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Contagion! Systemic Risk in Financial Networks

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Preface

This slim volume logs the development of a cascade of contagious ideas that has occupied my space, time and mind in recent years. There was a clear triggering event that occurred in April 2009. Late in that month, Michael Lynch and his colleagues at MITACS Canada brought together a host of scientists, mathematicians and finance industry participants for three days to brainstorm about underlying causes of the ongoing financial crisis and how mathematical thinking could be brought to bear on it. My role there was as gadfly to provoke discussion on a special topic no one at the meeting was very aware of, namely financial systemic risk.

Since that event introduced me to the subject, I have had many opportunities to present to a diversity of audiences an evolving view of how the architecture of the financial system can be described in terms of network science, and how such a network formulation can be made amenable to a certain type of mathematical analysis. This book is not intended to be a definitive work on the subject of financial systemic risk, and does not try to represent a broad consensus. Instead, it is a personal attempt to crystallize the early results of research that focuses on the basic modelling structure that ensures some kind of mathematical tractability, while allowing a great deal of both reality and complexity in the actual finance network specification. I owe a debt of thanks to a great number of people who have listened, commented, and added new nodes to this complex network of ideas, too many to list here in this preface.

My McMaster colleague, Matheus Grasselli, was instrumental in many ways, not least in providing the original impetus to write this SpringerBrief. Nizar Touzi encouraged and supported me in my first attempt at delivering a minicourse on Systemic Risk. The scope of this minicourse grew over time: Jorge Zubelli hosted me for an extended period at IMPA, where I delivered another version; Peter Spreij arranged a session for me to speak at the Winter School on Financial Mathematics in Lunteren; James Gleeson provided me with multiple invitations to Limerick. The Fields Institute for Research in Mathematical Sciences gave me encouragement and organized multiple events relevant to my work. The Global Risk Institute for Financial Services, in particular Michel Maila and Catherine Lubochinsky, have provided substantial financial and moral support for this research. I give my hearty thanks to

Mario Wüthrich and Paul Embrechts who hosted my extended stay at ETH Zürich in 2014 where I was extremely fortunate to be able to deliver a Nachdiplom lecture series based on the material contained in this book. Finally, to my wife, Rita Bertoldi, I offer my affectionate acknowledgment of her patient support throughout my lengthy exposure to this dangerous contagion.

Contents

Chapter 1
Systemic Risk Basics

Annual income twenty pounds, annual expenditure nineteen nineteen six, result happiness. Annual income twenty pounds, annual expenditure twenty pounds ought and six, result misery. The blossom is blighted, the leaf is withered, the god of day goes down upon the dreary scene, and—and, in short, you are for ever floored.[1]

Abstract Attempts to define systemic risk are summarized and found to be deficient in various respects. In this introductory chapter, after considering some of the salient features of financial crises in the past, we focus on the key characteristics of banks, their balance sheets and how they are regulated.

Bankruptcy! Mr. Micawber, David Copperfield's debt-ridden sometime mentor, knew first hand the difference between surplus and deficit, between happiness and the debtors' prison. In Dickens' fictional universe, and perhaps even in the real world of Victorian England, a small businessman's unpaid debts were never overlooked but always lead him and his loved ones to the unmitigated misery of the poorhouse. On the other hand, the aristocrats and upper classes, were treated more delicately, and usually given a comfortable escape.

For people, firms, and in particular banks, bankruptcy in modern times is more complicated yet it still retains some of the flavour of the olden days. When a bank fails, it often seems that the rich financiers responsible for its collapse and the collateral damage it inflicts walk away from the wreckage with intact bonuses and compensation packages. When a particularly egregious case arises and a scapegoat is needed, then a middle rank banker is identified who takes the bullet for the disaster. A cynic might say that despite the dictates of Basel I, II, III, ...∞, bank executives remain free to take excessive risks with their company, receiving a rich fraction of any upside while insulating themselves from any possible disaster they might cause.

As we learn afresh during every large scale financial crisis, society at large pays the ultimate costs when banks fail. Spiking unemployment leads to the poverty of

[1] Charles Dickens, *David Copperfield*, Chapter 12, p. 185 (1950). First published 1849–1850.

the less well-to-do, while salary freezes and imploded pension plans lead to belt-tightening and delayed retirement for the better-off. Those at the top of the pile, even those responsible, often do just fine. Banks that are too big to fail are propped up, while failed banks are bailed out by governments, their debts taken over and paid by the taxpayers.

If anything is different since the crisis of 2007-08, perhaps it is the widespread recognition that society needs to find ways and means to ensure that the responsible parties pay the downside costs of bank failure. New ideas on bank resolution, including contingent capital and bail-in regulation, aim to force the financial stakeholders, not the central bank, to pay much higher fractions of the costs of failure. Banks' creditors, bondholders and equity investors should in the future be forced to take their fair share of losses. When banking incentives and regulation are better aligned with the needs of society, then bank failures will be better anticipated, prepared for and managed to reduce their most catastrophic consequences.

1.1 The Nature of this Book

The title "Contagion! Systemic Risk in Financial Networks" is intended to suggest that financial contagion is analogous to the spread of disease, and that damaging financial crises may be better understood by bringing to bear ideas gained from studying the breakdown of other complex systems in our world. It also suggests that the aim of systemic risk management is similar to the primary aim of epidemiology, namely to identify situations when contagion danger is high, and then to make targeted interventions to damp out the risk.[2]

The primary goal of this book is to present a unified mathematical framework for the transmission channels for damaging shocks that can lead to instability in financial systems. Models in science and engineering can usually be described as either explanatory or predictive. In the early stages of research in a field, explanatory models may make dramatic oversimplifications or counterfactual assumptions that are only justifiable to the extent they highlight and explain the most critical mechanisms underlying the phenomenon of interest. Later, when guided by such improvements in understanding, predictive models become feasible. Certainly, predictive models will be more complex, and must be carefully calibrated to the details of the observed system in question. Since financial systemic risk is a rather new field, this book focuses on certain explanatory models developed by economists that aim to explore how disruptions can arise in large financial systems. We will therefore make certain dramatic oversimplifications in the hope of gaining mathematical clarity and analytic tractability that can improve understanding of the different ways financial instability can arise.

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[2] Interestingly, I found on Wikipedia that epidemiology has a code of nine principles, called the "Bradford Hill criteria", that should be considered to help assess evidence of a causal relationship between an incidence and a consequence. Perhaps, researchers can codify an analogous set of principles for assessing systemic risk.

This introductory chapter will develop the concepts and setting for systemic risk in financial networks. It provides a brief survey of how people have viewed and defined financial crises and systemic risk. It looks at how banks' balance sheets reflect the type of business they deal with, and the ways adverse shocks between banks can be transmitted and amplified. Finally, we review the key aspects of the new international regulatory regime for banks that are designed to safeguard global financial stability.

From Chapter Two onwards, we delve more deeply into the mechanics of the interactions between banking counterparties. Chapter Two puts a sharp focus on the type of bank behaviour that can negatively impact the functioning of the entire system, by surveying, dissecting and classifying a number of economic models for financial contagion that have been proposed in recent years. We will make the important discovery that a common mathematical structure unlies a variety of financial cascade mechanisms, namely such crises proceed through cascade mappings that approach a cascade equilibrium. To address the intrinsic opacity of financial institutions and their interconnections, we identify a particular point of view developed by Gai, Kapadia [42], Amini, Cont, Minca [7] and others that argues for the usefulness of *random financial networks*, a statistical representation of networks of banks, their interconnections and their balance sheets. The design of this concept reflects the type of models that network science, reviewed in the book [71], has already developed in other domains.

The remainder of the book is devoted to studying cascade models on asymptotically large random financial networks. Chapter Three provides the mathematical underpinning we need by developing and adapting the theory of random graphs which describes the *skeleton* structure at the heart of the random financial network. Two distinct classes of random graphs, the Assortative Configuration Graph model and the Inhomogeneous Random Graph model, are characterized in detail by their stochastic construction algorithms. The first class, which will form the framework underlying the cascade channels studied in the remaining chapters, is an extensive generalization of the well-known configuration graph model that incorporates *assortative wiring* between nodes that represent banks, which means wiring probabilities depend on banks' degree. It has not been well studied before so we spend time to develop its key mathematical properties, the most important of which we call the *locally tree-like property*. The second class of random graph extends the meaning of nodes to include *types* other than banks, such as asset classes or hedge funds. Chapter Four is devoted to understanding the relation between the Watts 2002 model of information cascades [82] and the concept of bootstrap percolation in random networks, studied recently in [10]. The Watts model can be fully analyzed from first principles, providing us with a template for results on more specific cascade mechanisms on financial networks. We shall learn that its properties can be determined using the mathematics of *percolation* which addresses the size distribution of connected network components. Chapter Five returns to focus on the zero recovery default cascade mechanism on due to Gai and Kapadia and studied by Amini, Cont, Minca [7]. It develops a purely analytical method for computing the large network asymptotics of cascade equilibria, based on the locally treelike property of assortative configuration graphs. The main theorem on the asymptotic form of the default cascade extends the work of Amini, Cont and Minca in certain respects, and requires new proof techniques not previously developed. This theory provides us with a computational methodology that is independent of and complementary to the usual Monte Carlo simulation techniques used everywhere in network science. Finally in Chapter Six we indicate some of the ways this theory can be extended to encompass more complex contagion channels.

Do there exist classes of mathematical systemic risk models that provide a degree of realism, but at the same time are sufficiently tractable that all critical parameters can be varied at will and resulting network characteristics computed? Can these model systems be tested for their systemic susceptibility? Are the mathematical conclusions robust and relevant to the real world of financial crisis regulation? We hope this book will be viewed as providing an emphatic "YES" in answer to these questions.

1.2 What is Systemic Risk?

First it is helpful to identify what systemic risk is not. Duffie and Singleton [32] identify five categories of risk faced by financial institutions: (i) market risk: the risk of unexpected changes in market prices; (ii) credit risk: the risk of changes in value due to unexpected changes in credit quality, in particular if a counterparty defaults on one of their contractual obligations; (iii) liquidity risk: the risk that costs of adjusting financial positions may increase substantially; (iv) operational risk: the risk that fraud, errors or other operational failures lead to loss in value; (v) systemic risk: the risk of market wide illiquidity or chain reaction defaults. To the extent that the first four risk categories are focussed on individual institutions, they are not deemed to be systemic risk. However, each of the four also has market wide implications: such market wide implications are wrapped up into the fifth category, systemic risk.

Kaufman and Scott [59], John B. Taylor [79] and others all seem to agree that the concept of systemic risk must comprise at least three ingredients. First, a triggering event. Second, the propagation of shocks through the financial system. And third, significant impact of the crisis on the macroeconomy. Possible triggers might come from outside the financial system, for example a terrorist attack that physically harms the system. Or triggers might come internally, such as the surprise spontaneous failure of a major institution within the system. Propagation of shocks may be through direct linkages between banks or indirectly, such as through the impact on the asset holdings of many banks caused by the forced sales of a few banks or through a crisis of confidence. The impact of systemic crises on the macroeconomy may take many forms: on the money supply, on the supply of credit, on major market indices, on interest rates, and ultimately on the production economy and the level of employment.

As Admati and Hellwig [3] have argued, ambiguity in the definition of systemic risk implies that mitigation of systemic risk might mean different things to different people. One approach might seek to reduce impact on the financial system, whereas a different approach might instead try to mitigate the damage to the economy at large. These aims do not necessarily coincide: the demise of Lehman Bros. illustrates that key components of the financial system might be sacrificed to save the larger economy during a severe crisis. It is therefore important to have an unambiguous definition of systemic risk supported by a widespread consensus.

1.2.1 Defining SR

The economics literature has used the term *systemic risk* in the context of financial systems for many years. Nonetheless, Kaufman and Scott, Taylor and many others argue that there is as yet no generally accepted definition of the concept, and furthermore, that without an agreed definition, it may be pointless and indeed dangerous to implement public policy that explicitly aims to reduce systemic risk. To see that there is as yet no consensus definition over the years, consider the following examples of definitions proposed in the past.

1. Mishkin 1995 [66]: “the likelihood of a sudden, usually unexpected, event that disrupts information in financial markets, making them unable to effectively channel funds to those parties with the most productive investment opportunities.”
2. Kaufman 1995 [58] “The probability that cumulative losses will accrue from an event that sets in motion a series of successive losses along a chain of institutions or markets comprising a system.... That is, systemic risk is the risk of a chain reaction of falling interconnected dominos.”
3. Bank for International Settlements 1994 [39] “ the risk that the failure of a participant to meet its contractual obligations may in turn cause other participants to default with a chain reaction leading to broader financial difficulties.”
4. Board of Governors of the Federal Reserve System 2001 [73] “In the payments system, systemic risk may occur if an institution participating on a private large- dollar payments network were unable or unwilling to settle its net debt position. If such a settlement failure occurred, the institution’s creditors on the network might also be unable to settle their commitments. Serious repercussions could, as a result, spread to other participants in the private network, to other depository institutions not participating in the network, and to the nonfinancial economy generally.”

In the light of the 2007-08 financial crisis, the above style of definitions, deficient as they are in several respects, can be seen to miss or be vague about one key attribute of any systemic crisis, namely that it also causes damage outside the network, through its failure to efficiently perform its key function of providing liquidity, credit and services. S. L. Schwarcz’ definition [75] of systemic risk explicitly includes this important aspect:

Systemic risk: a definition The risk that (i) an economic shock such as market or institutional failure triggers (through a panic or otherwise) either (X) the failure of a chain of

> markets or institutions or (Y) a chain of significant losses to financial institutions, (ii) resulting in increases in the cost of capital or decreases in its availability, often evidenced by substantial financial-market price volatility.

While the Schwarcz definition is hardly elegant in its phrasing, it has received support from a rather broad range of practitioners. We will therefore accept it as the closest thing we have to a concise definition of the spirit of systemic risk.

If this definition captures much of the spirit of systemic risk, it fails to address how to measure or quantify the level of systemic risk, and how it might be distributed over the network. Much of current research on systemic risk is dedicated to defining measures of systemic risk and identifying where it is concentrated. Some of the important concepts are *counterparty value at risk (CoVaR)* introduced by Brunnermeier and Pedersen [20]; and *systemic expected shortfall* introduced by Acharya, Pedersen, Philippon, and Richardson [2]. For a recent and comprehensive review of these and many other systemic risk measures, please see [12].

