figure 3 shows credit default spread curves for each rating class as computed in [Hurd and Kuznetsov(2004)].

Figures 4, 5, 6, 7 graph the dependencies of CDO tranches in the \( \rho = 0.1 \) model on the underlying parameters \( Z_0^1, Z_0^2, h_2, h_3 \) with remaining parameters fixed at their default values.

Figure 8 shows a simulation of the values of the six tranches in the \( \rho = 0.1 \) model over the entire duration \([0, 5]\) of the contract. In the particular sample path shown, the following sequence of credit migrations and defaults occurred:

6 Comparison with the normal copula model

To compare our model with the one factor normal copula model (see [Hull and White(2004)]) we use the following modeling assumptions: the Markov chains \( \tilde{Y}^i_t \) are just two-state \((K = 1)\) processes with Markov generator \( L_Y \) given by

\[
L_Y = \begin{pmatrix} 0 & 0 \\ \lambda & -\lambda \end{pmatrix},
\]

where the default intensity has the value \( \lambda = 0.01 \). Other parameters are: a constant recovery rate \( R_0 = 0.4 \); a constant interest rate \( r_t = 0.05 \).

The important difference between the two models is the correlation structure. While the copula model has a unique parameter \( \rho \) which explains all correlations between default events, in our model we have several parameters which are responsible for the default correlations (the most important ones are \( h_2, h_3, \lambda_2 \)). In order to compare the correlation structures of the two models we look at the term structure of default correlations given by the following function

\[
corr_{ij}(t) = corr(I\{t^*_i < t\}, I\{t^*_j < t\})
\]

where “corr” is the usual correlation between random variables. To match the correlation term structures we simply choose parameters of the time change process which give a reasonable fit. For a detailed discussion of the correlation issues see [Hurd and Kuznetsov(2004)].

<table>
<thead>
<tr>
<th>tranche</th>
<th>( \rho = .1, \text{ HW} )</th>
<th>( \rho = .1, \text{ AMC} )</th>
<th>( \rho = .3, \text{ HW} )</th>
<th>( \rho = .3, \text{ AMC} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0.03])</td>
<td>1740.6</td>
<td>1731.7</td>
<td>1206.6</td>
<td>1112.4</td>
</tr>
<tr>
<td>([0.03, 0.07])</td>
<td>251.1</td>
<td>268.7</td>
<td>323.5</td>
<td>344.2</td>
</tr>
<tr>
<td>([0.07, 0.10])</td>
<td>36.9</td>
<td>26.9</td>
<td>131.5</td>
<td>187.7</td>
</tr>
<tr>
<td>([0.10, 0.15])</td>
<td>5.33</td>
<td>1.79</td>
<td>57.3</td>
<td>64.6</td>
</tr>
<tr>
<td>([0.15, 0.30])</td>
<td>0.13</td>
<td>0.03</td>
<td>10.6</td>
<td>4.3</td>
</tr>
<tr>
<td>([0.30, 1.00])</td>
<td>1.5e-6</td>
<td>3.0e-3</td>
<td>10.0e-3</td>
<td>4.0e-3</td>
</tr>
</tbody>
</table>

We observe that although these two models are certainly different, they produce qualitatively similar results across all tranches.
7 Sensitivity

One good thing about an analytical or semi-analytical treatment of securities such as ours compared to Monte Carlo based methods is that sensitivity analysis is both conceptually and computationally straightforward. In our model, security prices are sensitive to the underlying dynamic risk factors $\tilde{Y}, Z^1, Z^2, Z^3$: these evolve in time, so they are the most important factors for risk management. Next in importance are the model parameters which are taken to be constant in time, but are subject to calibration error: these are $(\mathcal{L}Y, a, c, \lambda_2, h_2, h_3, M_r, M_r, m_r)$. In the present discussion we focus on hedging the sensitivities to the dynamic factors, and leave parameter hedging for future study.

