A framework for analyzing contagion in banking networks

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Abstract

A probabilistic framework is introduced that represents stylized banking networks and aims to predict the size of contagion events. Most previous work on random financial networks assumes independent connections between banks, whereas our framework explicitly allows for disassortative edge probabilities (an above average tendency for small banks to link to large banks). We give a new construction of general directed configuration graphs in the fully (dis)assortative setting and characterize the locally tree-like nature of such graphs. Using this locally tree-like independence assumption, we give a probabilistic analysis of the default cascade triggered by shocking the network. We find that the cascade can be understood as an explicit iterated mapping on a set of edge probabilities that converges to a fixed point. A cascade condition is derived that characterizes whether or not an infinitesimal shock to the network can grow to a finite size cascade, in analogy to the basic reproduction number R_0 in epidemic modelling. It provides an easily computed measure of the systemic risk inherent in a given banking network topology. An analytic formula is given for the frequency of global cascades, derived from percolation theory on the random network. Although the analytical methods are derived for infinite networks, large-scale Monte Carlo simulations are presented that demonstrate the applicability of the results to finite-sized networks. Two simple models are used to demonstrate that edge-assortativity can have a strong effect on the level of systemic risk as measured by the cascade condition. However, the effect of assortativity on systemic risk is subtle, and we propose a simple graph theoretic quantity, which we call "graph-assortativity", that seems to best capture systemic risk.

Key words: Systemic risk, banking network, random graph, cascade condition, credit risk, financial mathematics, assortativity.

AMS Subject Classification: 05C80, 91B74, 91G50

1 Introduction

The study of contagion in financial systems is very topical in light of the recent global credit crisis and the resultant damage inflicted on financial institutions. "Contagion" refers to the spread of defaults through a system of financial institutions, with each successive default causing increasing pressure on the remaining components of the system. The term "systemic risk" refers to the contagion-induced threat to the financial system as a whole, due to the default of one (or more) of its component institutions.

It is widely held that financial systems (see [14] and references therein), defined for example as the collection of banks and financial institutions in a developed country, can be modelled as a random network of *nodes* or *vertices* with stylized balance sheets, connected by directed links or edges that represent exposures ("interbank loans"), each edge with a positive weight that represents the size of the exposure. If ever a node becomes "insolvent" and ceases to operate as a bank, it will create balance sheet shocks to other nodes, creating the potential of chains of insolvency that we will call "default cascades". Financial networks are difficult to observe because interbank data is often not publicly available, but studies have indicated that they share characteristics of other types of technological and social networks, such as the world wide web and Facebook. For example, the degree distributions of financial networks are thought to be "fat-tailed" since a significant number of banks are very highly connected. A less studied feature observed in financial networks (and as it happens, also the world wide web) is that they are highly "disassortative" (see [13]). This refers to the property that any bank's counterparties (i.e. their graph neighbours) have a tendency to be banks of an opposite character. For example, it is observed that small banks tend to link preferentially to large banks rather than other small banks. Commonly, social networks are observed to be assortative rather than disassortative. Structural characteristics such as degree distribution and assortativity are felt (see [9]) to be highly relevant to the propagation of contagion in networks but the nature of such relationships is far from clear.

Our aim here is to develop a mathematical framework that will be able to determine the systemic susceptibility in a rich class of infinite random network models with enough flexibility to include the most important structural characteristics of real financial networks, in particular with general degree distributions and a prescribed degree of edge-assortativity. Our starting point will be the Gai-Kapadia cascade model ([5], hereafter referred to as GK for short) and the analytical methods developed there and in [6] for this model. The basic assumptions introduced in the GK model are:

1. The network is a large (actually infinite) random directed graph with a prescribed degree distribution;
2. Each node is labelled with a stylized banking balance sheet that identifies its external assets and liabilities, its internal (i.e. total interbank) assets and liabilities, and γ, its net worth or equity (i.e. its total assets minus its total liabilities).

Initially, the system is in equilibrium, meaning each node has positive net worth $\gamma > 0$.

3. Each directed edge is labelled with a deterministic weight that represents the positive exposure of one bank to another. These weights depend deterministically on the in-degree of the edge, and are consistent with the interbank (IB) assets and liabilities at each node;
4. A random shock is applied to the balance sheets in the system that triggers the default or insolvency of a fixed fraction of nodes;
5. The residual value available to creditors of a defaulted bank is zero, and thus the shock has the potential to trigger a cascade of further bank defaults.

The principle of *limited liability* for banks means that equity holders are never asked to cover a negative net worth of an insolvent firm. Instead, the insolvent firm is assumed to "default", meaning it ceases to operate as a going concern, and its creditors divide the residual value. Since this residual value is always less than the nominal liabilities, creditor banks thus receive a shock to their balance sheets, creating the potential for a default cascade. The GK model makes a very simple *zero recovery* assumption that residual values of defaulted banks will be zero, and thus every time a bank defaults a maximal possible shock will be transmitted to its creditors.

Our paper makes the following contributions towards developing a mathematical theory of systemic risk.

1. We generalize the GK model in an important respect, namely that the edge degree distribution Q is arbitrary, allowing for any desired amount of assortativity or disassortativity in the network.
2. We present a simple algorithm for constructing general assortative random directed graphs of the configuration class.
3. We identify the "locally-treelike independence assumption" that characterizes the mathematical nature of the locally tree-like structure of configuration graphs.
4. We offer a simple graph theoretical notation as a useful alternative to the more standard "generating function" approach.
5. We derive probabilistic formulas for the result of general cascades, including the fixed point equation, the spectral "cascade condition" and the frequency of global cascades.
6. We introduce the concept of "graph assortativity" for directed graphs.
7. We present numerical experiments in a family of network models to test the accuracy of our analysis compared to Monte Carlo. We also test whether graph assortativity correlates well with the degree of systemic risk.
8. We identify a number of "next steps" for research that will move our theory to a new level of applicability.

The remainder of this paper is structured as follows. In Section 2 the extended GK model is described in detail, and the analytical description of its solution is given in Section 3. Section 4 discusses the cascade condition, and Section 5 derives a formula for the frequency of large scale cascades arising from an infinitesimally small seed. Numerical results of large-scale Monte Carlo simulations are compared with the analytical predictions in Section 6. Section 7 concludes.

2 The Extended GK Model

In this section we completely specify the extended GK modelling framework for which our analytical techniques will apply. The specification will consist of three levels: first, the random directed graph model for the "skeleton" of the network; second, a specification of balance sheet values for all nodes and edges, representing the state of the system in equilibrium prior to a systemic shock; thirdly, a specification of the type of systemic shocks that will be considered. We shall work within a probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P})$ for some $N \leq \infty$, where a general outcome $\omega \in \Omega_N$ is a directed graph with N nodes, with specified balance sheet values and an initial shock.

2.1 The Skeleton Network

The first step in building a random financial network is to build the skeleton directed graph that labels the banks and their interbank connections. Our construction is an extension of the well-known configuration graph model (see [3]), and to describe it we introduce certain graph theoretic definitions and notation:

1. A node $v \in \mathcal{N} = \cup_{jk}\mathcal{N}_{jk}$ has type (j,k) means its in-degree is $\deg^-(v) = j$ and its out-degree is $\deg^+(v) = k$. We shall write $k_v = k, j_v = j$ for any $v \in \mathcal{N}_{jk}$ and allow degrees to be any non-negative integer.
2. An edge $\ell \in \mathcal{E} = \cup_{kj}\mathcal{E}_{kj}$ is said to have type (k,j) with out-degree k and in-degree j if it is an out-edge of a node with out-degree k and an in-edge of a node with in-degree j. We shall write $\deg^+(\ell) = k_\ell = k$ and $\deg^-(\ell) = j_\ell = j$ for any $\ell \in \mathcal{E}_{kj}$.
3. We write $\mathcal{E}^+_v$ (or $\mathcal{E}^-_v$) for the set of out-edges (respectively, in-edges) of a given node v and v^+_ℓ (or v^-_ℓ) for the node for which ℓ is an out-edge (respectively, in-edge).
4. We will always write j, j', j'', etc. to refer to in-degrees while k, k', k'', etc. refer to out-degrees.

Figure 1 illustrates the neighbourhood of a type $(3,2)$ bank.

Definition 1. *The directed configuration graph model on $(\Omega_N, \mathcal{F}_N, \mathbb{P})$ for any $N \leq \infty$ is defined by the following construction based on probability laws P, Q for nodes and edges:*

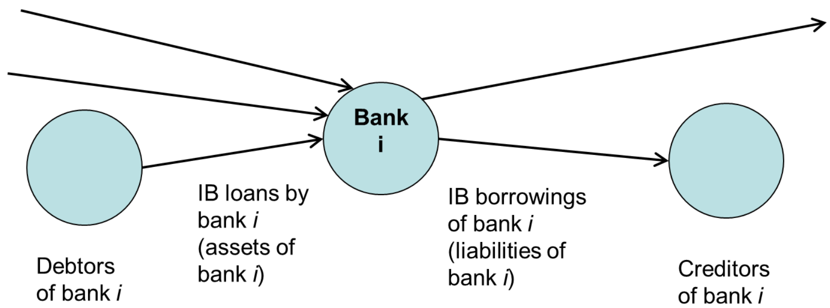


Figure 1: The skeleton structure of the network locality of a bank v. Bank v is in the $(j,k)=(3,2)$ class, since it has 3 debtors and 2 creditors in the interbank (IB) network.