1.2.2 Haldane's 2009 Speech

In 2009, in the aftermath of the crisis, Andrew G. Haldane, Executive Director of Financial Stability at the Bank of England, gave a provocative and visionary talk, entitled "Rethinking the Financial Network" [46]. In this brilliant summary of the nature of networks, he compares the 2002 SARS epidemic to the 2008 collapse of Lehman Bros, with the aim to inspire efforts to better understand the nature of systemic risk. For a very broad free thinking overview, we can't do better than summarize the high points of his speech.

In these two examples of contagion events he identifies the following pattern:

- an external event strikes;
- panic ensues and the complex system seizes up;
- *collateral damage* is wide and deep;
- in hindsight, the trigger event was modest;
- during the event itself, dynamics was chaotic.

He claims this type of pattern is a manifestation of any complex adaptive system, and should be the target where we need to direct our attention.

So, in more detail, what went wrong with the financial network in 2008? Haldane identifies two contributing trends: increasing complexity and decreasing diversity. In real world networks these two trends are observed to lead to fragility, and ring alarm bells for ecologists, engineers, geologists. Figure 1.1 illustrates how the global financial network has grown in complexity. Highly connected, heterogeneous networks may be *robust yet fragile*, by which he means that they may be resistant to average or typical shocks, yet highly susceptible to an attack that targets a highly connected or dominant node. In such networks, connections that we think of as shock absorbers may turn out to act as shock amplifiers during a crisis. There may be a sharp *tipping point* that separates normal behaviour from a crisis regime. Thus, a network with a fat-tailed *degree distribution* (i.e. where there is a significant number of highly connected nodes) may be robust to random shocks while vulnerable to shocks that preferentially target these highly connected nodes.

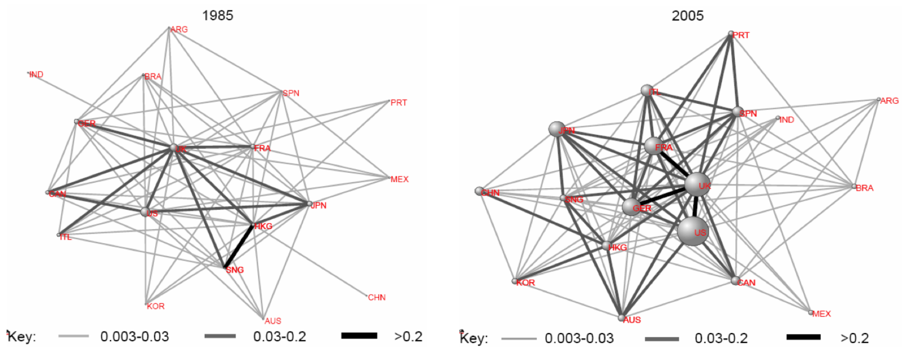

Fig. 1.1 The global financial network in 1985 (left) and 2005 (right). Here line thickness denotes link strength as fraction of total GDP. (figure taken from Haldane [46].)

In both of Haldane's examples of contagion events, agents exhibit a variety of behavioural responses that create feedback and influence the stability of the network. In epidemics, two classic responses, "hide" or "flee", may prevail and the virulence of the event is highly dependent on which behaviour dominates. In a financial crisis, two likely responses of banks are to hoard liquidity or to sell assets. Both responses are rational, but both make the systemic problem worse. Massive government intervention to provide liquidity and restore capital to banks in a timely manner may be needed in order to curtail systemic events.

Financial networks generate chains of claims and at times of stress, these chains can amplify uncertainties about true counterparty exposures. In good times, counterparty risk is known to be small, and thus "Knightian" uncertainty[3] is small, and in such times we might expect that stability will improve with connectivity. In bad times, counterparty risk can be large and highly uncertain, due to the complicated web and the nature of the links: we then expect stability to decline with connectivity. Financial innovation, particularly *securitization*, created additional instability. As CDOs, MBSs, RMBSs and similar high dimensional products proliferated internationally, they dramatically expanded the size and scope of the precrisis bubble (see [76]). The structure of these contracts was opaque not transparent. They dramatically increased the connectedness and complexity of the network, and moreover adverse selection made them hard to evaluate. As Haldane wrote:

[3] In *Knightian* terms, *uncertainty* describes modelling situations where probabilities cannot plausibly be assigned to outcomes. On the other hand, *risk* describes situations where uncertainty can be adequately captured in a probability distribution.

> Haldane 2009 [46]: “With no time to read the small-print, the instruments were instead devoured whole. Food poisoning and a lengthy loss of appetite have been the predictable consequences. ”

In ecosystems, many instances have been observed that show that biodiversity tends to improve stability. On the other hand, Haldane argues that during the *Great Moderation* prior to 2007, financial diversity has been reduced. Pursuit of returns led many major players, including global banks, insurance companies and hedge funds, to follow similar strategies leading to averaged portfolio correlations in excess of 90% during 2004-2007. Moreover, risk management regulation following Basel II led to similar risk management strategies for banks. As a result of such trends, bank balance sheets became increasingly homogeneous. Finance became almost a monoculture, in which all banks became vulnerable to infection by the same *virus*.

What one learns from Haldane’s analysis is that networks arising in ecology, engineering, the internet, and in finance, are complex and adaptive. Such networks are in a sense *robust yet fragile*. He asks “what properties of the financial network most influence stability?” and expresses the hope that the key determinants for financial stability can be deduced from studies of other types of networks.

1.2.3 A Lesson from Network Science: The Sandpile Model

Is there more specific guidance to understanding systemic risk that comes from other branches of the science of complex adapted systems? Consider the following thought experiment, first proposed by Bak, Tang and Wiesenfeld [9]. A very slow trickle of sand is allowed to fall in the middle of a large circular table. How do we expect the system to evolve? The growing accumulation of sand forms a pile on the table and our common experience tells us that the steepness of the pile cannot exceed a certain critical slope that depends on the microscopic and statistical properties of the sand. As more sand is added, the sandpile, still near its critical slope, eventually expands to cover the entire surface of the table. Having reached this maximal extent, the properties of the system take on a new character. On average, as sand is added near the centre of the table, an equal amount of sand must fall off the edge.

The interesting thing is the nature of the likelihood of n grains falling off, for each single grain added. BTW’s assertion, vindicated since by experiments, is that the frequency for between N and $2N$ grains to fall is roughly the same as for $2N$ to $4N$ grains. In other words, it is a power law or scale-invariant distribution similar to the Gutenberg-Richter frequency law for earthquakes, that carries the implication that disturbances of unbounded size can occasionally be triggered by a very small event. They coined the term *self-organized criticality*, or “SOC”, for this type of phenomenon, and boldly asserted that large scale driven systems have an innate tendency to build into a steady state that exhibits power law statistics that are *universal*, or insensitive to the microscopic details of the system.

Self-organized criticality has also been invoked to explain the widespread observation of fat-tailed *Pareto distributions* in economic contexts, such as the size of cities, the distribution of wealth, and the distribution of firm sizes (see[26]). Network scientists are thus not surprised to see results of [78], [27] and others that show evidence of Pareto tails in the size and connectivity of large financial networks, with highly connected *hub* banks that form a core within a periphery of intermediate and small banks.

It might sound naive to assert that something analogous to sand piling is happening in financial systems. However, as Minsky wrote in [65], “Stability–even of an expansion–is destabilizing in that more adventuresome financing of investment pays off to the leaders, and others follow.” Perhaps the financial system is like a sand pile near its maximal size, where unbounded disturbances are possible. The *Minsky moment* when a financial bubble bursts might then be analogous to one of these large scale disturbances. Adrian and Shin [5] provide a possible explanation. They demonstrate that in the 10 year period leading up to the 2007-08 crisis, financial institutions exhibited strongly pro cyclical investment strategies: as asset prices rose during the prolonged period of stability, so did the balance sheets and leverage ratios of banks, showing that they pursued ever more adventurous strategies. Eventually, as the financial system approached a critical state with little government oversight, only small triggers were needed to create the inevitable collapse.

1.3 Capital Structure of a Bank

Banks around the globe form a diverse family of firms, spanning a huge range of sizes and types. In addition to traditional retail and investment banks, financial network models need eventually to include a whole host of shadow banking institutions, including hedge funds, pension and investment funds, savings and credit institutions, and so on. As our systemic models evolve, we will include in our system more and more components of the wider production and retail economy. It will clearly be impossible to capture here in a brief overview the full range of holdings and liabilities that such institutions might have, and that should in principle be understood.

Quarterly financial reports of a firm’s balance sheet offer a snapshot at a moment in time of the details of a firm’s capital structure, that is, the valuation of the totality of their assets and liabilities. It is helpful to imagine these balance sheet entries as existing at all times, even if not observed by the market. One can imagine that the bank management maintains its accounting books, updating them daily, and only once a quarter makes them available to the public. Regulators, on the other hand, have the power to examine the books of any bank, at any moment.

Figure 1.3 shows the main classes of assets (what the bank owns) and liabilities (what the bank owes). All entries must be non-negative, and *equity*, which is the value of the firm to the share owners, is defined to be the difference $E = A - L$ between asset value A and liability value L. The most fundamental characteristic of a bank's balance sheet is its accounting ratio, *leverage* A/E. For banks, it has often exceeded 50 in the past. Its reciprocal, E/A, measures the bank's safety buffer to absorb adverse balance sheet shocks.

Assets	Liabilities and Equity
Loan portfolio: mortgages commercial loans credit cards	Deposits: retail deposits wholesale deposits certificates of deposit other banks' deposits
OTC Securities: bonds OTC derivatives	OTC Securities: bond issues OTC derivatives
Market Securities exchange traded derivatives	Market Securities exchange traded derivatives
Reverse Repos	Repos
Cash cash equivalents	Hybrid Capital preferred shares COCOs
Other Assets	Other Liabilities
	Equity

Table 1.1 The main components of a bank's balance sheet.

In studies of the "financial system", it is important to carefully define the boundary between the interior and exterior of the system. Correspondingly, for systemic analysis, assets and liabilities will always be separated into intra- and extra-network components. One important insight to keep in mind when considering capital structure is the formal duality between assets and liabilities. Almost any asset is someone else's liability and in many cases where a story can be told of asset side contagion, an analogous story can be told of liability side contagion.

1.3.1 Bank Asset Classes

Assets, what the firm owns, are entered on the left side of the balance sheet. The available classes of assets relevant in banking is extremely diverse, and this section gives only a schematic overview of some of the most important basic types. From a systemic risk perspective, assets' most relevant characteristics are: duration or maturity, credit quality (or collateral), interest rate and liquidity.

Loan portfolio: Banks, like any firm, invest heavily in endeavours for which they have a competitive advantage. Such *irreversible projects* are by their nature illiquid, and fail to recoup their full value when sold. For banks, this business line is called the *bank book*, and consists of a heterogeneous portfolio of loans and mortgages of all maturities, to counterparties ranging across the retail sector, small and medium enterprises (SMEs) and major corporates. Far from receding in importance since the crisis, [57] shows that real estate lending in the form of mortgages in particular accounts for an ever increasing percentage of bank assets. They comment: “Mortgage credit has risen dramatically as a share of banks’ balance sheets from about one third at the beginning of the 20th century to about two thirds today” and suggest that as in the past, real estate bubbles will continue to be the dominant factor in triggering future systemic risk events.

From an accounting perspective, loans are typically regarded as to be “held to maturity”, and are therefore assigned *book value*, which is typically the value at the time of origination. The book value may be marked down infrequently, if the asset is regarded as sufficiently stressed.

As mentioned above, in systemic risk analysis, assets placed within the financial system, called *interbank assets* are always distinguished from *external assets* placed outside the system.

Over-the-counter securities: Bonds, derivatives and swap contracts between banks are to a large extent negotiated and transacted in the OTC markets, sometimes bilaterally, but increasingly within a *central clearing party* (CCP) . Some of these exposures fluctuate rapidly in time, both in magnitude and in sign, and may or may not be collateralized by margin accounts to reduce counterparty risk. Between a pair of financial counterparties there may be many contracts, each with positive and negative exposures. To reduce risk, large counterparties often negotiate a bilateral *master netting agreement* (MNA), subject to the regulations stipulated by ISDA, that allows them to offset exposures of opposite signs. Entering into an MNA is a costly endeavour, and thus existence of an MNA is an indication of a strong network connection between two banks. Counterparty risk management, reviewed for example in [28], is a particular flavour of credit risk management that has developed rapidly since the crisis. As part of this methodology, banks now routinely forecast the *potential future exposure*, or PFE, for all their important counterparties. This is a high quantile of the positive part of their exposure to the given counterparty on a given date in the near future. In the event that one bank suddenly defaults, PFE is a pessimistic estimate of losses to which its counterparties are exposed.

A systemically important subclass of OTC securities are *total return swaps* (TRS) that exchange the random returns on an underlying asset for a fixed periodic payment. An example of a TRS is the credit default swap (CDS) that exchanges fixed quarterly payments for a large payment at the moment the underlier defaults, providing a form of default insurance. From a network perspective, complex and poorly understood effects are bound to arise when the underlier is itself part of the system, as would be the case of a CDS written on the default of another bank.

Cash and market securities: In addition to cash, the firm may hold other securities for which there is a liquid market, and low or moderate transaction costs. Such *cash equivalents* are used to pay depositors on demand. Examples include money-market lending that pays the over-night rate, stocks, T-bills, and exchange traded derivatives. Since cash equivalent assets are regarded as “available for sale”, instead of book value they are assigned their *mark-to-market* value. A new aspect of the

Basel III regulatory framework requires banks to exercise active and prudent liquidity management which means that a fraction of assets must be held in a portfolio of cash and market securities that can be easily liquidated when the bank needs to meet its short term debt obligations in a timely manner.

Reverse repo assets: As described in the next section on repos, a repo-lending bank receives collateral assets known as reverse repos. Such assets can be "rehypothecated", which means they can themselves be used as collateral for repo borrowing.

Other assets: Of lesser importance are a range of further asset classes: real estate, accounts receivable, goodwill and the like.