Since $Z^1, Z^2$ control the overall shape of credit spread and correlation curves, hedging these factors may be thought of as hedging general market risk. The most important hedge is thus to create delta-neutral combinations with respect to these risk factors. Fortunately, the requisite derivatives of both the premium and insurance legs are explicitly computable:

\[(\Delta_{V,1}, \Delta_{V,2}) = \partial_z V^U = \int_0^\infty H^U(\tau) \partial_z F^P(\tau, z) d\tau, \quad (31)\]

\[(\Delta_{W,1}, \Delta_{W,2}) = \partial_z W^S = \int_0^\infty H^S(\tau) \partial_z F^I(\tau, z) d\tau, \quad (32)\]

where $\partial_z F^P, \partial_z F^I$ are explicit in terms of the building blocks defined so far. Thus hedging general market risk is a tractable problem in our model. Figures 9,10,11,12 show graphs of $V^U, W^S$ against $Z^1_0, Z^2_0$ for the contract A tranches in the $\rho = 0.1$ model with all remaining parameters chosen as in section 5.

Next we consider hedging the risk factors $Y$. This amounts to protecting against the risk of any individual downgrade, upgrade or default, and such firm specific risks can only be delta hedged by holding additional credit securities on each name of the basket. Similarly hedging for jumps of $Z^3$ involves firm specific risks. For large scale baskets this type of hedging is of secondary importance.

It is interesting to observe from Figures 5-8 that the third tranche in contract A is highly sensitive to all parameters. This can be understood graphically because the degree of overlap of the functions $F^I, F^P$ with $H^S, H^U$ is intermediate, and varies rapidly with the parameters.

8 Conclusion

The AMC framework gives dynamical models of multifirm credit migration and default which fall in the class of reduced form or doubly stochastic models, yet it builds in some of the features of the structural models. The particular way of combining a continuous time Markov chain with an independent set of affine processes yields a flexible framework within which computations are very efficient.
In this paper and its companion [Hurd and Kuznetsov(2004)] we have demonstrated methods for computing credit spreads, correlation curves and CDO tranches, all of which pass visual inspection to be a plausible representation of real markets. In contrast to typical static copula models for large scale basket derivatives, our approach attempts to capture a resemblance to real market dynamics. This suggests using the AMC framework as the basis for scenario generation and stress testing of other pricing methodologies.

The increased realism of our framework does not lead to slower computation times. In fact, in our specification, the computation times are not particularly longer than those achievable in a simple one factor normal copula CDO model. In common with such models, the loss process is a sum over firms of conditionally independent random variables. In contrast, however, the conditioning variable in our modeling is a stochastic process \( \tau_t \) rather than a single random variable. Nonetheless, the computation speeds are similar because the affine structure of the conditioning process \( \tau_t \) reduces critical computations to one-dimensional integrals.

The version we have presented appears to be an example of a rather general dynamic approach to credit risk, one which can be extended in several distinct directions. We mention here two important types of improvements which are easy to add. The first is to replace the conditioning process \( \tau_t \) by \( m \)-dimensional processes which might include for example the stochastic recovery or a variety of time changes for different sectors of the economy. It appears that our CDO pricing framework extends easily to this setting and would yield formulas for CDOs involving \( m \)-dimensional integrals and Fourier transforms. Another type of extension would be to include idiosyncratic factors which are specific to individual firms: as long as these factors are deterministic this would avoid a curse of dimensionality of integrals in \( M \) dimensions.

We have shown that in the normal approximation, CDO prices have \( M \) dependence consistent with an \( O(1/M) \) error. Another direction for improvement of the method would be to improve the large \( M \) asymptotics of the errors \( |V^U - V^{U^*}|, |W^{S} - W^{S^*}| \) by including further correction terms. [Hurd et al.(2004)] derives a very general asymptotic expansion which can perform this task.

Our aim here has been to demonstrate that the AMC framework is flexible enough in principle to fit defaultable bond data and credit derivatives such as CDOs. A detailed study of the validity of the approach for modeling real data sets is clearly justified.
A Appendix: Formulas for main building blocks

For the reader’s convenience, we reproduce here computations given in [Hurd and Kuznetsov(2004)].