1. *For each $j,k\in\mathcal{K}$, $P_{jk}:=\mathbb{P}[\mathcal{N}_{jk}]$ is the probability of a type (j,k) node. This distribution has marginals $P^+_k:=\sum_j P_{jk}, P^-_j:=\sum_k P_{jk}$ and mean in and out degree $z=\sum_j jP^-_j=\sum_k kP^+_k$.*
2. *For each $j,k\in\mathcal{K}$, $Q_{kj}:=\mathbb{P}[\mathcal{E}_{kj}]$ is the probability of a type (k,j) edge. This distribution has marginals $Q^+_k:=\sum_j Q_{kj}, Q^-_j:=\sum_k Q_{kj}$.*
3. *To simplify the analysis that follows, we fix a finite set of possible degrees $\mathcal{K}=\{0,1,\dots,K-1\}$ for some K and assume that $P^-_j, P^+_k>0$ for all $j,k\in\mathcal{K}\setminus\{0\}$.*

The following construction can be used to define directed configuration graphs for any finite N provided "consistency conditions" hold. For each j,k:

- $NP_{jk}\in\mathbb{Z}^+=\{0,1,2,\dots\}\ ,\quad NzQ_{jk}\in\mathbb{Z}^+=\{0,1,2,\dots\}$;
- $$Q^+_k=kP^+_k/z;\quad Q^-_j=jP^-_k/z. \tag{1}$$

Under these conditions, the directed graph construction goes as follows

1. Make a list of N nodes of which exactly NP_{jk} are of type j,k and a list of zN edges of which exactly NzQ_{kj} have type k,j. Let the unpaired out (in) arrows of each node and edge be called "k-stubs" (or "j-stubs").
2. For $\ell=1,\dots,Nz$:
   - Pick a random edge from the unmatched edges. Let its type be (k,j);
   - Match its k-stub to a random unpaired k-stub of a node, chosen with equal probability from unmatched k-stubs.
   - Match its j-stub to a random unpaired j-stub of a node, chosen with equal probability from unmatched j-stubs.

One can readily check that the consistency condition (1) implies that for each j, all j- stubs will be matched, and likewise for the k-stubs, so the graph construction is consistent. However, it is important to recognize that this graph construction leads for finite N to multiple edges between node pairs with positive probability, as well as the positive probability of having self-edges. We allow such anomalies (normally ruled inadmissible in graph theory) since they do not seriously affect finance interpretations and more importantly occur with vanishing probability as N goes to ∞.

By following work of [10] and [1], we can consider consistent sequences $(N^i, P^i, Q^i)_{i=1,2,\dots}$ of finite random graphs such that $N^i \to \infty, P^i \to P$ and $Q^i \to Q$. We conjecture that one can define the $N = \infty$ configuration model for any pair P, Q of probability distributions that satisfy (1) (plus possible moment conditions in case $K \to \infty$), but leave the details of such a construction as an open problem. The important and highly restrictive property of configuration graphs we expect in the $N = \infty$ limit, called the locally tree-like (LT) property, is that cycles of any fixed finite length occur only with zero probability. The following condition is a heuristic (not proven) consequence of the LT property for configuration graphs with $N = \infty$, that we will assume throughout this paper:

The *LT* independence assumption (LTIA): Let $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}$ be any two subsets that share exactly one node $\mathcal{N}_1 \cap \mathcal{N}_2 = \{v\}$, and let the corresponding sigma-algebras[1] be denoted $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_v$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are statistically independent, conditioned on $\mathcal{G}_v$:

$$(\mathcal{G}_1 \perp\!\!\!\perp \mathcal{G}_2)\,|\mathcal{G}_v \ . \tag{2}$$

The LT independence assumption (2) leads to an important result that will be used throughout this paper[2]: For all j, j', j'', k, k', k'',

$$\mathbb{P}[v_\ell^+ \in \mathcal{N}_{jk}, v_\ell^- \in \mathcal{N}_{j'k'} | \ell \in \mathcal{E}_{k''j''}] = \delta_{j'j''}\delta_{kk''}\mathbb{P}[j_v = j | k_v = k]\ \mathbb{P}[k_v = k' | j_v = j'] \ . \tag{3}$$

Remark 2. (Independent edge condition) *The special case $Q_{kj} = kjP^-_j P^+_k / z^2 = Q^-_j Q^+_k$ corresponds to the usual notion of a random configuration graph and arises from a simpler construction. One lists the nodes and their types (but not the edges), and sequentially matches each node's in-stubs to node out-stubs selected uniformly from the collection of all unmatched out-stubs. We are interested in the general assortative case described above because observed financial networks do not appear to satisfy the independent edge condition. Apparently (see [13]) real financial networks have the "edge-disassortative property" that high degree banks attach preferentially to low degree banks.*

Remark 3. *A natural measure of edge-assortativity by degree is the "edge-assortativity coefficient" $r_Q \in [-1, 1]$ given by*

$$r_Q = \frac{\sum_{jk} jk[Q_{kj} - Q^-_j Q^+_k]}{\sqrt{\left(\sum_j j^2 Q^-_j - (\sum_j j Q^-_j)^2\right)\left(\sum_k k^2 Q^+_k - (\sum_k k Q^+_k)^2\right)}} \ . \tag{4}$$

[1]To any subset of nodes $\mathcal{N}' \subset \mathcal{N}$ we associate the sigma-algebra $\mathcal{G}'$ generated by the balance sheets, shocks and degrees of nodes in $\mathcal{N}'$ and edges in $\mathcal{N}' \times \mathcal{N}'$.

[2]A conditional independence structure more general than (3), not arising from the above graph construction, is analyzed in [2].

However, we will find some evidence that systemic risk of a network is related more strongly to a combination of edge-assortativity and node-assortativity (arising from the dependence between in- and out- degrees of nodes). We therefore define a measure we call the "graph-assortativity coefficient" $r \in [-1, 1]$ given by

$$r = \frac{\sum_{jj'} jj'[B_{jj'} - B_j^- B_{j'}^+]}{\sqrt{\left(\sum_j j^2 B_j^- - (\sum_j j B_j^-)^2\right)\left(\sum_{j'} j'^2 B_{j'}^+ - (\sum_{j'} j' B_{j'}^+)^2\right)}} \ . \tag{5}$$

where $B_{jj'}$ is the joint distribution of the in-degree of pairs of nodes connected by an edge:

$$\begin{aligned} B_{jj'} &= \mathbb{P}[j_v = j, j_{v'} = j' | v \textit{ is joined by a single out-edge } \ell \textit{ to } v'] \\ &= \sum_k \frac{P_{jk} Q_{kj'}}{P_k^+} \ , \end{aligned}$$

and $B_j^- = \sum_{j'} B_{jj'}, B_{j'}^+ = \sum_j B_{jj'}$ are the marginals. Here, the formula for $B_{jj'}$ is derived from (3)*.*

2.2 Balance Sheets

To build a financial network with full accounting information, consistent with a given skeleton graph, one specifies the external assets Y_v and external liabilities Z_v for each node v, and for each edge ℓ of the network, an exposure size or weight w_ℓ. All these quantities are positive. From this one defines the *net worth* or *capital buffer* of a node v to be

$$\gamma_v = Y_v + \sum_{\ell \in \mathcal{E}_v^-} w_\ell - Z_v - \sum_{\ell \in \mathcal{E}_v^+} w_\ell \ . \tag{6}$$

We will always assume that the system is initially in a "cascade equilibrium" (or "equilibrium" for short) in which all banks are solvent, which means that $\gamma_v > 0$ at every node v. Thus γ_v can be thought of as a buffer that keeps the bank solvent when subjected to balance sheet shocks up to a certain size.

The cascade dynamics in the GK framework do not depend on full accounting information, but only on the partial information

$$\{\gamma_v, v \in \mathcal{N}\} \cup \{w_\ell, \ell \in \mathcal{E}\}. \tag{7}$$

We adopt a deterministic rule for which buffers may depend on the node type (j, k) but the edge weights depend only on the in-degree $\deg^-(\ell)$:

$$\begin{cases} \gamma_v = \gamma_{jk}, & v \in \mathcal{N}_{jk} \\ w_\ell = w_j, & \ell \in \mathcal{E}_{kj} \end{cases} . \tag{8}$$

For the original GK model described in [5], which we will call the GK specification, the following choices are made:

$$\gamma_{jk} = \gamma := 0.035; \quad w_j = \frac{1}{5j} \ ,$$

but the analytical results of that paper clearly hold for general prescriptions of the form (8).