1.3.2 Debt and Liabilities

Debt and liability are regarded as the same thing, namely all obligations that the firm owes to others. Equity, on the other hand, even if entered on the right side of the balance sheet, is the value of firm ownership, and is *not* regarded as debt or liability.

Deposits: A large fraction of the debt of a traditional bank is in the form of deposits made by both institutional investors and small retail investors. Since there are many of them, with a diverse range of maturities, the collective of deposits can be thought of as a multiple of an asset that pays a constant dividend rate (for banks, we will assume it is less than the risk free rate). One important class of wholesale depositor is short-term money-market funds. Their widespread meltdown in early stages of the last financial crisis played an important contagion role. Small depositors are typically protected by deposit insurance in the event of the bank's default, while institutional depositors have no such protection. Banks in particular seek protection through collateralization. Uncollateralized lending between banks takes other forms: certificates of deposit and bankers' acceptances are variants banks use to lend to each other.

Bonds: Like most large firms, banks issue bonds as a primary means to raise long term debt capital, each bond being characterized by its notional amount, maturity and coupon rate, plus a variety of additional attributes. Sometimes a bank's bonds differ in seniority, meaning that in the event of default, the most senior bonds are paid in full before junior bonds. Typically, the firm cannot retire existing bonds or issue new bonds quickly.

Market securities Hedge funds and investment banks have the characteristic that they often take large short positions in market securities, such as stocks and derivatives. To a lesser extent, commercial banks also hold short positions for hedging and risk management reasons. Short positions can be thought of as holding negative amounts in market securities.

Collateralized Loans (Repos): Short for *repurchase agreements*, repos are an important class of collateralized short term debt issued between banks and other institutional investors. Typically, money is borrowed for a short term, often overnight, at an interest rate r called the *repo rate*. They are backed by assets (*repo assets*) whose value exceeds the loan amount by a percentage h called the *haircut*. The haircut reflects the liquidation value of the collateral in the event the money is not repaid. This haircut thus compensates the lender, also called the asset buyer, for the counterparty risk inherent in the contract when the borrower (or asset seller) defaults. To illustrate how repos are used, suppose a bank has an excess \$1 in cash. Then they might undertake an overnight repo of \$1 with another bank for collateral valued at $\$1/(1-h)$. The next day the contract is closed by *repurchase* of the collateral for the price $\$(1+r)$. While the lending bank holds the collateral, they may also elect to use it to finance a second repo with another counterparty: we then say the collateral has been *rehypothecated*.

Hybrid capital: This term denotes parts of a firm's funding that possess both equity and debt features. Preferred shares can be considered as hybrid capital: Like equity, it may pay tax-deductible dividends, and like debt it maintains seniority over equity in the event of default. There is now active interest in forms of hybrid capital issued by banks such as *contingent convertible bonds* (COCOs) that behave like bonds as long as the firm is healthy, but provide additional equity cushioning for the bank when its balance sheets weaken.

Other Liabilities: Accounts payable are analogous to accounts receivable. Investment banks act as prime brokers and hold their clients' securities in trust. They often act on the right to borrow such securities to be used as collateral for purchasing further assets. Such a debt can be understood as similar to a collateralized loan.

1.3.3 Equity

Equity, defined to be the value of a firm's assets minus its liabilities, what it owns minus what it owes, represents the total value conveyed by ownership of the firm. For a publicly owned firm, ownership is divided into shares that have an observable fluctuating market price: in this case, total market capitalization, the share price times the number of shares outstanding, is a market value of equity. For privately held firms, equity does not have a transparent market value: valuation of privately held firms can only be gleaned through investigation of the firm's accounts. Firms, especially banks, are typically highly leveraged, meaning A/E is large. In such situations, equity, being a small difference of large positive numbers, is inherently difficult to estimate and this uncertainty is reflected in the high volatility of the stock price.

Limited liability is the principle, applying to almost all publicly held companies, that share owners are never required to make additional payments. In the event the firm ceases to operate, the shareholders are not held responsible for unpaid liabilities. We can say this means equity can never be negative, and is zero at the moment the firm ceases to operate. This is called bankruptcy, and limited liability is a central principle in bankruptcy law worldwide.

Firms return profits to their shareholders two ways, either through the payment of regular dividends, or through the increase in share price when the value of firm equity rises. Share issuance and its opposite, the share buyback, are two further financing strategies firms can adopt. A firm near to default may attempt a share issuance, hoping that new investors will view the decline in its fortunes as a temporary effect before the firm's return to health.

1.4 Channels of Systemic Risk

Systemic contagion that causes the failure or impairment of a large number of banks will in reality always manifest itself through a multitude of different channels, with spillover or domino effects from one to another. In the language of network science, financial networks are multiplex, meaning there are interbank links of many different types, and a contagious event that starts with one type of link will likely quickly infect all other types of links. Nonetheless, it is important to identify the basic types of shock mechanisms that we expect to find activated during a financial crisis, either as the primary cause, or else as the result of spillover effects stemming from the initial shock. For an in-depth discussion of various channels of systemic risk, and in particular, contagion, please see [29].

Asset Correlation: Different banks tend to hold common assets in their portfolios. Haldane [46] has argued that banks' asset portfolios became increasingly similar during the Great Moderation, making them more and more susceptible to correlated asset shocks that can be considered as a channel of systemic risk. In 2007, most large banks around the world held significant positions in the US sub-prime mortgage market. The prolonged drawdown of US housing prices in that year acted as a huge downward asset shock that dramatically increased most banks' leverage and hence the vulnerability of their asset portfolios. Such systemic events undermine the health of the system, in the same way that famine impairs the health of a community. They make it vulnerable to other types of contagion, but do not exhibit the amplification effect that characterizes contagion.

Default Contagion: Bank deposits held in other banks can be considered as a form of interbank lending, but banking practise in modern times has dramatically extended the range of interbank exposures. There are now a multitude of linkage types between bank counterparties that range well beyond traditional interbank lending, to include swaps, derivatives and other securitized assets. At any moment, banks can at least in principle identify their exposures to all other banks and they work hard to identify their expected potential exposure (EPE) over different future time horizons. An insolvent bank, if it is not bailed out by a government agency, will be forced into bankruptcy thereby disrupting promised contractual payments. Its creditors, including other banks, will then experience severe losses given this default, possibly losing close to 100% of their total exposure in the short term aftermath. Such shocks to creditor banks' interbank assets at the time of default of a debtor bank are the channel for *default contagion*. Such shocks can in principle chain together like dominos to create a *default cascade*. Default cascades can only happen when interbank exposures are a high fraction of lending banks' equity, and [80] provides evidence that this was the case in Europe before and during the crisis, when many banks' interbank exposures exceeded their capital by factors of 5 or more. Despite being the first type of contagion most economists consider, default cascades are rare in practise because defaulting banks are often bailed out by government.

Liquidity Contagion: *Funding illiquidity* is the situation of a bank with insufficient access to short term borrowing. Such banks, being short of cash or other liquid assets, will adopt a variety of strategies that can be considered as shrinking their balance sheets. They will try to access the repo markets for untapped sources of collateralized borrowing. They will refuse to rollover short term loans and repo lending to other counterparties. The amplification characteristic of contagion occurs when banks respond to funding illiquidity by curtailing a large fraction of their interbank lending. The resulting funding shocks to other banks are the channel for *liquidity contagion* in the system.

Market Illiquidity and Asset Fire Sales: As [5] discussed, in good times banks tend to create upward asset price spirals by increasing their leverage through large scale asset purchasing. This pushes up prices, creating the illusion of even better times ahead and further increases in leverage. As they also discuss, the reverse is true in bad times. This tendency for distressed banks to sell assets into a depressed market creates the contagion mechanism known as an *asset fire sale*. A fire sale cascade proceeds through a double step mechanism: first, asset sales by distressed banks decreases prices, then marking-to-market leads to losses by other banks holding these assets.

Other Channels: Many authors have identified further channels for systemic risk. *Rollover risk* is the name given to the effect that lenders to a bank may fail to renew or "rollover" short term debt. [69] models this effect as a coordination game played by the lenders to a single counterparty: a player that perceives that other players are likely not to roll over their debt, will be more likely not to roll over their debt. Such a decision may be due either to a lending bank's assessment of the health of the counterparty (which was termed *structural uncertainty*), or to that bank's assessment of the lending behaviour of other banks (termed *strategic uncertainty*). [8] extend this picture to a network setting by considering a multitude of simultaneous coordination games, leading to runs in the interbank network. In [44], it is argued that the 2008 crisis was largely a crisis of confidence in the repo market that led to a drying up of banks' funding opportunities. In normal times, the repo market provides a huge source of short term funding for banks that is *information insensitive* in the sense that the lender has little incentive to be concerned about the health of its counterparty. During the crisis however, lenders became *information sensitive* and questioned the quality of counterparties and their underlying collateral. Consequently, they began to demand large increases in repo *haircuts*. In other words, they demanded collateral valued well above the loan amounts, and in consequence dramatically decreased the availability of repo funding at a time it was most needed. This effect was contagious: banks that raised haircuts imposed funding illiquidity on their counterparties, leading to further questioning of the quality of counterparties and their collateral.

1.5 Regulatory Capital and Constraints

The 1988 Basel Accord, also called Basel I, was a set of minimal capital requirements for banks that arose from the deliberations of international central bankers who formed the Basel Committee on Banking Supervision. Largely as a result of lessons hard learned during the 07-08 crisis, the world is now moving quickly beyond the Basel II regulatory regime, put in place in the early 2000s, that can be characterized as *microprudential* in emphasis, with regulations that were imposed bank by bank without taking into account the dependence of risks between banks. For example, the *capital adequacy ratio* (CAR) which stipulates

$$\text{Risk-weighted Assets} \leq 12.5 \times \text{Total Capital}$$

is the main requirement of Basel II, and is based only on the individual bank's balance sheets. The importance of network effects is now recognized at the core of Basel III in measures that are *macroprudential* in nature, meaning they try to account for the network and the interconnectivity of risks between banks.

An example of macroprudential regulation is the new requirement by Basel III that banks must report their large exposures to individual counterparties or groups of counterparties, both financial and non-financial. This is clear recognition that large interbank linkages are systemically important during a crisis. It has also become clear that the fixed regulatory capital ratios of Basel II were *procyclical* and can dangerously amplify the swings of the credit cycle. When the financial system enters a contraction phase, capital buffers of some banks will be squeezed below the regulatory minimum, leading naturally to fire sales and further asset shocks. Basel III seeks to ward off this tendency by making the capital requirements countercyclical. During normal periods, capital requirements have a surcharge which can be removed as the system begins to contract to provide banks with more flexibility. Yet another example of macroprudential regulation is that Basel III now recognizes the existence of *SIFI*s (for *systemically important financial institutions*), also called *G-SIB*s (for *global systemically important banks*), and subjects them to a regulatory capital surcharge that will hopefully make them more resilient. Clearly, the identification of SIFIs must be grounded in well established systemic risk theory, and the SIFIs themselves will demand a theoretical basis for what is to them a punitive measure.

The Basel II capital adequacy ratio, although it has been strengthened in Basel III, still leads to distortions of banking balance sheets through its use of *risk weights*.

For example, the risk weight for sovereign bonds of OECD countries remains zero, meaning these assets require no offsetting equity capital, allowing banks to operate with unsustainably high leverage ratios. Basel III provides a counterbalance to the CAR by requiring an additional constraint on bank leverage. Liquidity risk is for the first time explicitly addressed in Basel III through the implementation of two new regulatory ratios. The *Liquidity Coverage Ratio* (LCR) ensures that every bank's liquid assets will be sufficient to cover an extreme stress scenario that includes a 30 day run off of its short-term liabilities. The *Net Stable Funding Ratio* (NSFR) similarly seeks to ensure that enough long term (greater than one year) funding will be available to cover a stress scenario that hits the bank's long term assets. Considering the Lucas critique [62](p. 41) that "any change in policy will systematically alter the structure of econometric models", we must expect that the systemic ramifications of the LCR and NSFR will be subtle and far-reaching.

Critics, notably Haldane [47], argue that the Basel III regulatory framework has become excessively complex, and that the regulatory community must do an about-turn in strategy and operate by a smaller, less-detailed rulebook. Haldane writes: "As you do not fight fire with fire, you do not fight complexity with complexity. Because complexity generates uncertainty, not risk, it requires a regulatory response grounded in simplicity, not complexity."

Our task now is to begin exploring the channels of cascading systemic risks in detail, with an aim to closing the gap in understanding that exists between knowing bank-to-bank interactions and knowing how a large ensemble of banks will behave.

Chapter 2
Static Cascade Models

Happy families are all alike; every unhappy family is unhappy in its own way. [1]

Abstract Network effects such as default contagion and liquidity hoarding are transmitted between banks by direct contact through their interbank exposures. During asset fire sales, shocks are transmitted indirectly from a bank selling assets to other banks via the impact on the price of their common assets. Banks maintain safety buffers in normal times, but these may be weakened or fail during a crisis. Asset prices that are relatively stable in normal times may collapse during a crisis. Banks react to such stresses by making large adjustments to their balance sheets. Such adjustments send further shocks to their counterparties both directly through their exposures and indirectly via asset price impact, creating a cascade. All these cascade mechanisms can be modelled mathematically starting from a common framework. In such models, the eventual extent of a crisis is a fixed point or equilibrium of a cascade mapping. Towards the end of the chapter, a proposal is made that the properties of cascade mappings can be most clearly understood when implemented on very large random financial networks.

Keywords: Cascade mechanism, default and liquidity buffers, fixed point equations, cascade equilibrium, asset fire sales, random financial network.

If one takes the Anna Karenina Principle seriously, one imagines that stable banking systems must all be alike, while every type of financial instability has its own characteristics. This chapter will explore some of the different ways financial instability can propagate through the system. Paradoxically, we will find that while such channels are definitely distinct, they retain common features that we can exploit in the mathematical models developed in this book.