A.1 Computing $G_1$

Since $Z_1, Z_2$ are independent, we have

$$G_1(t; z; u, v) = E_{0,z} \left[ e^{-\int_0^t (uZ_s) ds} e^{-\langle v, Z_t \rangle} \right] = \prod_{i=1}^2 E_{0,z^i} \left[ e^{-u^i \int_0^t Z^i_s ds} e^{-v^i Z^i_t} \right] = \prod_{i=1}^2 G_1^{(Z^i)}(t; z^i; u^i, v^i)$$

Since $Z^1$ is a CIR process function, $G_1^{(Z^1)}$ is well known. Let $Z$ be a CIR process

$$dZ_t = (a - bZ_t) dt + c \sqrt{Z_t} dW_t.$$

Then $G_1^{(Z)}$ is given by

$$G_1^{(Z)}(t; z; u, v) = E_{0,z} \left[ e^{-u \int_0^t Z_s ds} e^{-v Z_t} \right] = e^{\phi(t,u,v) + z\psi(t,u,v)}, \quad (33)$$

and functions $\phi$ and $\psi$ are computed as

$$\begin{cases} 
\phi(t, u, v) = a\psi_2 t - \frac{u}{c} \log \left( e^{-\gamma t} + \frac{\xi}{\gamma} (v + \psi_1) (1 - e^{-\gamma t}) \right), \\
\psi(t, u, v) = \psi_2 - \frac{v + \psi_2}{1 + \frac{\xi}{\gamma} (v + \psi_1) (e^{\gamma t} - 1)}, \end{cases} \quad (34)$$

and constants $\psi_1, \psi_2$ and $\gamma$ are given by

$$\begin{cases} 
\gamma = \sqrt{b^2 + 4uc} \\
\psi_1 = \frac{b+\gamma}{2c} \\
\psi_2 = \frac{b-\gamma}{2c}. \quad (35) 
\end{cases}$$

Process $Z = Z^2$ with the Markov generator

$$\mathcal{L}_Z f(x) = \Lambda(f(x + h) - f(x)) - bx f'(x)$$

is again affine. Thus it’s function $G_1^{(Z)}$ is computed as

$$G_1^{(Z)}(t; z; u, v) = E_{0,z} \left[ e^{-u \int_0^t Z_s ds} e^{-v Z_t} \right] = e^{\phi(t,u,v) + z\psi(t,u,v)}, \quad (36)$$

and functions $\phi$ and $\psi$ are given as solutions to the following system of equations

$$\begin{cases} 
\frac{d\psi}{dt} = -b\psi - u, \\
\frac{d\phi}{dt} = \Lambda(e^{h\psi} - 1), \quad \phi(0, u, v) = 0. \quad (37) 
\end{cases}$$

This system can be solved explicitly to give the following expressions

$$\begin{cases} 
\psi(t, u, v) = \left( \frac{u}{b} - v \right) e^{-bt} - \frac{u}{b} \\
\phi(t, u, v) = \frac{\Lambda}{b} e^{-\frac{u}{b}} (Ei(h(\frac{u}{b} - v)) - Ei(h(\frac{u}{b} - v) e^{-bt})) - \Lambda t, \quad (38) 
\end{cases}$$

where $Ei(x)$ is the special function called the exponential integral and defined as a Cauchy principal value of the integral $\int_{-\infty}^x e^{u/y} dy$.  