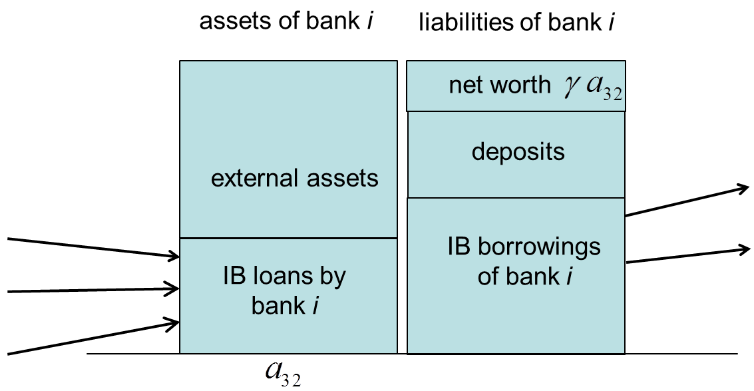


Figure 2: Schematic balance sheet of banks in the $(j, k) = (3, 2)$ class.

2.3 Shocks and the Solvency Condition

Insolvencies arise in a system initially in equilibrium only when a shock hits that is hard enough to cause at least one node to suffer a loss larger than its buffer γ_v. For simplicity, we suppose that such an initial shock to our system causes an initial set $\bar{\mathcal{M}} \subset \mathcal{N}$ of nodes to become insolvent (for example by hitting their external assets), but leaves other banks' balance sheets unchanged. The set $\bar{\mathcal{M}}$ is drawn randomly, with the fraction of type (j, k) nodes that are defaulted denoted by

$$\bar{\rho}_{jk} := \mathbb{P}[v \in \bar{\mathcal{M}} | v \in \mathcal{N}_{jk}] \ .$$

Under the GK "zero recovery" assumption that an insolvent bank can pay none of its interbank credit obligations, each insolvent node $v \in \bar{\mathcal{M}}$ triggers all its out-edges to have zero value. This triggering of edges to default is an instance of what we call an "edge update" step of the cascade: corresponding to any default node set $\mathcal{M}$ there is a default edge set $\mathcal{D} \subset \mathcal{E}$ defined by the condition $\ell \in \mathcal{D}$ if and only if $v_\ell^+ \in \mathcal{M}$.

Each such defaulted edge ℓ now transmits a maximal shock w_ℓ to the asset side of the balance sheet of its in-node v_ℓ^- (the creditor bank). If all balance sheets are determined by the reduced accounting information $\{\gamma_{jk}\} \cup \{w_j\}$, then when $\mathcal{D}$ is a set of defaulted edges, the solvency condition on a node $v \in \mathcal{N}_{jk}$ is[3]

$$\gamma_{jk} > \sum_{\ell \in \mathcal{E}_v^-} \mathbf{1}_{\{\ell \in \mathcal{D}\}} \, w_j \ .$$

[3]The indicator function $\mathbf{1}_A$ of any set A is the random variable that is 1 on the set and 0 on its complement.

We call this triggering of nodes to default a "node update" step of the cascade: corresponding to the default edge set $\mathcal{D}$ there is a default node set $\mathcal{M}'$ defined by the condition $v \in \mathcal{M}'$ if and only if

$$\#\{\mathcal{E}_v^- \cap \mathcal{D}\} \geq M_{jk} := \lceil \gamma_{jk}/w_j \rceil \ , \tag{9}$$

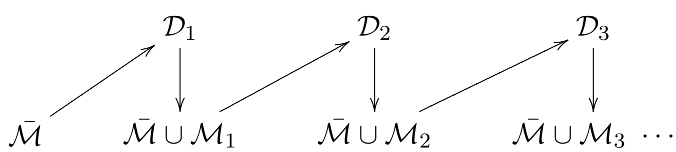
where (j,k) is the type of v. Here $\lceil x \rceil$ denotes the "ceiling" function, i.e. the smallest integer greater than or equal to x, and so M_{jk} is the threshold for the number of defaulted in-edges that will cause a type (j,k) node to default.

To summarize, a specification $(N, P, Q, \gamma, w, \bar{\rho})$ of an extended GK financial system is the following information: (i) a skeleton directed graph structure, defined by the probabilities P_{jk}, Q_{kj} over all node and edge types; (ii) reduced accounting information, denoted by $\{\gamma_{jk}\} \cup \{w_j\}$ and (iii) an initial shocked set $\bar{\mathcal{M}}$ with the default probabilities $\bar{\rho}_{jk}$ for each node type.

Given any realization of a shocked financial system so specified, the complete default cascade will be an alternating sequence of edge and node updates (finite if N is finite), beginning with the initial shocked set $\bar{\mathcal{M}}$. The cascade can now be resolved, and the expected fraction of total defaulted nodes and edges (and other statistics) will be determined by the inductive analysis given in the next section.

3 Default Cascade Steps

The proofs of all the results of this section depend strongly on the LT independence assumption (2), which in our setting requires $N = \infty$. Given any realization of a shocked financial system $(N, P, Q, \gamma, w, \bar{\rho})$ as specified above, with an initial shocked set $\bar{\mathcal{M}}$, the default cascade can be thought of as a sequence of updates:



Note that the above set unions are assumed to be disjoint, since we prefer to distinguish the initial default set $\bar{\mathcal{M}}$ from the sets of "newly defaulted" nodes. Inductively, we have nondecreasing sequences of sets:

$$\mathcal{D}_n \ := \ \text{defaulted edges "triggered" by nodes in } \bar{\mathcal{M}} \cup \mathcal{M}_{n-1} \tag{10}$$

$$\mathcal{M}_n \ := \ \text{defaulted nodes not in } \bar{\mathcal{M}} \text{ "triggered" by edges in } \mathcal{D}_n. \tag{11}$$

for $n = 1, 2, \ldots$, and we take $\mathcal{D}_0, \mathcal{M}_0 = \emptyset$. The initial default set $\bar{\mathcal{M}}$ has probabilities

$$\bar{\rho}_{jk} \ := \ \mathbb{P}[v \in \bar{\mathcal{M}} | v \in \mathcal{N}_{jk}] \tag{12}$$

and we define probabilities for $n = 0, 1, 2, \dots$

$$\sigma_{kj}^{(n)} := \mathbb{P}[\ell \in \mathcal{D}_n | \ell \in \mathcal{E}_{kj}] \tag{13}$$

$$a_j^{(n)} := \mathbb{P}[\ell \in \mathcal{D}_n | j_\ell = j] \tag{14}$$

$$\rho_{jk}^{(n)} := \mathbb{P}[v \in \mathcal{M}_n | v \in \mathcal{N}_{jk}] \; . \tag{15}$$

Now the set $\mathcal{D}_1$ is determined from $\bar{\mathcal{M}}$ by an edge update step. Similarly, for each $n > 1$ the set $\mathcal{D}_n$ is determined from $\bar{\mathcal{M}} \cup \mathcal{M}_{n-1}$ by an edge update step. In all these cases the probabilities $\sigma_{kj}^{(n)}$ are determined by the following general lemma.

Lemma 4. *(Edge update) Suppose $\bar{\mathcal{M}} \cup \mathcal{M} \subset \mathcal{N}, \bar{\mathcal{M}} \cap \mathcal{M} = \emptyset$ denotes a set of defaulted nodes and for all j, k let $\rho_{jk} := \mathbb{P}[v \in \mathcal{M} | v \in \mathcal{N}_{jk}]$. If the set of defaulted edges triggered by $\bar{\mathcal{M}} \cup \mathcal{M}$ is denoted $\mathcal{D} \subset \mathcal{E}$ then for all j, k*

$$\sigma_{kj} := \mathbb{P}[\ell \in \mathcal{D} | \ell \in \mathcal{E}_{kj}] = \frac{\sum_{j'} (\bar{\rho}_{jk} + \rho_{j'k}) P_{j'k}}{P_k^+} \; .$$

We note that σ_{kj} does not depend on j.

Proof: Using the LTIA (2) we compute that

$$\begin{aligned}
\mathbb{P}[\ell \in \mathcal{D} | \ell = \mathcal{E}_{kj}] &= \sum_{j'} \mathbb{P}[v_\ell^+ \in (\bar{\mathcal{M}} \cup \mathcal{M}) \cap \mathcal{N}_{j'k} | \ell = \mathcal{E}_{kj}] \\
&= \sum_{j'} \mathbb{P}[v \in \bar{\mathcal{M}} \cup \mathcal{M} | v \in \mathcal{N}_{j'k}] \; \mathbb{P}[v_\ell^+ \in \mathcal{N}_{j'k} | \ell = \mathcal{E}_{kj}] \\
&= \sum_{j'} (\bar{\rho}_{jk} + \rho_{j'k}) \mathbb{P}[j_v = j' | k_v = k] \\
&= \frac{\sum_{j'} (\bar{\rho}_{jk} + \rho_{j'k}) P_{j'k}}{P_k^+}
\end{aligned}$$

□

Given any set $\mathcal{D}$ of defaulted edges, each of the nodes (excluding the initially defaulted nodes in $\bar{\mathcal{M}}$) recomputes its balance sheet and the node update step leads to a subset $\mathcal{M} \subset \mathcal{N} \setminus \bar{\mathcal{M}}$ of defaulted nodes triggered by $\mathcal{D}$. We separate out the originally defaulted nodes $\bar{\mathcal{M}}$ since these were not triggered by defaulted edges. The probabilities associated to $\mathcal{M}$ are characterized by the following result.