Contagion, meaning the transmission of a disease by direct or indirect contact, is an appropriate term for the damaging effects that can be transmitted through the network of banks (and possibly other financial entities) linked by the contracts they

[1] Leo Tolstoy, *Anna Karenina*, Part 1, Chapter 1. First published 1874–1877.

exchange. This chapter will develop various mathematical frameworks for contagion, or cascading instability, both direct and indirect, that can arise in hypothetical financial networks. The essential picture will be to treat banks and their behaviour under stress as determined by their balance sheets, and to concentrate on damaging shocks that can be transmitted through interbank links. In addition to direct bank-to-bank effects, we will find that indirect effects can also be included by extending the definition of node to include non-banks.

The cascade models of this chapter all follow a common script:

1. At the onset of the crisis, the system that was previously in a normal or quiescent state is hit by a damaging shock. This shock impacts banks' balance sheets sufficiently hard that one or more fail or become stressed;
2. Failed or stressed banks transmit balance sheet shocks to their network counterparties;
3. Thereafter, the network undergoes a sequence of updates as banks respond to the balance sheet shocks they receive, thereby inflicting further shocks to their counterparties.

The precise form of updating, called the *cascade mechanism*, amounts to a set of behavioural rules that banks are assumed to follow. Each updating step can be thought of as leading to a *cascade mapping* of the system state into its new state. In our models, the cascade mapping is always monotonically increasing in the damage it causes, which is sufficient to guarantee that its iterations converge to a fixed point called the *cascade equilibrium*. The total damage inflicted during the crisis on both the financial system and the economy at large is determined from quantitative risk measures computed in the equilibrium.

This script is sufficiently general to cover a wide variety of economic narratives. For example, the stylized facts of the 2007-2009 US financial crisis can be mapped schematically to this script: (1) prior to the active phase of the crisis, the year long collapse of the US real estate market acted as a non contagious *correlated asset shock* that brought the whole financial system to an unhealthy, susceptible state; (2) the September 2008 collapse of Lehman Bros. provided the trigger for the contagious phase of the crisis; (3) during the subsequent months, rounds of the crisis lead to the defaults of other financial institutions, fire sales in the CDO markets, a freezing of the repo market, liquidity hoarding and other elements that can be viewed collectively as a cascade mechanism; (4) by Spring 2009, the contagion had slowed down, leaving the US financial system close to a new, more quiescent cascade equilibrium.

Obviously the cascade mechanism that underlay the most contagious period of this crisis was an extremely complex interweaving of different effects. In this chapter, we separate out three different contagion channels for mathematical study: default cascades, liquidity cascades and asset fire sales. Later, one can investigate more complex models of higher dimensional cascades that combine two or more different contagion transmission mechanisms, each with some of these basic characteristics.

The basic cascade models of this chapter concern a financial system assumed to consist of N "banks", labelled by $v \in \{1,2,\ldots,N\} := [N]$ (which may include non-regulated, non-deposit taking, leveraged institutions such as hedge funds, or other regulated financial institutions such as insurance companies). Their balance sheets can be characterized schematically as in Table 2.1.

Assets	Liabilities
interbank assets $\bar{\mathrm{Z}}$	interbank debt $\bar{\mathrm{X}}$
external fixed assets $\bar{\mathrm{Y}}^F$	external debt $\bar{\mathrm{D}}$
external liquid assets $\bar{\mathrm{Y}}^L$	equity $\bar{\mathrm{E}}$

Table 2.1 An over-simplified bank balance sheet.

At the outset, the entries in these banks' balance sheets refer to *nominal values* of assets and liabilities, and give the aggregated values of contracts, valued as if all banks are solvent. Nominal values, denoted by upper case letters with bars, can also be considered *book values* or *face values*. Assets and liabilities are also decomposed into interbank and external quantities depending on whether the loan or debt counterparty is a bank or not. Banks and institutions such as foreign banks that are not part of the system under analysis are deemed to be part of the exterior, and their exposures are included as part of the external debts and assets. It is also convenient to separate *fixed assets*, which comprises the assets such as the bank's loan portfolio that cannot be sold without high liquidation costs, from *liquid assets*, such as cash and cash equivalents.

Definition 1. The *nominal value of assets* of bank v at any time consists of *nominal external assets*, both fixed and liquid, denoted by $\bar{\mathrm{Y}}_v = \bar{\mathrm{Y}}_v^F + \bar{\mathrm{Y}}_v^L$, plus *nominal interbank assets* $\bar{\mathrm{Z}}_v$. The *nominal value of liabilities* of the bank includes *nominal external debt* $\bar{\mathrm{D}}_v$ and *nominal interbank debt* $\bar{\mathrm{X}}_v$. The bank's *nominal equity* is defined by $\bar{\mathrm{E}}_v = \bar{\mathrm{Y}}_v + \bar{\mathrm{Z}}_v - \bar{\mathrm{D}}_v - \bar{\mathrm{X}}_v$. The *nominal exposure* of bank w to bank v, that is the amount v owes w, is denoted by $\bar{\Omega}_{vw}$. We define interbank loan fractions to be $\bar{\Pi}_{vw} = \bar{\Omega}_{vw}/\bar{\mathrm{X}}_v$ as long as $\bar{\mathrm{X}}_v > 0$. Interbank assets and liabilities satisfy the constraints:

$$\bar{\mathrm{Z}}_v = \sum_w \bar{\Omega}_{wv}, \quad \bar{\mathrm{X}}_v = \sum_w \bar{\Omega}_{vw}, \quad \sum_v \bar{\mathrm{Z}}_v = \sum_v \bar{\mathrm{X}}_v, \quad \bar{\Omega}_{vv} = 0 .$$

The combined balance sheets of all banks in the network are shown in Table 2.2.

Such a schematic financial system can be pictured as a network of N nodes representing banks, connected by $E = |\{(v,w) : \Omega_{vw} > 0\}|$ directed edges that point from debtor banks to creditor banks (a direction we sometimes call "downstream").

Economic cascade models invoke the notion of limited liability, and define a firm to be defaulted when its mark-to-market equity is non-positive which means its aggregated assets are insufficient to pay its aggregated debt. Analogously, we regard a bank without liquid assets available to pay demand depositors as subject to *liquidity stress*.

Definition 2. A *defaulted bank* is a bank with $\mathrm{E} = 0$. A *solvent bank* is a bank with $\mathrm{E} > 0$. A *stressed bank* is a bank with $\mathrm{Y}^L = 0$.

	1	2	$\cdots$	N	$\bar{\mathrm{X}}$	$\bar{\mathrm{D}}$	$\bar{\mathrm{E}}$
1	0	$\bar{\Pi}_{12}\bar{\mathrm{X}}_1$	$\cdots$	$\bar{\Pi}_{1N}\bar{\mathrm{X}}_1$	$\bar{\mathrm{X}}_1$	$\bar{\mathrm{D}}_1$	$\bar{\mathrm{E}}_1$
2	$\bar{\Pi}_{21}\bar{\mathrm{X}}_2$	0	$\cdots$	$\bar{\Pi}_{2N}\bar{\mathrm{X}}_2$	$\bar{\mathrm{X}}_2$	$\bar{\mathrm{D}}_2$	$\bar{\mathrm{E}}_2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$\bar{\Pi}_{N1}\bar{\mathrm{X}}_N$	$\bar{\Pi}_{N2}\bar{\mathrm{X}}_N$	$\cdots$	0	$\bar{\mathrm{X}}_N$	$\bar{\mathrm{D}}_N$	$\bar{\mathrm{E}}_N$
$\bar{\mathrm{Z}}$	$\bar{\mathrm{Z}}_1$	$\bar{\mathrm{Z}}_2$	$\cdots$	$\bar{\mathrm{Z}}_N$			
$\bar{\mathrm{Y}}^F$	$\bar{\mathrm{Y}}_1^F$	$\bar{\mathrm{Y}}_2^F$	$\cdots$	$\bar{\mathrm{Y}}_N^F$			
$\bar{\mathrm{Y}}^L$	$\bar{\mathrm{Y}}_1^L$	$\bar{\mathrm{Y}}_2^L$	$\cdots$	$\bar{\mathrm{Y}}_N^L$			

Table 2.2 The first N rows of this table represent different banks' liabilities and the first N columns represent their assets. The matrix of interbank exposures contains the values $\bar{\Omega}_{vw} = \bar{\Pi}_{vw}\bar{\mathrm{X}}_v$.

When a bank v is known to be insolvent or defaulted, creditors of v will naturally mark down their exposure to less than their nominal values. Similarly, the response of a bank to liquidity stress on its liability side will naturally include reducing their interbank lending. We denote by symbols $\mathrm{Z}, \mathrm{Y}^F, \mathrm{Y}^L, \mathrm{X}, \mathrm{D}, \mathrm{E}, \Omega$ without upper bars, the changing *actual* or *mark-to-market* values of balance sheets that typically decrease during the steps of the cascade.

With these common definitions, we are now in a position to discuss models of the three basic contagion channels: default contagion, liquidity contagion and asset fire sales. These models will be called "static" because they describe cascades whose end result is a deterministic function of the initial balance sheets and exposures. In typical applications, static cascades proceed from a *random* initial configuration $\bar{\mathrm{Z}}, \bar{\mathrm{Y}}^F, \bar{\mathrm{Y}}^L, \bar{\mathrm{X}}, \bar{\mathrm{D}}, \bar{\mathrm{E}}, \bar{\Omega}$ through a *cascade mechanism* that generates a series of deterministic steps until a steady state or *cascade equilibrium* is reached. Of course as a deterministic function of a random variable, the cascade equilibrium is a random variable, and one is interested to compute a variety of *systemic risk measures* defined as certain expectations over the cascade equilibrium.

2.1 Default Cascades

Basic default cascades depend on $\bar{\mathrm{Y}}$ but not on $\bar{\mathrm{Y}}^F$ and $\bar{\mathrm{Y}}^L$ separately. In such models, the triggering event can be taken to be an initial shock that leaves some banks with nonpositive nominal equity $\bar{\mathrm{E}} = 0$ and therefore insolvent. As the cascade progresses, the market value of equity of all banks generally decreases, potentially leading to secondary defaulted banks. The relative claims by creditors in the event a debtor bank defaults are determined by the nominal amounts $\bar{\mathrm{Y}}, \bar{\mathrm{Z}}, \bar{\mathrm{D}}, \bar{\mathrm{X}}, \bar{\Omega}$. The rule by which defaulted claims are valued distinguishes the two different approaches we now examine.

2.1.1 The Eisenberg-Noe 2001 Model

A slightly extended version of the famous model of default contagion introduced by Eisenberg and Noe in[33] makes two additional assumptions that determine a precise default resolution mechanism:

Assumptions 1.

1. External debt is senior to interbank debt and all interbank debt is of equal seniority;
2. There are no losses due to bankruptcy charges.

The original model discussed only the case when external assets $\bar{\text{Y}}$ exceed external debt $\bar{\text{D}}$ for all banks. As discussed in [35], the justification for making this restriction was based on a faulty argument, and therefore we treat only the general case here.

In tandem with the limited liability assumption, the first assumption means that on default a bank's equity is valued at zero, and none of its interbank debt is paid before its external debt is paid in full. A variant of this assumption is if some or all of the external debt has the same seniority as interbank debt. This can be incorporated without additional modelling complexity by adding a fictitious bank labelled by $v = 0$ that lends to but does not borrow from other banks and can never default.

The no-bankruptcy costs assumption is somewhat optimistic in the context of systemic risk and has the strong consequence that when the system is viewed as a whole, no system-wide equity is lost during the crisis. That is, the system equity, defined as total assets minus total liabilities, is independent of the payments within the interbank sector:

$$\bar{\text{E}}_{sys} = \sum_v (\bar{\text{Y}}_v + \bar{\text{Z}}_v - \bar{\text{D}}_v - \bar{\text{X}}_v) = \sum_v (\bar{\text{Y}}_v - \bar{\text{D}}_v) \ .$$

Let us suppose the banking network, previously in equilibrium with no defaulted banks, experiences a catastrophic event, such as the discovery of a major fraud in a bank or a system wide event, whereby the nominal assets of some banks suddenly contract. If one or more banks are then found to be in a state of *primary default*, they are assumed to be quickly liquidated, and any proceeds go to pay off these banks' creditors, in order of seniority. We let $p_v^{(n)}, v \in [N]$ denote the (mark-to-market) amount available to pay v's *internal debt* at the end of the nth step of the cascade, and $\text{p}^{(n)} = [p_1^{(n)}, \ldots, p_N^{(n)}]'$. Similarly, we let $q_v^{(n)}, v \in [N]$ denote the (mark-to-market) amount available to pay v's *total debt* and $\text{q}^{(n)} = [q_1^{(n)}, \ldots, q_N^{(n)}]'$. We draw the reader's attention to the vector and matrix notation described in Appendix A.1.

By Assumption 1, the value $p_v^{(n)}$ is split amongst the creditor banks of v in proportion to the fractions $\bar{\Pi}_{vw} = \bar{\Omega}_{vw}/\bar{\text{X}}_v$ (when $\bar{\text{X}}_v = 0$, we define $\bar{\Pi}_{vw} = 0, w \in [N]$). Therefore, at step $n \geq 1$ of the cascade, every bank w values its interbank assets as $\text{Z}_w^{(n)} = \sum_v \bar{\Pi}_{vw} p_v^{(n-1)}$. Since by assumption there are no bankruptcy charges, we find for all $v \in [N]$:

$$q_v^{(n)} = \min(\bar{\mathrm{Y}}_v + \sum_w \bar{\Pi}_{wv} p_w^{(n-1)}, \bar{\mathrm{D}}_v + \bar{\mathrm{X}}_v) \,, \tag{2.1}$$

$$p_v^{(n)} = (q_v^{(n)} - \bar{\mathrm{D}}_v)^+ \,. \tag{2.2}$$

This can be written compactly in terms of p alone:

$$\mathrm{p}^{(\mathrm{n})} = \mathrm{F}^{(EN)}(\mathrm{p}^{(\mathrm{n-1})}) \tag{2.3}$$

where $\mathrm{F}^{(EN)} = [F_1^{(EN)}, \ldots, F_N^{(EN)}]'$ with

$$F_v^{(EN)}(\mathrm{p}) = \min(\bar{\mathrm{X}}_v, \max(\bar{\mathrm{Y}}_v + \sum_w \bar{\Pi}_{wv} p_w - \bar{\mathrm{D}}_v, 0)) \,. \tag{2.4}$$

Now we let $\mathrm{p} = [p_1, \ldots, p_N]'$ denote the vector of banks' internal debt values at the end of the cascade. This *clearing vector* satisfies the *clearing condition* or *fixed point condition*

$$\mathrm{p} = \mathrm{F}^{(EN)}(\mathrm{p}) \;:=\; \min(\bar{\mathrm{X}}, \max(\bar{\mathrm{Y}} + \bar{\Pi}' \cdot \mathrm{p} - \bar{\mathrm{D}}, 0)) \,. \tag{2.5}$$

The main theorem of Eisenberg and Noe is the following:

Theorem 1. *Corresponding to every financial system* $(\bar{\mathrm{Y}}, \bar{\mathrm{Z}}, \bar{\mathrm{D}}, \bar{\mathrm{X}}, \bar{\Omega})$ *satisfying Assumption 1 there exists a greatest and a least clearing vector* p^+ *and* p^-.