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A.2 Computing $G_2$

Here again we can use the fact that processes $Z^1, Z^2, Z^3$ are independent to simplify function $G_2$ as

$$G_2(t, z; u, v, w) = E_{0,z} \left[ e^{-\int_0^t (w^T Z_s) ds} (w \cdot Z_t) e^{-(v \cdot Z_t)} \right]$$

$$= w^1 G_2^{(Z^1)}(t, z^1; u^1, v^1) G_2^{(Z^2)}(t, z^2; u^2, v^2) + w^2 G_1^{(Z^1)}(t, z^1; u^1, v^1) G_2^{(Z^2)}(t, z^2; u^2, v^2).$$

Here for a process $Z$ function $G_2^{(Z)}(t, z; u, v)$ is defined as

$$G_2^{(Z)}(t, z; u, v) = E_{0,z} \left[ e^{-\int_0^t u Z_s ds} Z_t e^{-(v \cdot Z_t)} \right] = -\frac{\partial}{\partial v} G_1^{(Z)}(t, z; u, v).$$

Since we have explicit expression for individual functions $G_1$ we can easily find explicit formulas for $G_2$:

$$G_2^{(Z)}(t, z; u, v) = -\frac{\partial}{\partial v} G_1^{(Z)}(t, z; u, v) = -\frac{\partial}{\partial v} e^{\phi(t, u, v) + z\psi(t, u, v)}$$

$$= -\left( \frac{\partial}{\partial v} \phi(t, u, v) + x \frac{\partial}{\partial v} \psi(t, u, v) \right) G_1^{(Z)}(t, z; u, v).$$

Thus to compute individual functions $G_2$ we just need to compute the partial derivative in $v$ of $\phi$ and $\psi$. In the case of CIR process they can be easily found from the formula (34):

$$\begin{cases} 
\frac{\partial}{\partial v} \psi(t, u, v) = -e^{-bt} \frac{e^{\gamma t}}{1 + \frac{\gamma}{e^{\gamma t} - 1}} \\
\frac{\partial}{\partial v} \phi(t, u, v) = -\frac{a}{\gamma} e^{\gamma t} - 1 \frac{e^{\gamma t} - (\epsilon + \psi_1) (e^{\gamma t} - 1)}{1 + \frac{\gamma}{e^{\gamma t} - 1}} 
\end{cases}$$ (39)

For the process $Z^2$ these partial derivatives are easily found from the formula (38)

$$\begin{cases} 
\frac{\partial}{\partial v} \psi(t, u, v) = -e^{-bt} \\
\frac{\partial}{\partial v} \phi(t, u, v) = -\frac{\Lambda}{u - bv} e^{-\frac{u}{b}} \left( e^{h(\frac{u}{b} - v)} - e^{h(\frac{u}{b} - v) e^{-bt}} \right) 
\end{cases}$$ (40)

References


Figure 1: Simulation of the time change factors. The first graph shows typical sample paths of the diffusion process $Z^1$, the jump process $Z^2$ and the jump process $Z^3$ as well as the integrated time change generated by $Z^1$ alone. The second graph shows the total integrated time change resulting from the three sample paths combined together.
Figure 2: Computing the insurance leg: The heavy curve shows a typical graph of the function $F^I(\tau)$; the light curves show the $H^S(\tau)$ functions for the junior (leftmost) to senior (rightmost) tranches. Integrating $H^S(\tau)$ against $F^I(\tau)$ yields the expected value of the insurance leg.
Figure 3: Hazard rates: These show typical hazard rate curves over a 10 year period for firms of all rating classes. Parameter values are the default values from section 5.
Figure 4: Dependence of CDO spreads on the initial value $Z_0^1$. Parameter values are the default values from section 5.

Figure 5: Dependence of CDO spreads on the initial value $Z_0^2$. Parameter values are the default values from section 5.
Figure 6: Dependence of CDO spreads on the jumpsize $h_2$. Parameter values are the default values from section 5.

Figure 7: Dependence of CDO spreads on the jumpsize $h_3$. Parameter values are the default values from section 5.
Figure 8: Dependence of premium leg on the initial value $Z_0^1$. Parameter values are the default values from section 5.

Figure 9: Dependence of premium leg on the initial value $Z_0^2$. Parameter values are the default values from section 5.
Figure 10: Dependence of insurance leg on the initial value $Z_0^1$. Parameter values are the default values from section 5.

Figure 11: Dependence of insurance leg on the initial value $Z_0^2$. Parameter values are the default values from section 5.