Lemma 5. *(Node update) Suppose $\mathcal{D} \subset \mathcal{E}$ denotes a set of defaulted edges with associated probabilities $\sigma_{kj} := \mathbb{P}[\ell \in \mathcal{D} | \ell \in \mathcal{E}_{kj}]$. Then for all j, $a_j = \mathbb{P}[\ell \in \mathcal{D} | j_\ell = j]$ is given by*

$$a_j = \frac{\sum_k (Q_{kj} \sigma_{kj})}{Q_j^-} \; .$$

If $\mathcal{M}$ denotes the subset of $\mathcal{N}\setminus\bar{\mathcal{M}}$ of defaulted nodes triggered by $\mathcal{D}$, then for all j,k, $\rho_{jk} := \mathbb{P}[v\in\mathcal{M}|v\in\mathcal{N}_{jk}]$ is given by

$$\rho_{jk} = (1-\bar{\rho}_{jk})\sum_{m=M_{jk}}^{j}\binom{j}{m}a_j^m(1-a_j)^{j-m}$$

where the default thresholds are defined as in (9) *by $M_{jk} = \lceil \gamma_{jk}/w_j\rceil$.*

Proof: First we compute that

$$a_j = \mathbb{P}[\ell\in\mathcal{D}|j_\ell = j] = \frac{\sum_k \mathbb{P}[\ell\in\mathcal{D}\cap\mathcal{E}_{kj}]}{\mathbb{P}[j_\ell = j]} = \frac{\sum_k (Q_{kj}\sigma_{kj})}{Q_j^-}\ . \tag{16}$$

Note that $\mathbb{P}[\ell\in\mathcal{D}|v_\ell^-\in\mathcal{N}_{jk}] = \mathbb{P}[\ell\in\mathcal{D}|j_\ell = j]$ and also that

$$\begin{aligned}\mathbb{P}[v\in\mathcal{M}|v\in\mathcal{N}_{jk}] &= \mathbb{P}[v\in\mathcal{M}|v\in\mathcal{N}_{jk}\setminus\bar{\mathcal{M}}]\ \mathbb{P}[v\notin\bar{\mathcal{M}}|v\in\mathcal{N}_{jk}]\\ &= (1-\bar{\rho}_{jk})\mathbb{P}[v\in\mathcal{M}|v\in\mathcal{N}_{jk}\setminus\bar{\mathcal{M}}]\ .\end{aligned} \tag{17}$$

Under our assumptions, a node $v\in\mathcal{N}_{jk}\setminus\bar{\mathcal{M}}$ will be in default if and only if at least M_{jk} in-edges to v are in $\mathcal{D}$. One notes that LTIA (2) implies that the random variables $\mathbf{1}_{\ell\in\mathcal{D}}$ for all $\ell\in\mathcal{E}_v^-$, under the condition that $v\in\mathcal{N}_{jk}\setminus\bar{\mathcal{M}}$, are a collection of j identical independent Bernoulli random variables with probability a_j. Putting these facts together gives

$$\mathbb{P}[v\in\mathcal{M}|v\in\mathcal{N}_{jk}\setminus\bar{\mathcal{M}}] = \sum_{m=M_{jk}}^{j}\binom{j}{m}a_j^m(1-a_j)^{j-m}$$

which combined with (17) leads to the required result.

□

Using these lemmas and the definitions (13)-(15), it is straightforward to piece together the steps of the default cascade and obtain the main result of the paper.

Theorem 6. *Consider the extended GK financial network $(N,P,Q,\gamma,w,\bar{\rho})$ with $N=\infty$. Then*

1. *For $n=1,2,\dots,$ the quantities $\sigma^{(n)}, a^{(n)}, \rho^{(n)}$ satisfy the recursive formulas*

$$\sigma_{kj}^{(n)} = \frac{\sum_{j'}(\bar{\rho}_{j'k}+\rho_{j'k}^{(n-1)})P_{j'k}}{P_k^+}\ , \tag{18}$$

$$a_j^{(n)} = \frac{\sum_k(Q_{kj}\sigma_{kj}^{(n)})}{Q_j^-}\ , \tag{19}$$

$$\rho_{jk}^{(n)} = (1-\bar{\rho}_{jk})\sum_{m=M_{jk}}^{j}\binom{j}{m}(a_j^{(n)})^m(1-a_j^{(n)})^{j-m}\ , \tag{20}$$

where $M_{jk} = \lceil\gamma_{jk}/w_j\rceil$ and $\rho_{jk}^{(0)} = 0$. The total probabilities for defaulted (j,k) edges and nodes at step $n\geq 1$ are $\sigma_{kj}^{(n)}$ and $\bar{\rho}_{jk}+\rho_{jk}^{(n)}$.

2. *The new probabilities* $\vec{a}^{(n)} = \{a_j^{(n)}\}$ *are a function* $G(\vec{a}^{(n-1)})$ *which is explicit in terms of the specification* $(N, P, Q, c, w, \bar{\rho})$.
3. G *maps* $[0,1]^K$ *onto itself, and is monotonic under the partial ordering relation defined by* $\vec{a} \le \vec{b}$ *if and only if* $a_j \le b_j$ *for all* $j \in \mathcal{K}$. *That is,* $G(\vec{a}) \le G(\vec{b})$ *whenever* $\vec{a} \le \vec{b}$.

Proof: In Part 1, (18) follows from Lemma 4 while (19) and (20) follow from Lemma 5. Part 2 is simply a composition of (18), (19), (20). Part 3 follows by inspection.

□

Remark 7. *In the case of edge independence when* $Q_{kj} = Q_k^+ Q_j^-$, *the quantities* $a_j^{(n)} = a^{(n)}$ *no longer depend on* j. *Then the fixed point equation simplifies to the scalar equation* $a = G(a)$ *where*

$$G(a) = \sum_{j,k} \frac{jk}{z} P_{jk}(\bar{\rho}_{jk} + \rho_{jk}(a)), \quad \rho_{jk}(a) = (1 - \bar{\rho}_{jk}) \sum_{m=M_{jk}}^{j} \binom{j}{m} a^m (1-a)^{j-m} \tag{21}$$

4 The Cascade Condition

The expected size of global cascades in a given extended GK financial network has essentially been reduced to solving the vector valued fixed point equation

$$\vec{a} = G(\vec{a}) \tag{22}$$

by iteration of the mapping G. Scalar equations similar to (21), giving the expected size of cascades on directed networks, have been previously derived in various contexts without assortativity. In [8], the main focus is on percolation-type phenomena (see also the undirected networks case [7]), while [1] considers more complicated dynamics but takes the limit $\bar{\rho} \to 0$. The case considered in [6], where initial default fractions can be different for each (j,k) class, has not, to our knowledge, been considered previously. In the current work, we include for the first time (through Q_{kj}) the effect of non-trivial correlations between the degrees of nodes at either end of a randomly chosen edge.

As a consequence of Part 3 of Theorem 6 and the Knaster-Tarski Fixed Point Theorem, equation (22) always has at least one solution $\vec{a}^{\infty}$ and this will be a vector of probabilities $a_j^{\infty} \in [0,1]$ for all j. Let us consider initial probabilities $\epsilon\bar{\rho}$ for a fixed set of default probabilities $\bar{\rho}$ and small $\epsilon > 0$. We write $G(\vec{a}^{(n-1)}) = G(\vec{a}^{(n-1)}, \epsilon)$ to highlight the dependence on the parameter ϵ. One important question is to ask whether the fixed points $\vec{a}^{(\infty)}(\epsilon)$ are of order ϵ or of order 1 as $\epsilon \to 0$. In other words, what is the "cascade condition" that determines if an infinitesimally small seed fraction will grow to a large-scale cascade? It turns out this depends on the spectral radius of the derivative matrix $D = \{D_{jj'}\}$ with $D_{jj'} = \partial G_j / \partial a_{j'}|_{\vec{a}=\mathbf{0},\epsilon=0}$.

Recall that the spectral radius of D, $\|D\| := \max_{\vec{a}:\|\vec{a}\|=1} \|D\vec{a}\|$, is the largest eigenvalue of D in absolute value. In our framework, the derivatives $D_{jj'}$ are easy to calculate:

$$D_{jj'} = \sum_k \frac{j' Q_{kj} P_{j'k} \mathbf{1}_{\{\gamma_{j'k} \le w_{j'}\}}}{Q^-_j P^+_k} \ . \tag{23}$$

Note each component of D is non-negative: To enable an elementary proof of the following result, we assume each component is strictly positive.