Proof of Theorem 1: The result follows immediately from the Knaster-Tarski Fixed Point Theorem[2] once we verify certain characteristics of the mapping $\mathrm{F}^{(EN)}$. We note that $\mathrm{F}^{(EN)}$ maps the hyperinterval $[\mathbf{0}, \bar{\mathrm{X}}] := \{x \in \mathbb{R}^N : 0 \le x_v \le \bar{\mathrm{X}}_v\}$ into itself. We also note that it is monotonic: $x \le y$ implies $\mathrm{F}^{(EN)}(x) \le \mathrm{F}^{(EN)}(y)$. Finally, note that $[\mathbf{0}, \bar{\mathrm{X}}]$ is a complete lattice. We therefore conclude that the set of clearing vectors, being the fixed points of the mapping $\mathrm{F}^{(EN)}$, is a complete lattice, hence nonempty, and with maximum and minimum elements p^+ and p^-. □

Finding natural necessary and sufficient conditions on E-N networks to ensure the uniqueness of the clearing vector proves to be more challenging. Figure 2.1 shows an example of non-uniqueness of the clearing vector.

We present here a complete characterization of the clearing vector set in the E-N model without the condition $\bar{\mathrm{Y}} \ge \bar{\mathrm{D}}$. First, we identify groups of banks called *in-subgraphs* (these are essentially the same as the *surplus sets* in [33]), that do not lend outside their group.

Definition 3.

1. *In-subgraph*: any subgraph $\mathscr{M} \subset \mathscr{N}$ (possibly $\mathscr{M} = \mathscr{N}$) with no out-links to its complement $\mathscr{M}^c$.
2. *Irreducible in-subgraph*: an in-subgraph $\mathscr{M} \subset \mathscr{N}$ with at least 2 nodes that does not contain a smaller in-subgraph.

[2] A statement and proof of this result can be found at
`http://en.wikipedia.org/wiki/Knaster-Tarski-theorem`

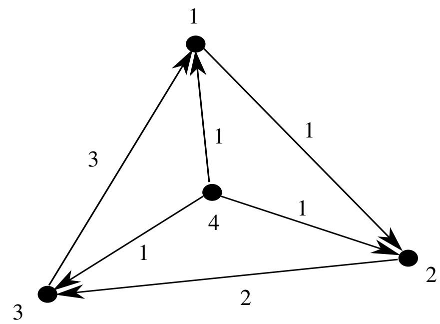

Fig. 2.1 This $N = 4$ bank network with $\bar{\mathrm{Y}} - \bar{\mathrm{D}} = 0$ has multiple clearing vectors $\mathrm{p} = \lambda[1,1,1,0]$ with $\lambda \in [0,1]$.

The exposure matrix $\bar{\Pi}'$ is always a substochastic matrix, and when projected onto any irreducible in-subgraph $\mathscr{M}$, is always an irreducible substochastic matrix, where we recall the standard definitions:

Definition 4.

1. A *stochastic (substochastic) matrix* has non-negative entries and columns that sum to 1 (respectively, ≤ 1).
2. An *irreducible* substochastic matrix Γ has the property that for every $v, w \in [N]$ there is $k \geq 1$ such that $(\Gamma^k)_{vw} > 0$.

Having identified all the irreducible in-subgraphs of a given network, we can simplify the following discussion by removing all singleton *in-banks* (i.e. that do not lend to other banks). Any such bank v will have $p_v = \bar{\mathrm{X}}_v = 0$ and its state has no effect on other banks. We then consider the exposure matrix $\bar{\Pi}'$ restricted to the *reduced network* $\tilde{\mathscr{N}}$ without such in-banks, which may then be strictly substochastic in some columns.

The theorem that characterizes the set of clearing vectors in the E-N model is:

Theorem 2. *Let the reduced network $\tilde{\mathscr{N}}$ have exactly $K \geq 0$ non-overlapping irreducible in-graphs $\mathscr{M}_1, \mathscr{M}_2, \ldots \mathscr{M}_K$, and a decomposition $\tilde{\mathscr{N}} = \mathscr{M}_0 \cup \left(\cup_{k=1}^{K} \mathscr{M}_k\right)$. Let the possible fixed points be written in block form $\mathrm{p}^* = [\mathrm{p}_0^*, \mathrm{p}_1^*, \ldots, \mathrm{p}_K^*]'$. Then:*

1. *In case $K = 0$, the clearing vector $\mathrm{p}^* = \mathrm{p}_0^*$ is unique;*
2. *In case there are exactly $K \geq 1$ (non-overlapping) irreducible in-graphs $\mathscr{M}_1, \mathscr{M}_2, \ldots \mathscr{M}_K$, then the multiplicity of the clearing vectors is characterized as follows: p_0^* is unique, and each p_k^* is either unique or of the form $\mathrm{p}_k^* = \alpha_k v_k$ where the vector v_k is the unique normalized 1-eigenvector of the matrix $P_k \cdot \bar{\Pi}' \cdot P_k'$ projected onto $\mathscr{M}_k$ and $\alpha_k \in [0, \bar{\alpha}_k]$ for some $\bar{\alpha}_k$. The precise form of each p_k^* is shown in the proof.*

Remark 1. This theorem demonstrates that non-uniqueness in this model is a highly non-generic property: only very special arrangements of the network lead to multiplicity of solutions.

The proof of the theorem involves extending the following lemma to the most general kind of E-N fixed point equation.

Lemma 1. *Consider the system*

$$\mathrm{p} = \min(\mathrm{X}, \max(\mathrm{Y} + \Gamma \cdot \mathrm{p}, 0)) \tag{2.6}$$

where Γ is substochastic and irreducible, X *is a positive vector and* Y *is arbitrary.*

1. *If Γ is strictly substochastic, then there is a unique fixed point.*
2. *If Γ is stochastic, with y the unique eigenvector such that $\Gamma \cdot y = y$ and $\mathbf{1} \cdot y = 1$ where $\mathbf{1} = [1, \ldots, 1]$, then one of two possible cases holds:*
   a. *If $\mathbf{1} \cdot \mathrm{Y} = 0$, there is a one-parameter family of solutions that have the form $\mathrm{p}^* = \lambda y$, $\lambda \in [0, \lambda_{max}]$.*
   b. *If $\mathbf{1} \cdot \mathrm{Y} \neq 0$, there is a unique fixed point, of which at least one component is either 0 or X.*

Proof of Lemma: Note that for irreducible substochastic matrices, the largest eigenvalue is simple and 1 if it is stochastic or less than one if it is strictly substochastic. Moreover, every submatrix obtained by deleting one or more nodes has largest eigenvalue less than one. Thus in Part 1, uniqueness is guaranteed because $I - \Gamma$ has largest eigenvalue strictly less than one and hence an explicit inverse given by the convergent matrix power series $(I - \Gamma)^{-1} = \sum_{k=0}^{\infty} \Gamma^k$.

Under the conditions of Part 2, by dotting (2.6) with the vector $\mathbf{1} = [1, \ldots, 1]$ and noting that $\mathbf{1} \cdot \Gamma = \mathbf{1}$, we see that the system $\mathrm{p} = \mathrm{Y} + \Gamma \cdot \mathrm{p}$ has a solution if and only if $\mathbf{1} \cdot \mathrm{Y} = 0$. If $\mathbf{1} \cdot \mathrm{Y} = 0$ one can check that case 2(a) of the lemma holds. If $\mathbf{1} \cdot \mathrm{Y} \neq 0$, at least one component of any fixed point of (2.6) must be X or 0. Substituting in this component value, and reducing the system by one dimension now leads to a new fixed point equation of the same form (2.6) but where the submatrix matrix $\tilde{\Gamma}$ has largest eigenvalue less than one. Such systems have a unique fixed point by part 1. □

Proof of Theorem 2: We must now deal with the system (2.6) when $\Gamma = \bar{\Pi}'$ is not irreducible. In general, it is easy to see that $\mathscr{N}$, itself an in-subgraph, contains within it a maximal number of irreducible non-overlapping in-subgraphs $\mathscr{M}_1, \mathscr{M}_2, \ldots \mathscr{M}_K$, $K \geq 0$ plus the possibility of additional single non-lending nodes with $\bar{\mathrm{X}}_v = 0$. As discussed above, as a first step we can eliminate all non-lending nodes and consider the reduced piece-wise linear fixed point problem on the subgraph $\tilde{\mathscr{N}}$. The case when $\tilde{\mathscr{N}}$ has no irreducible in-subgraphs, i.e. $K = 0$, has a unique clearing vector because then, by Part 1 of the Lemma, $\bar{\Pi}'$ must be substochastic with largest eigenvalue less than one.

If $K > 0$, we decompose into $\tilde{\mathscr{N}} = \mathscr{M}_0 \cup \left(\cup_{k=1}^{K} \mathscr{M}_k\right)$, and after reordering nodes write the matrix $\bar{\Pi}'$ in block form with respect to this decomposition:

$$\bar{\Pi}' = \begin{pmatrix} A & 0 & \cdots & 0 \\ B_1 & \Pi_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ B_K & 0 & 0 & \Pi_K \end{pmatrix}$$

with the column sums less than or equal to one. We shall characterize the possible fixed points written in block form $\mathrm{p} = [\mathrm{p}_0^*, \mathrm{p}_1^*, \ldots, \mathrm{p}_K^*]'$. We note that A is strictly substochastic with largest eigenvalue less than 1. Therefore the fixed point equation for p_0^* is a closed equation that has a unique solution by part 1 of Lemma 1. Each of the remaining pieces of p^*, $\mathrm{p}_k^*, k \geq 1$, is a solution of a piecewise linear equation in the following form:

$$\mathrm{p}_k^* = \min\left(\bar{X}_k, \max\left(B_k \cdot \mathrm{p}_0^* + \bar{\mathrm{Y}}_k - \bar{\mathrm{D}}_k + \Pi_k \cdot \mathrm{p}_k^*, 0\right)\right) \ .$$

Now we note that each Π_k is an irreducible stochastic matrix, and by Part 2 of Lemma 1, p_k^* is unique if $\mathbf{1}_k \cdot (B_k \cdot \mathrm{p}_0^* + \bar{\mathrm{Y}}_k - \bar{\mathrm{D}}_k) \neq 0$ and a point in a one-dimensional interval if $\mathbf{1}_k \cdot (B_k \cdot \mathrm{p}_0^* + \bar{\mathrm{Y}}_k - \bar{\mathrm{D}}_k) = 0$. □

2.1.2 Reduced Form E-N Cascade Mechanism

In models such as this, cascades of defaults arise when primary defaults trigger further losses to the remaining banks. Theorem 1 proves the existence of an “equilibrium” clearing vector, which is usually unique, that gives the end result of cascades in the E-N framework. Sometimes different balance sheet specifications lead to identical cascades, and we can characterize the cascade mechanism and resultant clearing vectors in terms of a reduced set of balance sheet data. It turns out that the most important information to track is something we call the *default buffer*, which extends the notion of equity. We assume the initial default buffer $\Delta_v^{(0)}$ of bank v is its nominal equity, but possibly negative:

$$\Delta_v^{(0)} = \bar{\Delta}_v := \bar{\mathrm{Y}}_v + \sum_w \bar{\Omega}_{wv} - \bar{\mathrm{D}}_v - \bar{\mathrm{X}}_v \ . \tag{2.7}$$

As before, define $p_v^{(n)}$ to be the amount available to pay $\bar{\mathrm{X}}_v$ at the end of cascade step n, initialized to $p_v^{(0)} = \bar{\mathrm{X}}_v$ at $n = 0$. Introduce the normalized *threshold function* h that maps the extended real line $[-\infty, \infty]$ to the unit interval $[0, 1]$:

$$h(x) = (x+1)^+ - x^+ = \max(0, \min(x+1, 1)) \ . \tag{2.8}$$

Then an important but straightforward calculation shows that equations (2.1) and (2.2) for $n > 0$ the nth step of E-N cascade are expressible in terms of the default buffers $\Delta^{(n-1)}$:

$$p_v^{(n)} = \bar{X}_v\, h(\Delta_v^{(n-1)}/\bar{X}_v) \quad (2.9)$$

$$q_v^{(n)} = (\bar{D}_v^- + \bar{X}_v)\, h(\Delta_v^{(n-1)}/(\bar{D}_v + \bar{X}_v)) \quad (2.10)$$

whereas the default buffers themselves satisfy a set of closed equations

$$\Delta^{(n)} = F^{(EN)}(\Delta^{(n-1)}); \quad F_v^{(EN)}(\Delta) := \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}(1 - h(\Delta_w/\bar{X}_w)) \ . \quad (2.11)$$

Thus the cascade mapping boils down to a vector-valued function $\Delta^{(n-1)} \mapsto \Delta^{(n)} = F^{(EN)}(\Delta^{(n-1)}|\bar{\Delta}, \bar{\Omega})$ that depends parametrically only on the initial default buffers $\bar{\Delta}$ and the interbank exposures $\bar{\Omega}$. The mark-to-market equity is the positive part of the default buffer, $E_v^{(n)} = (\Delta_v^{(n)})^+$, and default of bank v occurs at the first step that $\Delta_v^{(n)} \le 0$. As $n \to \infty$, the monotone decreasing sequence $\Delta^{(n)}$ converges to the maximal fixed point of $\Delta = F^{(EN)}(\Delta)$ which is a well-defined function of the reduced balance sheet data

$$\Delta^+ = G^+(\bar{\Delta}, \bar{\Omega}) \ ,$$

The corresponding maximal clearing vectors p^+, q^+ are given by

$$p_v^+ = \bar{X}_v\, h(\Delta_v^+/\bar{X}_v) \ ,$$
$$q_v^+ = (\bar{Y}_v + \bar{X}_v)\, h(\Delta_v^+/(\bar{Y}_v + \bar{X}_v)) \ .$$

If instead of starting the cascade at the optimistic initial values $\Delta_v^{(0)} = \bar{\Delta}_v$, we had begun with the most pessimistic values $\Delta_v^{(0)} \le \bar{\Delta}_v - \bar{Z}_v$, we would obtain a monotone *increasing* sequence $\Delta^{(n)}$ that converges to the minimal fixed point $\Delta^- := G^-(\bar{\Delta}, \bar{\Omega})$.