Proposition 8. *Suppose the extended GK financial network $(N = \infty, P, Q, \gamma, w, \bar{\rho})$ is such that D and $\bar{\rho}$ are positive matrices (i.e. all components are positive).*

1. *If $\|D\| > 1$, then there is $\bar{\epsilon} > 0$ and $\delta > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$, $\|\vec{a}^{(\infty)}(\epsilon)\| > \delta$. That is, in this network, an infinitesimal seed will trigger a large scale cascade almost surely.*
2. *If $\|D\| < 1$, then there is $\bar{\epsilon} > 0$ and C'' such that for all $0 < \epsilon < \bar{\epsilon}$, $\|\vec{a}^{(\infty)}(\epsilon)\| \le C''\epsilon$. That is, this network will not exhibit large scale cascades for any infinitesimal seed.*

Proof: See the Appendix.

In Section 6, we shall see that the cascade condition is indeed a strong measure of systemic risk in simulated networks. One can check that in the setting of independent edge probabilities $Q_{kj} = Q^+_k Q^-_j$, the spectral radius becomes

$$\|D\| = \sum_{jk} \frac{jk}{z} P_{jk} \mathbf{1}_{\{\gamma_{j'k} \le w_{j'}\}} \ ,$$

a result that has been derived in a rather different fashion in [5] and [1]. [5] also extends the [15] percolation theory approach from undirected networks to the case of directed nonassortative networks. We will see in the next section that the percolation approach to the cascade condition also extends to our directed assortative networks.

We can understand the cascade condition more clearly by introducing the notion of *vulnerable node*, that is any node that becomes insolvent if any one of its debtors defaults. In our specifications, a (j, k) node is thus vulnerable if and only if $\gamma_{jk} \le w_j$. The matrix element $D_{jj'}$ has a simple explanation that gives more intuition about the nature of the cascade condition: it is the expected fraction of edges ℓ' with $j_{\ell'} = j'$ that connect through a vulnerable node to an edge ℓ with $j_\ell = j$.Then for small values of $\vec{a}$, one has a linear approximation for the change in $\vec{a}$ in a single cascade step:

$$a^{m+1}_j - a^m_j = \sum_{j'} D_{jj'} \ (a^m_{j'} - a^{m-1}_{j'}) + O(\|a\|^2) \ . \tag{24}$$

The condition for a global cascade starting from an infinitesimal seed is that the matrix D must have an expanding direction, i.e. an eigenvalue with magnitude bigger than 1.

5 Frequency of global cascades and the giant vulnerable cluster

The previous argument does not tell us directly about the frequency of global cascades. However, it is well-known [11, Chapter 13.11] that the frequency of global cascades in infinite random graphs is related to (in fact, bounded from below by) the fractional size of the so-called in-component associated to the giant vulnerable cluster.

To facilitate the discussion we make the following further definitions:

- $\mathcal{V} \subset \mathcal{N}$ is the set of vulnerable nodes.
- $\mathcal{S}_s$ is the giant strongly connected set of vulnerable nodes (the "giant vulnerable cluster");
- the "in-component" of the giant vulnerable cluster $\mathcal{S}_i$ is the set of (possibly not vulnerable) nodes that are connected to $\mathcal{S}_s$ by a path of in-edges and vulnerable nodes;
- $\Gamma_{jk} = \mathbf{1}_{\{\gamma_{jk} \le w_j\}}$.

We are interested in the following probabilities $\vec{b} = \{b_k\},\ b_k := \mathbb{P}[v \notin \mathcal{S}_i | k_v = k]$ and note that $v \in \mathcal{S}_i^c$ (i.e. the complement of $\mathcal{S}_i$) is equivalent to the condition that all the downstream nodes $v_\ell^-, \ell \in \mathcal{E}_v^+$ are in the set $(\mathcal{S}_i^c \cap \mathcal{V}) \cup \mathcal{V}^c$. Thus, letting v' denote any node one edge downstream from v, one has:

$$b_k = (c_k)^k \tag{25}$$

where

$$c_k = \sum_{j',k'} \mathbb{P}[v' \in (\mathcal{S}_i^c \cap \mathcal{V}) \cup \mathcal{V}^c | v' \in \mathcal{N}_{j'k'}, k_\ell = k] \mathbb{P}[v' \in \mathcal{N}_{j'k'} | k_\ell = k] \ .$$

The LTIA implies

$$\begin{aligned} &\mathbb{P}[v' \in (\mathcal{S}_i^c \cap \mathcal{V}) \cup \mathcal{V}^c) | v' \in \mathcal{N}_{j'k'}, k_\ell = k] = \Gamma_{j'k'} b_{k'} + (1 - \Gamma_{j'k'}) \\ &\mathbb{P}[v' \in \mathcal{N}_{j'k'} | k_\ell = k] = \frac{P_{j'k'} Q_{kj'}}{P_{j'}^- Q_k^+} \end{aligned}$$

and thus

$$c_k = \sum_{j',k'} \left(\Gamma_{j'k'} b_{k'} + (1 - \Gamma_{j'k'}) \right) \frac{P_{j'k'} Q_{kj'}}{P_{j'}^- Q_k^+} \tag{26}$$

Since $b_k = (c_k)^k$ it follows that $\vec{c} = \{c_k\}$ satisfies the fixed point equation $\vec{c} = h(\vec{c})$ where

$$h_k(\vec{c}) = \sum_{j',k'} \left(\Gamma_{j'k'} (c_{k'})^{k'} + (1 - \Gamma_{j'k'}) \right) \frac{P_{j'k'} Q_{kj'}}{P_{j'}^- Q_k^+} \tag{27}$$

Note that the equation $\vec{c} = h(\vec{c})$ has a trivial fixed point $\vec{e} = (1, 1, \dots)$ that corresponds to the set $\mathcal{S}_i$ having probability zero. We now verify that the cascade condition $\|D\| > 1$ is equivalent to the condition that e is an unstable fixed point, in which case there will be a nontrivial fixed point $0 \le \vec{c}_\infty < \vec{e}$. A sufficient (and almost necessary) condition for $\vec{e}$ to be an unstable fixed point is that $\|\tilde{D}\| > 1$ where the derivative $\tilde{D}_{kk'} = (\partial h_k/\partial c_{k'})|_{\vec{c}=\vec{e}}$ is given by

$$\tilde{D}_{kk'} = \sum_{j'} \frac{k' Q_{kj'} P_{j'k'} \Gamma_{j'k'}}{Q^+_k P^-_{j'}} \tag{28}$$

One can verify directly that

$$\tilde{D} = \left(\Lambda B A \Lambda^{-1}\right)^T, \quad D = AB$$

for matrices

$$A_{jk} = \frac{Q_{kj}}{Q^-_j},\ B_{kj'} = \frac{j' P_{j'k} \Gamma_{j'k}}{P^+_k},\ \Lambda_{kk'} = \delta_{kk'} k P^+_k$$

and from this it follows that the spectrum, and hence the spectral radii and spectral norms, of $\tilde{D}$ and D are equal. Hence $\|D\| > 1$ if and only if $\|\tilde{D}\| > 1$.

As long as the cascade condition is satisfied, the cascade frequency f is bounded by the fractional size of the in-component $\mathcal{S}_i$:

$$f \ge \sum_k \mathbb{P}[v \in \mathcal{S}_i | k_v = k]\ \mathbb{P}[k_v = k] = \sum_k (1 - c_{k,\infty}) P^+_k\ . \tag{29}$$

Repeating this type of argument to determine the size of the out-component of the giant vulnerable cluster, one also obtains an upper bound on the mean size of the global cascade.

6 Numerical Results

In this section we present results from large-scale Monte Carlo simulations on random networks, and show that the analytical theory of Section 3 matches well to the numerical results when N, the number of nodes in the network, is sufficiently large.

6.1 A Simple Random Network Model

We consider networks constructed with nodes of types $(3,3), (3,12), (12,3), (12,12)$ and edges of the same types. We fix the marginal probabilities $P^+_3 = P^+_{12} = 1/2$ which lead to an average degree $z = 15/2$ and the marginals $Q^+_3 = 1/5, Q^+_{12} = 4/5$. For parameters $a \in [0, 1/2]$ and $b \in [0, 1/5]$ the following P and Q probabilities are consistent:

$$\begin{pmatrix} P_{3,3} & P_{3,12} \\ P_{12,3} & P_{12,12} \end{pmatrix} = \begin{pmatrix} 1/2 - a & a \\ a & 1/2 - a \end{pmatrix}; \quad \begin{pmatrix} Q_{3,3} & Q_{3,12} \\ Q_{12,3} & Q_{12,12} \end{pmatrix} = \begin{pmatrix} 1/5 - b & b \\ b & 4/5 - b \end{pmatrix}. \tag{30}$$

We first fix the value of a to be 0.5, which means that the in- and out-degrees of all nodes are negatively correlated: nodes with in-degree 3 have out-degree 12, and vice versa. We examine three different values of the parameter b: the independent connections case $b = 0.16$, the (almost) maximally assortative case $b = 0.01$ and the (almost) maximally disassortative case $b = 0.19$. Note that the independent edge condition has been assumed in all previous work on such problems. We also note that with $b = 0$, edges are maximally assortative and link nodes of out-degree 3 to nodes of in-degree 3 only, and nodes of out-degree 12 to nodes of in-degree 12 only. In this case, the network falls into two disjoint pieces.

The balance sheet quantities are those of [5] (except for the percentage net worth γ, which we vary over the range 0% to 10%), while the initial shock distribution is taken to be $\bar{\rho}_{jk} = 1/N$ for all types (j, k), corresponding to the shocking of a single, randomly-chosen, bank.

Figure 3 compares theory curves for cascade size (found by iterating equations (18)–(20) to convergence) as well as cascade frequency (assuming equality in (29)) with results from numerical simulations on random networks with $N = 10^4, 10^3$ and 200 nodes. The nodal correlation parameter is fixed at $a = 0.5$, while the edge correlation parameter takes the values $b = 0.01, 0.16, 0.19$. Results are plotted as functions of the percentage net worth parameter γ. In each case, 500 realizations are used to find the extent of global cascades (a global cascade is defined, similarly to [5, 6], as one in which more than 5% of nodes default), and the frequency with which such global cascades occur. As expected, the analytical approach accurately predicts the size of the global cascades. Some discrepancies may be noted in Figure 3, where the theory does not predict some global cascades, but note that these occur with only very small frequencies.

The cascade condition (23) predicts that the critical values of the buffer parameter γ are: $\gamma_c = 0.067$ for the parameters of Figure 3(a), and $\gamma_c = 0.017$ for the case of Figure 3(b). These values match very accurately to the locations of the dramatic transitions in the theory curve (and in the expected size of cascades in numerical experiments): for buffer values in excess of γ_c global cascades are extremely rare, while for γ values less than γ_c the entire financial system is likely to fail following a single bank's default. These result indicate the potential usefulness of the cascade condition as a measure of systemic risk.

We consider in Figure 4 the joint dependences on a, b of various theoretical quantities in the infinite limit. In the top figures, the critical value of γ and cascade size are seen to be discontinuous, and not directly related to edge-assortativity (parametrized by b). On the other hand (see bottom figures), the frequency of cascades is continuously varying, and does appear to correlate somewhat with the graph assortativity coefficient r given by (5).

6.2 Another Simple Random Network Model

Now we have shown that the infinite N theory meets our expectations, we can further explore the implications of the analytical method. Since the specification of extended

GK networks has many components, one must be rather careful in the questions one wishes to address: we choose here to try by means of a simple network specification to shed some additional light on the role the assortative properties of a network play in its susceptibility to systemic risk as measured by the cascade frequency.

We consider stylized networks with many small banks and a few large banks. The set of node types will be $\{(2,2),\ (4,4),\ (8,8),\ (16,16)\}$ with a diagonal node probability matrix:

$$P := (P_{jk}) = \text{diag}(8,\ 4,\ 2,\ 1)/15$$

The following edge probability matrices $Q = (Q_{kj})$ are consistent with P:

$$Q^{(1)} = \frac{1}{4}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ Q^{(2)} = \frac{1}{4}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

$$Q^{(3)} = \frac{1}{4}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\ Q^{(4)} = \frac{1}{4}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

and their convex combinations $q_1Q^{(1)} + q_2Q^{(2)} + q_3Q^{(3)} + q_4Q^{(4)}$, $q_1 + q_2 + q_3 + q_4 = 1, q_i \geq 0$ span a simplex of possible edge probability matrices. We can see that as measured by r_Q, $Q^{(1)}$ is maximally assortative, while $Q^{(3)}$ and $Q^{(4)}$ are maximally disassortative, and the independent case is $Q^{(0)} := [Q^{(1)} + Q^{(2)} + Q^{(3)} + Q^{(4)}]/4$.

For the remaining components of the specification we adopt the default GK balance sheet values but with γ a variable parameter and consider shocking a single randomly selected node (this is an infinitesimal shock in the infinite network limit). We then compute the critical γ_c using the cascade formula (23), the cascade size from Proposition 8, and the default frequency using equation (29).

Figure 5 shows how the theoretical values of γ_c and cascade size depend on the particular Q matrix. Figure 6 shows how the theoretical values of the graph assortativity coefficient and cascade frequency depend on Q. In both figures, the four rows correspond to the simplices of Q matrices with $q_4 = 0, q_3 = 0, q_2 = 0, q_1 = 0$ respectively.

We see again in these networks that r and f vary continuously, while γ_c and cascade size take on only discrete values. Of particular interest is the discernable covariation of f with r. Since r depends only on the skeleton graph and not on the balance sheet data, we cannot expect a one-to-one relationship between the quantities f and r. However, we conjecture that r is in some sense the best possible purely graph theoretic measure of systemic susceptibility. Heuristically, we might expect that systemic risk is lowered if, all else being equal, the network is such that the correlation between in-degrees of neighbouring nodes is lowered.

7 Concluding Remarks

In summary, we have described here a rigorous analytical framework which can predict the systemic risk of "deliberately simplified models" such as [5]. The qualitative type of networks one can address has been extended compared to most existing work, in particular by the inclusion of the non-independent connections between nodes. In this more general setting we find the cascade is described by a vector-valued fixed point problem that reduces to well-understood scalar problems in special cases. The examples of Section 6 demonstrate that the finite size effects that break the LT independence assumption do not appear to dramatically impact systemic risk as long as $N \gtrsim 100$. More subtly, we also observed that graph-assortativity, rather than edge- or node-assortativity, can strongly affect the course of contagion cascades, and hence show the importance of incorporating assortativity in numerical and analytical treatments of banking network models. Our analytic framework will enable extensive studies of alternative network topologies. In these studies the cascade condition and cascade frequency provide two easily computed and useful measures of systemic risk by which to compare different network topologies. However, the daunting range of network variables means that both analytical and numerical studies must be carefully framed to address specific issues, for example, to uncover other key determinants of systemic risk. Finally, we anticipate that future work can show how the approach described here may be further extended to include partial recovery models (such as [12]) and stochastic balance sheets.

8 Acknowledgements

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Appendix: Proof of Proposition 8

Part 1: By continuous dependence, there are values $\bar\epsilon > 0$ and $\lambda > 1$ such that the matrix $D_\epsilon = \partial G/\partial a|_{\vec a = \mathbf{0}, \epsilon}$ is positive and has spectral radius $\|D_\epsilon\| \geq \lambda$ for all $0 < \epsilon < \bar\epsilon$. Let us fix any such ϵ.

By the Perron-Frobenius Theorem for positive matrices, there is a unique normalized eigenvector $\vec v$ such that $D_\epsilon \vec v = \|D_\epsilon\| \vec v$: it has all positive entries and normalization $\|v\| = 1$. Taylor's Theorem implies that for $\bar\epsilon$ small enough there is $C > 0$ such that

$$G(a, \epsilon) = G(0, \epsilon) + D_\epsilon a + R(a), \quad \|R(a)\| \leq C\|a\|^2$$

for all a with $\|a\| \leq \bar\epsilon$ (note we drop the $\vec\cdot$ notation in the following).

Now we show that the sequence $a^{(1)} = G(0, \epsilon), a^{(n+1)} = G(a^{(n)}, \epsilon)$ leaves the set $\|a\| \leq \bar\epsilon$ provided $\bar\epsilon$ is chosen small enough (independently of ϵ). For this, since $\bar\rho > 0$

there is $\beta_1 > 0$ and a non-negative vector y_1 such that $a^{(1)} = \beta_1 v + y_1$. Assuming inductively that $a^{(n)} = \beta_n v + y_n$ for some $\beta_n > 0$ and a non-negative vector y_n and that $\|a^{(n)}\| \le \bar\epsilon$ the monotonic property of G combined with Taylor's Theorem implies

$$\begin{aligned} a^{(n+1)} &= G(a^{(n)}, \epsilon) \ge G(\beta_n v, \epsilon) \\ &= G(0, \epsilon) + \beta_n D_\epsilon v + R(\beta_n v) \\ &\ge \beta_1 v + y_1 + \frac{1}{2}(1+\lambda)\beta_n v + \left(\frac{1}{2}(\lambda - 1)\beta_n v + R(\beta_n v)\right) \end{aligned}$$

Let $\beta_{n+1} = \beta_1 + \frac{1}{2}(1+\lambda)\beta_n$ and note that $y_{n+1} = a^{(n+1)} - \beta_{n+1} v \ge 0$ provided $\bar\epsilon \le \frac{1}{2C}(\lambda - 1)\min_j v_j$. Since the sequence β_n increases without bound, we can iterate the inductive argument only a finite number of steps before $\|a^{(n+1)}\| > \epsilon$.