The scaled variable $\Delta/\bar{X}$, or alternatively $\Delta/(\bar{D} + \bar{X})$, has the interpretation of a bank's *distance-to-default*, and the threshold function h determines both the fractional recovery on interbank debt and on total debt when Δ is negative. Other possible threshold functions h are an important characteristic of different cascade models.

Some simple systemic risk measures of the total damage caused by the crisis are computable in terms of the cascade equilibrium (Δ^+, p^+, q^+):

1. Default probability: $DP = \frac{1}{N}\sum_v \mathbf{1}(\Delta_v^+ \le 0)$;
2. Default cascade impact on the financial sector: $DCI^{(1)} = \sum_v(\bar{X}_v - p_v^+)$;
3. Default cascade impact on the entire economy: $DCI^{(2)} = \sum_v(\bar{D}_v + \bar{X}_v - q_v^+)$.

2.1.3 The Gai-Kapadia 2010 Default Model

The threshold function $h(x)$ for the E-N 2001 model encodes a *soft* type of default in which the interbank debt of a defaulted bank with $\Delta/\bar{X} = x \sim 0$ recovers almost all its value. In their 2010 paper [42], Gai and Kapadia offer a model with *hard defaults*: interbank debt on defaulted banks recovers zero value. They justify their zero recovery assumption with the statement[3]: "This assumption is likely to be realistic in the midst of a crisis: in the immediate aftermath of a default, the recovery rate and the timing of recovery will be highly uncertain and banks' funders are likely to assume the worst-case scenario."

The G-K cascade mechanism boils down to the following assumptions:

Assumptions 2.

1. At step 0 of the cascade, one or more banks experience asset shocks that make their *default buffers* $\Delta_v^{(0)} = \bar{\Delta}_v \le 0$ go negative.
2. A defaulted bank v's interbank liabilities recover zero value and thus a default shock of magnitude $\bar{\Omega}_{vw}$ is sent to each of v's creditor banks w.
3. At each step $n \ge 0$ of the crisis, bank v marks to zero any interbank asset $\bar{\Omega}_{wv}$ from a newly defaulted counterparty bank w.

This cascade mechanism turns out to be precisely of the E-N type, but with a *zero-recovery* threshold function

$$\tilde{h}(x) = \mathbf{1}(x > 0) \,, \tag{2.12}$$

and exactly as in Section 2.1.2 it defines the sequence of vectors $\mathrm{p}^{(n)}$ and buffers $\Delta^{(n)}$ satisfying equations (2.9) and (2.11), with h replaced by $\tilde{h}$. That is,

$$p_v^{(n)} = \bar{\mathrm{X}}_v \tilde{h}(\Delta_v^{(n-1)}) \,, \tag{2.13}$$

$$\Delta_v^{(n)} = \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}(1 - \tilde{h}(\Delta_w^{(n-1)})) \,. \tag{2.14}$$

The clearing vector condition is now

$$\mathrm{p} = \bar{\mathrm{X}} \tilde{h}(\Delta)$$

where Δ is any fixed point of

$$\Delta = \mathrm{F}^{(GK)}(\Delta); \quad F_v^{(GK)}(\Delta) := \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}(1 - \tilde{h}(\Delta_w)) \,. \tag{2.15}$$

The existence of fixed points $\Delta = \mathrm{F}^{(GK)}(\Delta)$ follows as in the E-N model by repeating the proof of Theorem 1. On the other hand, uniqueness of fixed points has not been carefully studied for this model, and appears to be considerably more complicated. Like the E-N model, there are examples in which non-uniqueness arises associated to irreducible in-graphs, but now the multiplicity of clearing vectors is discrete. For example, the $N = 4$ bank network with $\bar{\Omega}$ as in Figure 2.1, and with $\bar{\mathrm{Y}} = [0,0,0,4], \bar{\mathrm{D}} = [1,1,1,0]$ has exactly two clearing vectors of the form $\mathrm{p} = \lambda[1,1,1,0]'$ with $\lambda = 0$ or $\lambda = 1$. As in the E-N model in case of non-uniqueness, our cascades typically start from $\mathrm{p}^{(0)} = \bar{\mathrm{X}}$ and therefore reach the maximal clearing vector p^+ with default buffers Δ^+.

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3 [42], footnote 9.

As before, the simplest systemic risk measure is default probability, given in terms of the fixed point Δ^+ by

$$\mathrm{DP} = \frac{1}{N}\sum_v \mathbf{1}(\Delta^+_v \le 0)\ .$$

The zero-recovery assumption implies there is a large cost to the financial system for banks that default given by *default cascade impact*:

$$\mathrm{DCI} = \sum_v \bar{\mathrm{X}}_v\ \mathbf{1}(\Delta^+_v \le 0)\ . \tag{2.16}$$

Recall that bankruptcy charges are ruled out in the E-N 2001 model and are maximal in the G-K model. We can interpolate between these two extreme cases with a single parameter $\tau \in [0,1]$ that represents the fraction of interbank debts that are paid as bankruptcy charges at the time any bank defaults. The cascade mapping is again given by equations (2.9) and (2.11), now with h replaced by the interpolated threshold function

$$h^{(\tau)}(x) = (1-\tau)h\Big(\frac{x}{1-\tau}\Big) + \tau\tilde{h}(x)\ . \tag{2.17}$$

The simple static default cascades just investigated can be summarized as follows:

1. They are characterized by shocks that are transmitted *downstream* from defaulting banks to the asset side of their creditor banks' balance sheets;
2. At each cascade step, banks update their default buffers by determining the current amount lost given default of its counterparties;
3. Different default recovery assumptions arise through the choice of a threshold function $h, \tilde{h}$ or $h^{(\tau)}$.

2.2 Liquidity Cascades

A funding liquidity cascade is a systemic phenomenon that occurs when stressed banks hoard liquidity, that is they curtail lending to each other on a large scale. In such a cascade, shocks are transmitted *upstream*, from creditor banks to their debtors as they act to reduce their interbank lending. A fundamental treatment of the connection between funding liquidity and market liquidity by Brunnermeier and Pedersen [20] proposes a picture of how the funding constraints on a bank impact the liquidity of its market portfolio, that its external assets. One finds the idea that when a bank's capital is reduced to below a threshold where a funding liquidity constraint becomes binding, that bank will reduce its assets by a discontinuous amount, and experience a discontinuous increase in its margin and collateral requirements. If, as is natural, we assume that at this threshold the bank will also reduce its interbank lending by a discontinuous amount, then this picture provides the seed of a cascade mechanism that is transmitted through the interbank network, from creditors to debtors. It turns out that our schematic default cascade models, when shocks are reversed and reinterpreted, become basic models of funding liquidity cascades.

The first paper to introduce a network model of funding liquidity cascades is a companion paper [41] to the default model by Gai-Kapadia in [42]. We now describe the G-K liquidity cascade model and two variations, all based on banks that have stylized balance sheets given as in Table 2.1.

2.2.1 Gai-Kapadia 2010 Liquidity Cascade Model

This systemic risk model aims to account for the observation that starting in August 2007 and continuing until after September 2008, interbank lending froze around the world as banks hoarded cash and curtailed lending to other banks. As Gai and Kapadia explain, during the build up of credit risk prior to 2007, some banks that held insufficiently liquid assets began to face funding liquidity difficulties. Such banks moved to more liquid positions by hoarding liquidity, in some cases reducing their interbank lending almost entirely.

What would a counterparty bank do when impacted by such a hoarding bank? Of course they might seek funding elsewhere, but in a climate of uncertainty they might themselves elect to become liquidity hoarders, thereby propagating further liquidity shocks.

The following liquidity cascade model assumes that prior to the crisis, banks hold assets and liabilities as shown in Figure 2.2. On the asset side we have: $\bar{Y}^F_v$ (external fixed assets, namely the bank book of loans to the economy at large), $\bar{Z}_v$ (interbank assets assumed to be short term unsecured loans to other banks) and liquid assets $\bar{Y}^L_v$. When non-zero, the liquid assets $\bar{Y}^L_v$ are used as a *stress buffer* $\bar{\Sigma}_v$ from which to pay liabilities as they arise. In analogy to the default buffers Δ_v, Σ_v can become negative: such a bank is called a *stressed bank*. On the liability side we have as before external debt $\bar{D}_v$, interbank debt $\bar{X}_v$ and the default buffer $\bar{\Delta}_v$.

In the new field of financial systemic risk, the network of interbank counterparty relationships can be described as a directed random graph. In "cascade models" of systemic risk, this "skeleton" acts as the medium through which financial contagion is propagated. It has been observed in real networks that such counterparty relationships exhibit negative assortativity, meaning that a bank's counterparties are more likely to have unlike characteristics. This paper introduces and studies a general class of random graphs called the assortative configuration model, parameterized by an arbitrary node-type distribution P and edge-type distribution Q. The first main result is a law of large numbers that says the empirical edge-type distributions converge in probability to Q. The second main result is a formula for the large N asymptotic probability distribution of general graphical objects called "configurations". This formula exhibits a key property called "locally tree-like" that in simpler models is known to imply strong results of percolation theory on the size of large connected clusters. Thus this paper provides the essential foundations needed to prove rigorous percolation bounds and cascade mappings in assortative networks.

Fig. 2.2 The stylized balance sheet of a bank v with three debtor banks w_1, w_2, w_3 and two creditor banks w'_1, w'_2.

Assumptions 3. 1. At step 0 of the cascade, one or more banks experience funding liquidity shocks or *stress shocks* that make their *stress buffers* $\Sigma_v^{(0)} = \bar{\Sigma}_v$ go negative.
2. Banks respond at the moment they become stressed by preemptively hoarding a fixed fraction $\lambda \leq 1$ of interbank lending. This sends a stress shock of magnitude $\lambda\bar{\Omega}_{wv}$ to each of the debtor banks w of v. Stressed banks remain stressed for the duration of the crisis.
3. At each step $n \geq 0$ of the crisis, bank v pays any interbank liabilities $\lambda\bar{\Omega}_{vw}$ that have been recalled by newly stressed banks w.

The assumption of a fixed hoarding fraction λ across the network is clearly a gross oversimplification that can be refined later. The essential point is that under stress, banks act preemptively to shrink their balance sheets by a fraction close to one. In order to disentangle default and stress, the third assumption implies that unstressed banks are always able to meet recalled interbank liabilities without negatively impacting their default buffers. In other words, this is a model of funding liquidity with zero market illiquidity effects.

These simple behavioural rules lead to a cascade mechanism (CM) that can be expressed succinctly as the recursive updating of the stress buffers of all banks starting from an initial state with $\Sigma_v^{(0)} = \bar{\Sigma}_v$. Given the collection of stress buffers $\Sigma_v^{(n-1)}$ at step $n-1$ of the cascade, the updated stress buffers are given by

$$\Sigma_v^{(n)} = \bar{\Sigma}_v - \lambda \sum_w \bar{\Omega}_{vw} \left(1 - \tilde{h}(\Sigma_w^{(n-1)})\right) . \tag{2.18}$$

In this proposed cascade mechanism, the parameter λ represents the average strength of banks' collective stress response. Under our assumptions, the iterated cascade mapping converges to an equilibrium set of buffers given by the maximal fixed point Σ^+. Two simple systemic risk measures quantify the effect of the crisis:

1. Stress probability: $\mathrm{SP} = \frac{1}{N}\sum_v \mathbf{1}(\Sigma_v^+ \leq 0)$;
2. Liquidity cascade impact on the financial system, that is, the total amount of interbank assets frozen during the crisis: $\mathrm{LCI} = \lambda \sum_v \mathrm{Z}_v \mathbf{1}(\Sigma_v^+ \leq 0)$.

It should not be a surprise that (2.18) is identical in form to (2.14), but with shocks going upstream from creditors to debtors instead of downstream from debtor banks to creditors. The pair of models [42] and [41] by Gai and Kapadia is a first instance of a formal symmetry of financial networks under interchange of assets and liabilities.

2.2.2 The Liquidity Model of S. H. Lee 2013

As a second illustration of how liquidity cascades are the mirror image of default cascades, we now show how a simple liquidity cascade model proposed by S. H. Lee [61] is formally identical to a version of the E-N cascade. This model is again based on banks with balance sheets as shown in Figure 2.2. The essential assumption of the model is:

Assumptions 4. Banks pay all withdrawals from external and interbank depositors first by selling liquid external and interbank assets in constant proportion, and then when these assets are depleted, from the illiquid external assets.

To put this model into an E-N form while preserving the labelling of Lee's paper, we introduce a fictitious "source" bank labelled by 0 that borrows from but does not lend to other banks and arbitrarily define $\bar{Z}_0 = \bar{D}_0 = 0$. We now label the interbank exposures as in [61]:

$$\bar{\Omega}_{vw} = \begin{cases} b_{vw} & v,w \neq 0 \\ q_w & v = 0, w \neq 0 \\ 0 & w = 0\,. \end{cases}$$

As before, let $\bar{Z}_v = \sum_w \bar{\Omega}_{wv}$, $\bar{X}_v = \sum_w \bar{\Omega}_{vw}$ and identify $\bar{Y}^F_v = z_v$, $\bar{Y}^L_v = q_v = \bar{\Omega}_{0v}$, $\bar{D}_v = d_v$.

At time 0, each bank experiences deposit withdrawals (a liquidity shock) $\Delta d_v \geq 0$. These withdrawals are paid immediately by each bank v in order of seniority: first from the liquid interbank assets $\bar{Z}_v$ (which now includes lending to the fictitious bank $\bar{\Omega}_{0v}$) until these are depleted, and then by selling fixed external assets $\bar{Y}^F_v$. Let us now define the initial *stress buffer* to be $\Sigma^{(0)}_v = \bar{\Sigma}_v = -\Delta d_v$, and then at the nth step of the liquidity cascade each buffer $\Sigma^{(n)}_v$, which is the negative of bank v's total liquidity needs $\ell^{(n)}_v$, will have accumulated shocks as follows

$$\Sigma^{(n)}_v = \bar{\Sigma}_v - \sum_w \bar{\Omega}_{vw}\left(1 - h(\Sigma^{(n-1)}_w/\bar{Z}_w)\right)\,.$$

This equation, being formally identical to (2.11), reveals that the Lee model is a special case of the E-N model provided $\bar{Z}$ and $\bar{X}$ are interchanged, and the exposures are reversed. However, in the Lee model, we begin with buffers $\Sigma^{(0)}_v \leq 0$ for all v except $v = 0$. For completeness, at step n we can define $p^{(n)}_v = \bar{Z}_v h(\Sigma^{(n-1)}_v/\bar{Z}_v)$, the amount of liquid and interbank assets remaining unsold, and $q^{(n)}_v = (\bar{Y}^F_v + \bar{Z}_v)h(\Sigma^{(n-1)}_v/(\bar{Y}^F_v + \bar{Z}_v))$, the total assets remaining unsold. We can call a bank *illiquid* when $\Sigma_v \leq -\bar{Z}_v$. Two natural measures of systemic liquidity risk are determined by the maximal cascade fixed point Σ^+:

1. Illiquidity probability: $\mathrm{LP} = \frac{1}{N}\sum_v \mathbf{1}(\Sigma^+_v \leq -\bar{Z}_v)$;
2. Liquidity cascade impact on the entire economy, that is, the total amount of fixed external assets sold during the crisis: $\mathrm{LCI} = \sum_v(-\Sigma^+_v - \bar{Z}_v)^+$.

2.2.3 Generalized Liquidity Cascades

The liquidity cascade model of S.H. Lee supposes that deposit withdrawals are funded in equal proportion by interbank assets and liquid external assets. A reasonable alternative picture is that each bank keeps a first line reserve of liquid external assets (or simply "cash") $\bar{Y}^L$ to absorb liquidity shocks. We now think of this as the stress buffer, labelled by Σ, to be kept positive during normal banking business. When the stress buffer goes zero or negative, the bank becomes *stressed* and must meet further withdrawals by liquidating first interbank assets $\bar{Z}$, and finally illiquid fixed assets $\bar{Y}^F$.

As for the Lee model, we may also add a fictitious sink bank $v = 0$ to represent external agents that borrow amounts $\bar{\Omega}_{0v}$ where in terms of liquidation priority, these external loans will be considered a component of a bank's interbank assets: $\bar{Z}_v = \sum_{w=0}^N \bar{\Omega}_{wv}$.

Let us suppose that just prior to an initial withdrawal shock that hits any or all of the banks, the banks' balance sheets are given as in Figure 2.2 by notional amounts $(\bar{Y}^F, \bar{Z}, \bar{Y}^L, \bar{D}, \bar{X}, \bar{E}, \bar{\Omega})$. At the onset of the liquidity crisis, all banks are hit by withdrawal shocks ΔD_v that reduce the initial stress buffers $\Sigma_v^{(0)} = \bar{Y}_v^L - \Delta D_v$ of at least some banks to below zero, making them stressed. Stressed banks then liquidate assets first from $\bar{Z}$, inflicting additional liquidity shocks to their debtor banks' liabilities, and then from $\bar{Y}^F$. A stressed bank that has depleted all of $\bar{Z}$ will be called *illiquid*, and must sell external fixed assets $\bar{Y}^F$ to meet further liquidity shocks.

Let $p_v^{(n)}$ be the amount of bank v's interbank assets remaining unsold after n steps of the liquidity cascade, starting with $p_v^{(0)} = \bar{Z}_v$. Illiquid banks have $p_v^{(n)} = 0$, stressed banks are those with $0 < p_v^{(n)} < \bar{Z}_v$ while normal, unstressed banks have $p_v^{(n)} = \bar{Z}_v$. If $\Sigma_v^{(n)}$ is the stress buffer after n steps and each stressed bank liquidates exactly enough additional interbank assets at each step to meet the additional liquidity shocks, the update rule is

$$p_v^{(n)} = \max(0, \min(\bar{Z}_v, (D_v - \Delta D_v) - \bar{Y}^F + \sum_w \bar{\Omega}_{vw}(p_w^{(n-1)}/\bar{Z}_w)) \ . \qquad (2.19)$$

We note that

$$\Sigma_v^{(n)} = \Sigma_v^{(0)} - \sum_w \bar{\Omega}_{vw}(1 - p_w^{(n-1)}/\bar{Z}_w) \ , \qquad (2.20)$$

and that (2.19) can be written

$$p_v^{(n)} = \bar{Z}_v h(\Sigma_v^{(n-1)}/\bar{Z}_v)$$

with the threshold function h of (2.8) used before.

Comparison of these equations with (2.9) and (2.11) reveals that our model is precisely equivalent to the full E-N 2001 model, with the role of assets and liabilities, and stress and default buffers, interchanged: $\bar{Y}^F \leftrightarrow \bar{D}, \bar{Z} \leftrightarrow \bar{X}, \bar{Y}^F \leftrightarrow \bar{E}$, $\Delta \leftrightarrow \Sigma$. We recover the Lee model simply by taking $\bar{Y}_v^L = 0$, which also has the effect of making all the banks initially stressed since the initial stress buffers are $\Sigma_v^{(0)} = -\Delta D_v \le 0$. We also recover the G-K 2010 liquidity model by replacing h by $\tilde{h}$.

To keep various cascade mechanism separated, the funding liquidity cascade models we have just described neglect *market illiquidity*, which of course is the very important systemic effect that large scale selling of assets will drive asset prices down. The next type of cascade turns the focus on this effect.

2.3 Asset Fire Sales

Certainly one of the basic triggers of financial crises is when a major asset class held by many banks is beset by bad news, resulting in a major devaluation shock that hits these banks. We identify this as an *asset correlation shock* described in Section 1.4 in which the external assets Y_v held by many banks exhibit a sharp one-time decline. If this shock is sufficient to cause the default of some banks, we face the possibility of a pure default cascade of the same nature as we have described already.

Of a different nature are *asset fire sales*, in which banks under stress (of which there will be many during a crisis) react by selling external assets on a large scale, driving their prices down. As described in detail in the 2005 paper by Cifuentes, Ferrucci and Shin [25], an asset fire sale creates a negative feedback loop in the financial network: large scale selling of an asset class by banks leads to strong downward pressure on the asset price, which leads to market-to-market losses by all banks holding that asset, to which they respond by selling this and other assets.

Of course, small and medium scale versions of such selling spirals are an everyday occurrence in financial markets, sometimes leading to an asset correlation shock. In the present context, we will focus on large scale selling spirals that form during and as a result of the crisis and are essential amplifiers of financial distress. Our aim in this section is to provide a stylized modelling framework that highlights the network cascade aspects of the fire sale mechanism.

2.3.1 Fire Sales of One Asset

The basic network picture of asset fire sales is most clearly explained by the CFS model of Cifuentes, Ferrucci and Shin [25]. The baseline CFS model consists of a network of N banks with balance sheets with the same components $(\bar{\mathrm{Y}}^F, \bar{\mathrm{Z}}, \bar{\mathrm{Y}}^L, \bar{\mathrm{D}}, \bar{\mathrm{X}}, \bar{\Omega})$ as shown in Figure 2.2 for funding liquidity cascade models. Since liquidity and solvency are both considered in this model, it can be regarded as a generalization of the Eisenberg-Noe model. In the one asset model, all banks hold their fixed assets $\bar{\mathrm{Y}}^F_v$ in the same security, which we might view as the market portfolio. We set the initial price of the asset to be $\bar{p} = p^{(0)} = 1$ so that each bank v holds $s_v^{(0)} = \bar{\mathrm{Y}}^F_v$ units.

The essential new feature is to include a *capital adequacy ratio* (CAR) as a regulatory constraint: For some fixed regulatory value r^* (say 7%), the bank must maintain the lower bound[4]

[4] We deviate from [25] at this point by omitting liquid assets $\bar{\mathrm{Y}}^L$ from the denominator of the CAR.

$$\frac{\Delta_v}{Y_v^F + Z_v} \geq r^* \ . \tag{2.21}$$

As soon as this condition is violated, any bank is compelled to restore the condition by selling fixed illiquid assets (but, in contrast to the Gai-Kapadia and Lee liquidity cascade models, not interbank assets[5]), which then triggers a downward impact on asset prices.

The detailed assumptions of [25] are as follows:

Assumptions 5.

1. A bank with $r^* Z_v \leq \Delta_v < r^*(Y_v^F + Z_v)$ is called *non-compliant* but solvent, and must sell fixed assets, but not interbank assets, to restore the CAR condition. [6]
2. A bank with $r^* Z_v > \Delta_v$ is insolvent, and must be fully liquidated. The picture to have is that such a bank can never achieve the CAR condition, and hence must be terminated, even if its default buffer may still be positive.
3. In the event of insolvency, the defaulted interbank assets are distributed at face value proportionally among the bank's creditors, and the bank ceases to function. External deposits have equal seniority to interbank debt and thus defaulted liabilities are valued in proportion to $\bar{\Pi}_{vw} = \bar{\Omega}_{vw}/(\bar{X}_v + \bar{D}_v)$.
4. The asset price when sold is determined by an inelastic supply curve and a downward sloping inverse demand function $d^{(-1)}(\cdot)$. That is, the asset price is $p = d^{(-1)}(s)$ when $s = \sum_v s_v$ is the aggregated amount sold. For the inverse demand function, [25] works with the family of exponential functions $d^{(-1)}(s) = e^{-\alpha s}$ for a specific value of α.

The crisis unfolds starting at step $n = 0$ from an initial balance sheet configuration $(\bar{Y}^F, \bar{Z}, \bar{Y}^L, \bar{X}, \bar{\Omega})$ with default buffers

$$\Delta^{(0)} = \bar{\Delta} = \bar{Y}^F + \bar{Z} + \bar{Y}^L - \bar{X} \tag{2.22}$$

in which at least one bank is found to be in violation of its CAR bound. In view of the equal seniority assumption on the debt, we adopt here the usual trick of replacing external debt by interbank debt owed to a non-borrowing fictitious bank $v = 0$ that has lent $\bar{D}_w := \bar{\Omega}_{w0}$ to each bank w. Recall we set $p^{(0)} = 1$ so the number of fixed assets is $s^{(0)} = \bar{Y}^F$. Then, the following recursive steps for the balance sheets of each bank $(Y^{F(n)}, Z^{(n)}, Y^{L(n)}, \Delta^{(n)})$ and the asset price $p^{(n)}$ for $n = 1, 2, \ldots$ are consistent with the underlying model assumptions:

1. Each bank v adjusts its fixed asset holdings by selling[7]

$$\delta s_v = \min(s_v^{(n-1)}, \max(0, s_v^{(n-1)} + Z_v^{(n-1)}/p^{(n-1)} - \Delta_v^{(n-1)}/(r^* p^{(n-1)})) \tag{2.23}$$

[5] It is interesting that these modelling frameworks make essentially contradictory assumptions at this point. The assumption of [25] removes the need to consider how interbank assets are liquidated.

[6] As [25] explains, "interbank loans normally cannot be expected to be recalled early in the event of default of the lender."

[7] This is a slight modification of [25] who assume these units are sold at an n dependent equilibrium price somewhat lower than $p^{(n-1)}$.

units at the price $p^{(n-1)}$. Note that $\delta s_v = 0$ corresponds to a compliant bank, while $\delta s_v = s_v^{(n-1)}$ corresponds to an insolvent bank. When $\delta s_v > 0$, the sale of the fixed asset increases the bank's liquid assets to $\mathrm{Y}_v^{L(n)} = \mathrm{Y}_v^{L(n-1)} + \delta s_v p^{(n-1)}$ and the number of shares held becomes $s_v^{(n)} = s_v^{(n-1)} - \delta s_v$.

2. In case $s_v^{(n)} = 0$, the insolvent bank v must be liquidated in the manner described above and the mark-to-market value of its debt adjusted: $\mathrm{X}_v^{(n)} = \bar{\mathrm{X}}_v + \min(0, \Delta_v^{(n-1)})$.
3. After all banks have completed their asset sales, the market price moves downwards according to the aggregate amount sold. Thus $p^{(n)} = d^{(-1)}(\sum_v (s_v^{(0)} - s_v^{(n)}))$ and the interbank assets are updated to account for new default losses, $\mathrm{Z}_v^{(n)} = \sum_w \bar{\Pi}_{wv} \mathrm{X}_w^{(n)}$.
4. The updated default buffer of bank v becomes

$$\Delta_v^{(n)} = s_v^{(n)} p^{(n)} + \mathrm{Y}_v^{L(n)} + \sum_w \bar{\Pi}_{wv} \mathrm{X}_w^{(n)} - \bar{\mathrm{X}}_v \ . \tag{2.24}$$

Just as the E-N framework could be simplified into a reduced form cascade mapping by focussing on the default buffers $\Delta_v^{(n)}$, it turns out the above recursion simplifies in a very similar way if we focus on the pairs $\Delta_v^{(n)}, s_v^{(n)}$. The key fact to recognize is that once the bank becomes noncompliant, it can never become compliant, nor can a defaulted bank recover. Having seen this, one can easily verify that the result of the n-th step of the CFS cascade is given by

$$\mathrm{X}_v^{(n)} = \bar{\mathrm{X}}_v \, h(\Delta_v^{(n-1)}/\bar{\mathrm{X}}_v) \tag{2.25}$$

$$\Delta_v^{(n)} = \bar{\Delta}_v - \sum_{m=1}^{n} (p^{(m-1)} - p^{(m)}) \, s_v^{(m)} - \sum_w \bar{\Omega}_{wv}(1 - h(\Delta_w^{(n-1)}/\bar{\mathrm{X}}_w)) \tag{2.26}$$

$$s_v^{(n)} = \max(0, \min(s_v^{(n-1)}, \frac{1}{p^{(n-1)}} \big[\frac{\Delta_v^{(n-1)}}{r^*} - \sum_w \bar{\Omega}_{wv} h(\Delta_w^{(n-1)}/\bar{\mathrm{X}}_w) \big]) \tag{2.27}$$

$$p^{(n)} = d^{(-1)} \big(\sum_v (s_v^{(0)} - s_v^{(n)} \big) \ . \tag{2.28}$$

The third of these equations corresponds to the trichotomy of possibilities of the bank being compliant, noncompliant but solvent, and insolvent.