Part 2: By continuous dependence, there are now values $\bar\epsilon > 0$ and $\lambda = \frac{1}{2}(1 + \|D\|) < 1$ such that the matrix $D_{a,\epsilon} = \partial G/\partial a|_{\vec{a},\epsilon}$ has spectral radius $\|D_{a,\epsilon}\| \le \lambda$ for all $0 \le \epsilon < \bar\epsilon$ and $\|a\| \le \bar\epsilon$. Fix any such ϵ. Now we note that for vectors a, b with $\|a\|, \|b\| \le \bar\epsilon$ we can use Taylor's Theorem again to write

$$G(a, \epsilon) - G(b, \epsilon) = D_\epsilon(a - b) + R(a, b)$$

where the remainder has bound $C\|a - b\|^2$ for some $C > 0$. Then provided $\|a^{(n+1)}\|, \|a^{(n)}\| \le \bar\epsilon$ and $\bar\epsilon \le \frac{1-\|D\|}{4C}$

$$\begin{aligned} \|a^{(n+1)} - a^{(n)}\| &= \|G(a^{(n)}, \epsilon) - G(a^{(n-1)}, \epsilon)\| \\ &\le \frac{1}{2}(\lambda+1)\|a^{(n)} - a^{(n-1)}\| + \left(\frac{1}{2}(\lambda - 1)\|a^{(n)} - a^{(n-1)}\| + \|R(a^{(n)}, a^{(n-1)})\|\right) \\ &\le \frac{1}{2}(\lambda+1)\|a^{(n)} - a^{(n-1)}\| \end{aligned}$$

for all $n \ge 1$. Since $\|G(0, \epsilon)\| \le C'\epsilon$ for some $C' > 0$, we can iterate this inequality to show $\|a^{(\infty)}\| \le C''\epsilon$ with $C'' = \frac{4C'}{1-\|D\|}$.

□

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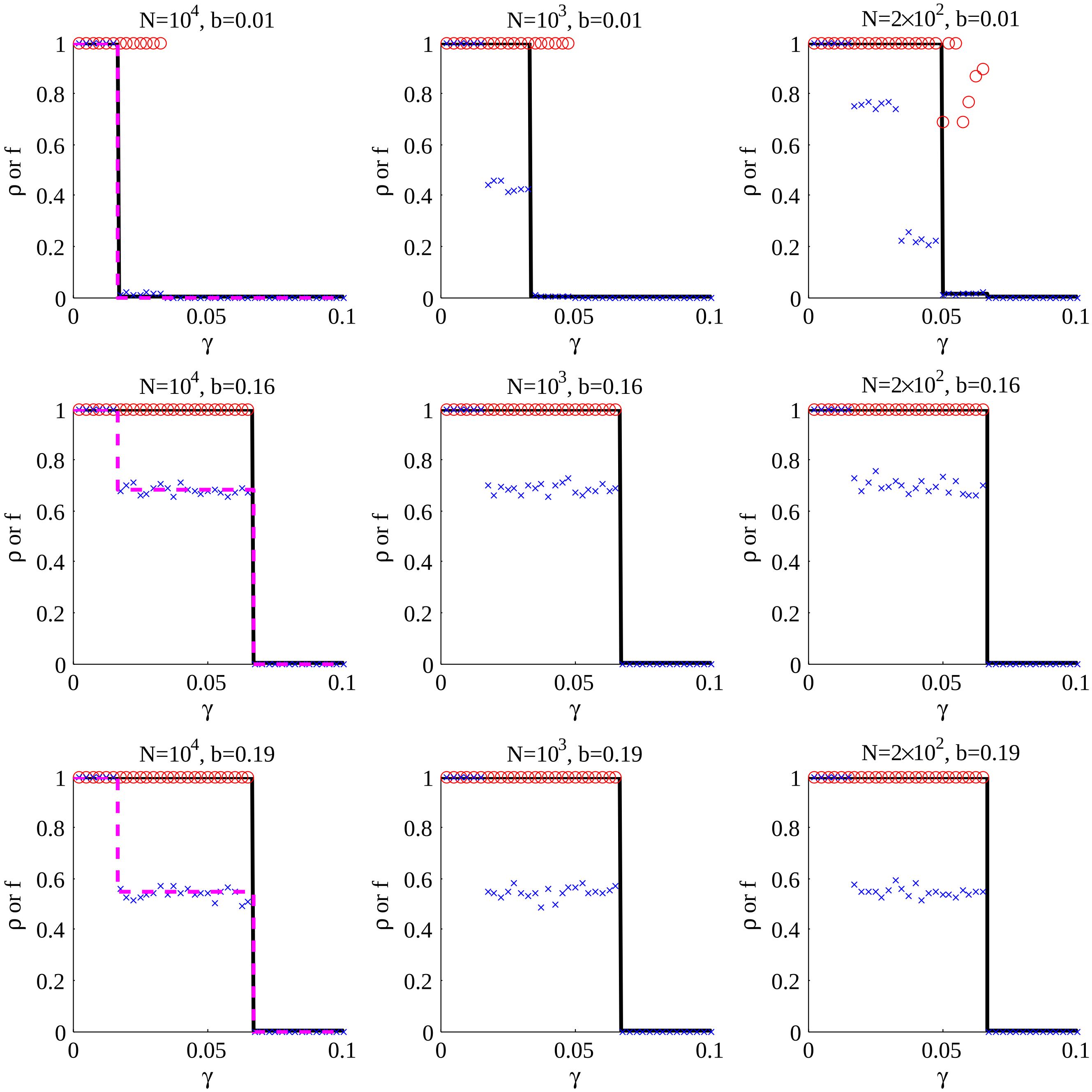

Figure 3: Numerical simulation results (symbols) and theoretical results (curves) for the random network model of Equation (30), on networks of N nodes with parameter $a = 0.5$, as functions of the net worth γ. The average size and frequency of global cascades in simulations are shown by red circles and blue crosses, respectively. Theoretical results for the expected cascade size (black solid curve) are from Section 3; those for the frequency of cascades (dashed magenta curve) are from Section 5. Each column shows results for a different network size N, and the parameter b takes a different value on each row of the figure.

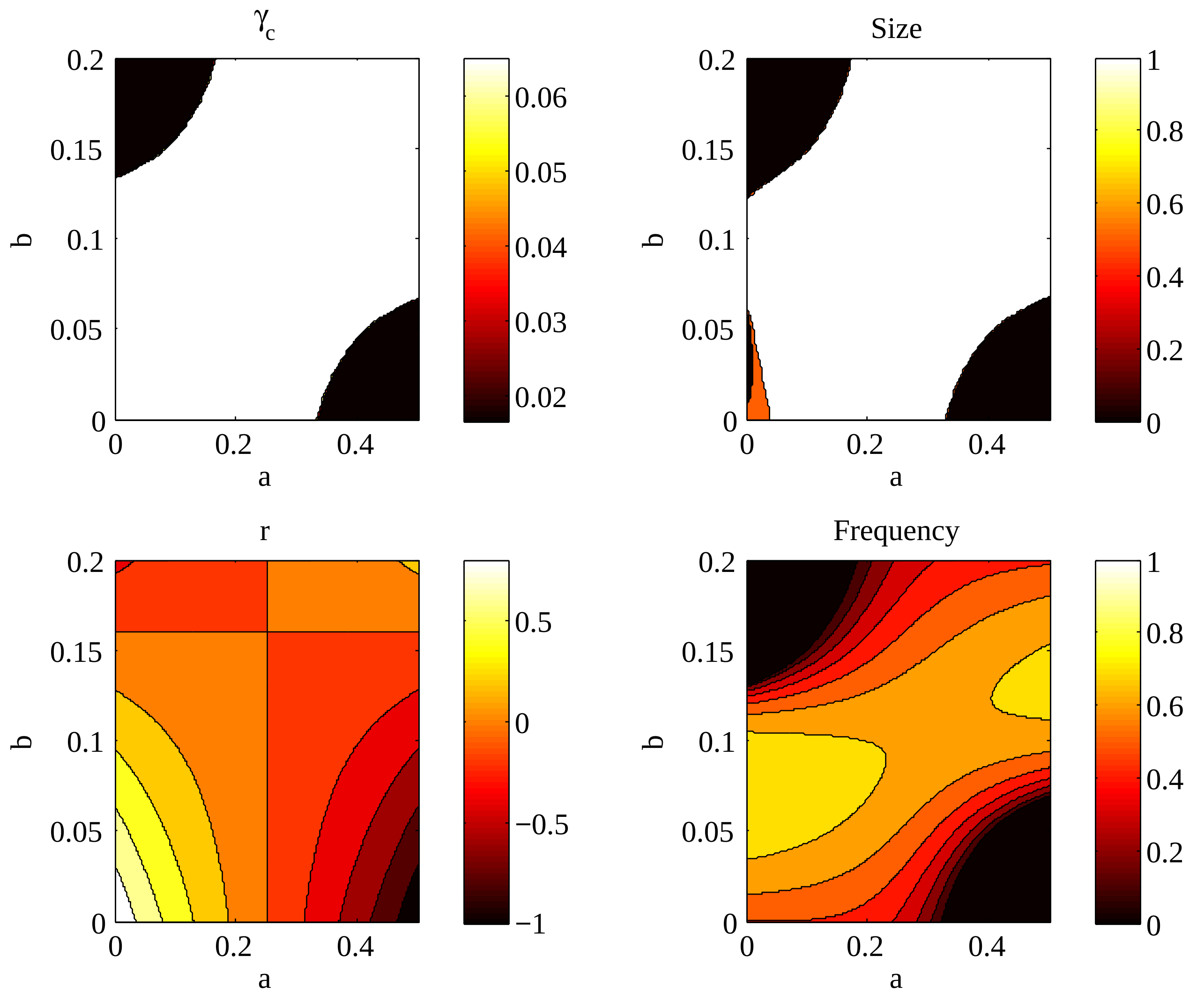

Figure 4: Theoretical joint dependences on the graph parameters (a, b). Top left: critical γ value for the random network model of Equation (30). Top right: Expected size of cascades (from Section 3) when $\gamma = 0.05$ and $\bar{\rho}_{jk} = 10^{-4}$. Bottom left: the graph assortativity parameter r. Bottom right: Frequency of cascades (from Section 5) when $\gamma = 0.05$.

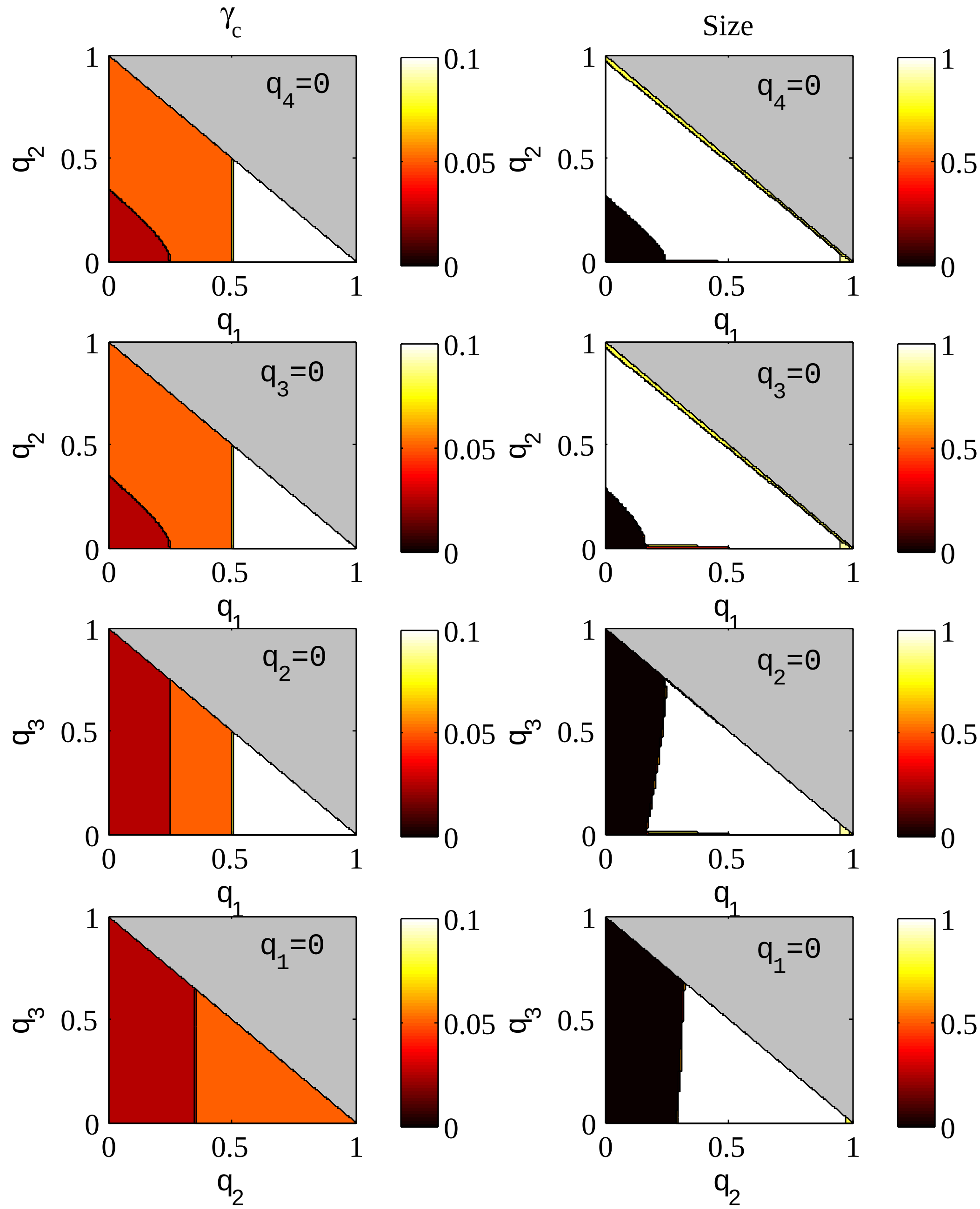

Figure 5: Critical value of γ (left column) and expected size (right column) for cascades on the random network model of Section 6.2 with $\gamma = 0.0375$ and $\bar{\rho}_{jk} = 10^{-4}$. The triangles shown correspond to Q matrices with (from top to bottom) $q_4 = 0$, $q_3 = 0$, $q_2 = 0$, and $q_1 = 0$.

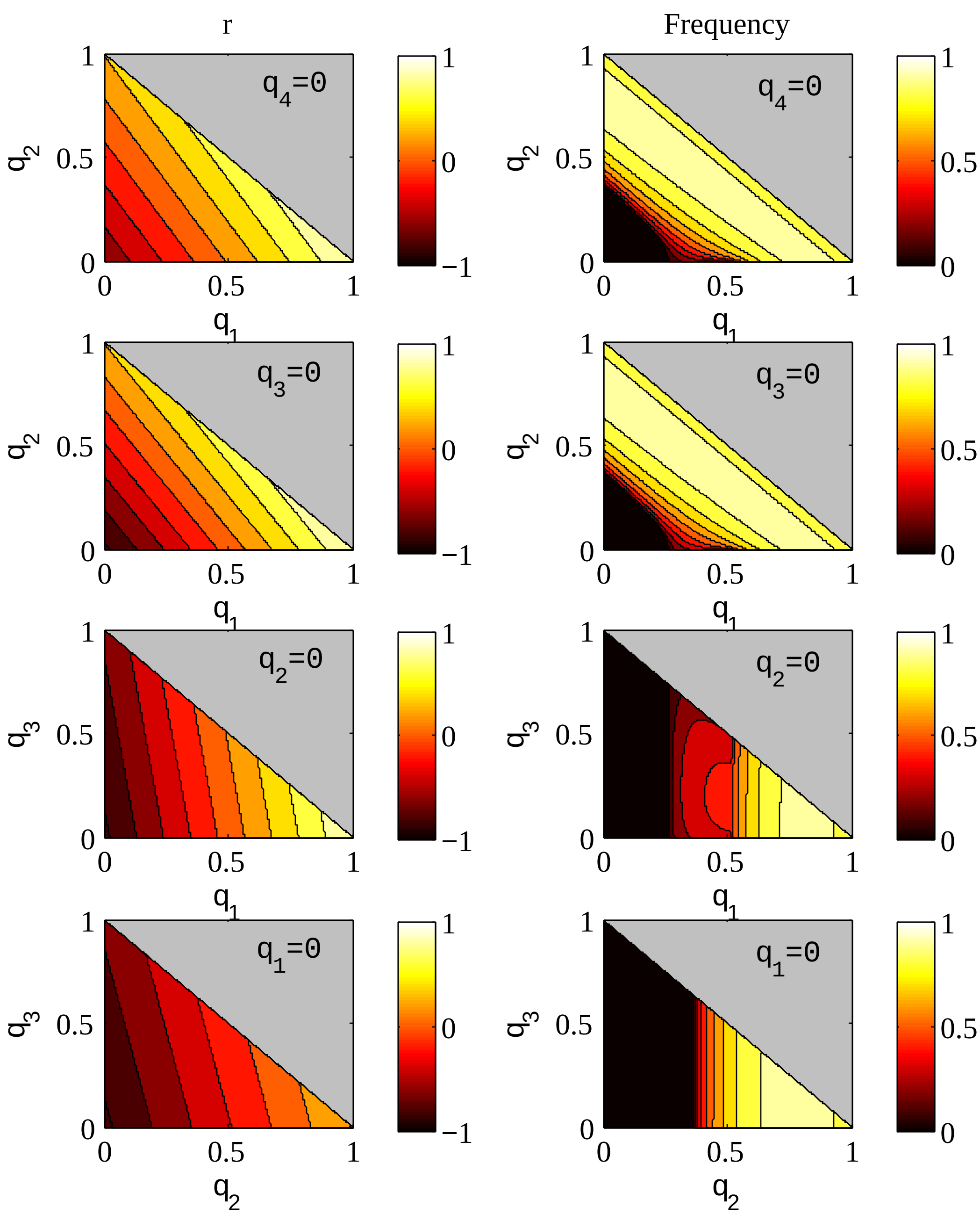

Figure 6: Graph assortativity parameter r (left column) and frequency (right column) for the same parameters as Fig. 5.