By comparison to the E-N cascade dynamics given by (2.9) and (2.11), we note that the key effect of the fire sale on the cascade is to provide additional shocks that further reduce the default buffers, thereby amplifying the contagion. Structurally, the fire sale cascade mechanism can be expressed as a monotonically decreasing mapping $(\Delta^{(n)}, p^{(n)}) = \mathbf{F}^{CFS}(\Delta^{(n-1)}, p^{(n-1)})$ from $\mathbb{R}^{N+1}$ to itself, which leads as usual to a cascade equilibrium. In case the price impact is omitted by assuming $d^{(-1)}(\cdot) = 1$, one reproduces the E-N model.

An interesting special case of the model emerges if we set the compliancy parameter $r^* = 0$ and take equation (2.27) to mean

$$s_v^{(n)} = s_v^{(0)}\, \mathbf{1}(\Delta_v^{(n-1)} > 0)\ . \tag{2.29}$$

We then interpret the breach of this regulatory condition as the insolvency of the bank in two senses: first, the bank has insufficient assets to repay its liabilities; second, the bank has insufficient equity to support its assets and thus needs to liquidate all fixed assets. Thus the bank must sell all fixed assets at this moment. However, as in the E-N model, the bank continues to lose money after it defaults, further eroding the value of $\mathrm{X}_v^{(n)}$. Thus, when $r^* = 0$ this model looks very similar to the E-N 2001 model, albeit with a G-K-like condition to determine the amount of fixed assets sold.

The simplification $r^* = 0$ also provides a simpler setting to address a different question: how do fire sales create contagion when banks hold different portfolios of a multiplicity of assets. As it turns out, this variation has been studied already, in a paper [21] we shall now describe.

2.3.2 Fire Sales of Many Assets

A model due to Caccioli et al [21] addresses the question: How do fire sales create contagion when banks hold different but overlapping portfolios of a multiplicity of assets? Their paper is a variation of the CFS approach in which N banks have balance sheets $(\bar{\mathrm{Y}}^F, \bar{\mathrm{Y}}^L, \bar{\mathrm{D}})$ with the interbank sector set to zero $\bar{\mathrm{Z}}_v = \bar{\mathrm{X}}_v = 0$ for simplicity, and the fixed assets are portfolios of M non-bank assets labelled by $a \in \{1,2,\ldots,M\} := [M]$. One can describe their model in terms of a bipartite graph with nodes of two colours, "blue" nodes that represent banks and "red" nodes that represent non-bank assets, and links connecting banks to assets when the bank has a significant holding of that asset. Figure 2.3 shows a typical network with 5 banks and 4 assets.

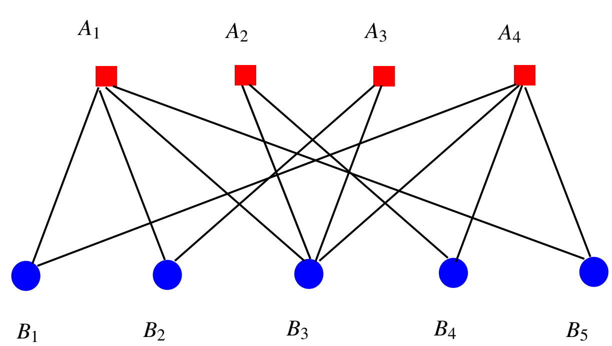


Fig. 2.3 A bipartite graph with 5 banks (blue nodes) co-owning 4 assets (red nodes).

Again, banks are constrained by the regulatory CAR condition which says

$$\frac{\Delta_v}{\mathrm{Y}_v^F} \geq r^* \ . \tag{2.30}$$

The paper [21] deals only with the case $r^* = 0$, but here we consider the general case with $r^* \geq 0$. The assumptions for the many asset fire sale are slight modifications of Assumptions 5, but with some additional points: The solvency condition is now $\Delta_v > 0$, and ownership of insolvent banks goes to external debtholders; there is an inverse demand function $d_a^{(-1)}$ for each asset; a portfolio policy must be specified for each bank that determines its portfolio weights during rebalancing.

The cascade picture that arises from these assumptions is that non-compliant banks that fail the CAR condition (2.30) are forced to liquidate some fixed assets, and while insolvent banks sell all their remaining fixed assets to pay their external creditors. At any cascade step, forced asset sales by a bank creates shocks that drive down the prices of the assets it holds, causing a shock to be transmitted along each link from the bank to its assets. Each asset price drop creates mark-to-market shocks acting in the reverse direction, from the asset to the banks that hold it. As usual, the iterated cascade steps converge monotonically to a cascade equilibrium.

Without loss of generality, we assume the initial share prices are $\bar{p}_a = 1$. The initial external fixed asset value of bank v is then $\bar{\mathrm{Y}}_v^F = \sum_a \bar{s}_{av}$ where $s_{av}^{(0)} = \bar{s}_{av}$ denotes the initial number of shares of asset a owned by v. If after n cascade steps each bank v holds $s_{av}^{(n)}$ shares of asset a, each asset share price will have been driven down to the value

$$p_a^{(n)} = d_a^{(-1)}\Big(\sum_v (s_{av}^{(0)} - s_{av}^{(n)})\Big) \tag{2.31}$$

determined by the total amount of selling and its inverse demand function. Various portfolio selection rules can be proposed to determine in which proportion banks choose to sell assets during the cascade. For illustrative purposes, we assume that non-compliant banks follow a *fixed-mix trading strategy* that keeps the *number of shares* in different assets in proportion to the initial ratios

$$\bar{\phi}_{av} := \frac{\bar{s}_{av}}{\sum_b \bar{s}_{bv}} \ . \tag{2.32}$$

Based on these assumptions, recursively for $n = 1, 2, \ldots$ the balance sheets of each bank $(\mathrm{Y}_v^{F(n)}, \mathrm{Y}_v^{L(n)}, \Delta_v^{(n)})$, the portfolio allocations $s_{av}^{(n)}$ and asset prices $p_a^{(n)}$ are updated according to the following steps, starting from the initial values $\mathrm{Y}^{F(0)} = \bar{\mathrm{Y}}^F, \mathrm{Y}^{L(0)} = \bar{\mathrm{Y}}^L, \Delta^{(0)} = \bar{\mathrm{Y}}^F + \bar{\mathrm{Y}}^L - \bar{\mathrm{D}}$:

1. Each bank v adjusts its fixed asset holdings by selling

$$\delta s_{av} = s_{av}^{(n-1)} \min\big(1, \max(0, 1 - \Delta_v^{(n-1)}/(r^* \mathrm{Y}_v^{F(n-1)})\big) \tag{2.33}$$

units at the price $p^{(n-1)}$. Note that $\delta s_{av} = 0$ for a compliant bank, while $\delta s_v = s_{av}^{(n-1)}$ corresponds to an insolvent bank. When $\delta s_v > 0$, the sale of the fixed assets increases the bank's liquid assets to $Y_v^{L(n)} = Y_v^{L(n-1)} + \sum_a \delta s_{av} p_a^{(n-1)}$ and the number of shares held becomes $s_{av}^{(n)} = s_{av}^{(n-1)} - \delta s_{av}$.

2. After all banks have completed their asset sales, the market prices move downwards according to the aggregate amount sold: $p_a^{(n)} = d_a^{(-1)}(\sum_v (s_{av}^{(0)} - s_{av}^{(n)}))$.
3. The updated default buffer and fixed assets of bank v decrease:

$$\Delta_v^{(n)} = \Delta_v^{(n-1)} - \sum_a s_{av}^{(n)} (p_a^{(n-1)} - p_a^{(n)}) \, , \tag{2.34}$$

$$Y^{F(n)} = \sum_a s_{av}^{(n)} p_a^{(n)} \, . \tag{2.35}$$

We observe the formal similarity between this model and the original E-N 2001 model, in the sense that the cascade mechanism can be expressed as a cascade mapping

$$(\Delta^{(n)}, p^{(n)}) = \mathbf{F}^{CSMF}(\Delta^{(n-1)}, p^{(n-1)}) \tag{2.36}$$

from $\mathbb{R}^{N+M}$ to itself. This means asset prices $p_a^{(n)}$ behave as if they were "asset buffers" attached to the red nodes, in analogy to the buffers $\Delta_v^{(n)}$ attached to blue nodes. In addition, the dynamics depends explicitly on the initial balance sheets only through $\bar{\Delta}$ and the fixed-mix ratios $\bar{\phi}_{av}$ that are analogous to the edge weights $\bar{\Omega}_{wv}$. The inverse demand functions d_a^{-1} play a role similar to the $h, \tilde{h}$ threshold functions in our default cascade models. Finally, the cascade mapping from step $n-1$ to n is monotonic and bounded below, and thus its iterates converge to a fixed point.

2.4 Random Financial Networks

This chapter has explored stylistic features of various types of shocks that can be transmitted through the interbank exposures (or, in one case, through bank-to-asset exposures), and how they might potentially cascade into large scale disruptions of the sort seen during the 2007-08 financial crisis. We have seen how to build these shock channels into a variety of different financial network models, all of which boil down to a monotone cascade mapping whose iterates converge to a cascade equilibrium.

The real world financial systems in most countries are of course far from behaving like these models. Bank balance sheets are hugely complex. Interbank exposure data are never publicly available, and in many countries nonexistent even for central regulators. Sometimes, the only way to infer exposures is indirectly, for example, through bank payment system data as done in [40]. Interbank exposures are of a diversity of types and known to change rapidly day to day. In a large jurisdiction like the US, the banking sector is highly heterogeneous, and the systemic impact due to the idiosyncrasies of individual banks will likely overwhelm anything one might predict from their average properties.

Nonetheless, a large and rapidly growing web of economics and complex systems research continues to address the real world implications of theoretical network cascades. The conceptual tools that we will explore in the remainder of this book come partly from the experience gained by modelling large complex networks that arise in other areas, such as the world wide web, Facebook and power grids.

The central theme of this book is that something fundamental about financial systems can be learned by studying very large stochastic networks. There are at least three important reasons why stochastic network models are a good way to approach studying real world financial systems. The first and most fundamental reason comes from over a century of statistical mechanics theory, which has discovered that the macroscopic properties of matter, for example crystals, are determined by the ensemble averages of the deterministic dynamics of constituent microscopic particles. Even a completely known deterministic system, if it is large enough, can be well described by the average properties of the system. From this fact we can expect that for large N, an E-N model with fully specified parameters will behave as if it were a stochastic model with averaged characteristics.

The second important reason is that in a concrete sense, the true networks are to be thought of as stochastic at any moment in time. The balance sheets of banks, between reporting dates, are not observed even in principle. Moreover, they change so quickly that last week's values, if they were known, will have only limited correlation with this week's values. This fact is especially true for the interbank exposures that provide the main channels for transmitting network cascades: even in jurisdictions where they are reported to central regulators, they comprise a diversity of different securities, including derivatives and swaps whose mark-to-market valuations fluctuate dramatically on intraday time-scales.

A third important reason is that a hypothetical financial system, with all balance sheets completely known, will be hit constantly by random shocks from the outside, stochastic world. A deterministic system, subjected to a generic random shock, becomes after only one cascade step a fully stochastic system.

For all these reasons, and more, the next chapters will consider the stochastic nature of financial networks, and the extent to which the large scale properties of cascades might be predictable from models of their stochastic properties. From now on in this book, our various contagion channels will usually take their dynamics within the framework of "random financial networks", defined provisionally as follows:

Definition 5. A *random financial network* or RFN is a random object representing a possible state of the financial network at an instant in time. It consists of three layers of mathematical structure. At the base structural level, the *skeleton* is a random directed graph $(\mathscr{N}, \mathscr{E})$ whose nodes $\mathscr{N}$ represent "banks" and whose directed edges or links $\mathscr{E}$ represent the presence of a non-negligible "interbank exposure" between a debtor bank and its creditor bank. Conditioned on a realization of the skeleton, the second structural layer is a collection of random *balance sheets*, one for each bank. In our simple models this is usually a coarse-grained description, listing for example the amounts $(\bar{\mathrm{Y}}_v, \bar{\mathrm{Z}}_v, \bar{\mathrm{D}}_v, \bar{\mathrm{X}}_v)$ for each $v \in \mathscr{N}$ as in Section 2.1.1. Finally, conditioned on a realization of the skeleton and balance sheets, the third level is a collection of positive random variables $\bar{\Omega}_\ell$ for each link $\ell = (w,v) \in \mathscr{E}$ that represent the *exposure* of w to v, that is, what v owes w. The interbank assets and liabilities are constrained to equal the aggregated exposures:

$$\bar{\mathrm{Z}}_v = \sum_w \bar{\Omega}_{wv}, \quad \bar{\mathrm{X}}_v = \sum_w \bar{\Omega}_{vw} \ . \tag{2.37}$$

Typically in cascade scenarios, we consider the RFN at the instant that a crisis triggering event unexpectedly occurs. We will combine the choice of RFN with the choice of a cascade mechanism such as the ones described in this chapter to describe what happens next. Depending on the cascade mechanism, only reduced balance sheet information in the form of buffer random variables $\bar{\Delta}_v$ and exposures $\bar{\Omega}_\ell$ is needed to follow the progress of the cascade. In that case, and we can work with a minimal parametrization of the RFN by the quadruple $(\mathscr{N}, \mathscr{E}, \bar{\Delta}, \bar{\Omega})$.

This schematic definition will prove to be acceptable for the description of simple contagion models. But more than that, it scales conceptually to much more complex settings. Nodes may have additional attributes or "types" beyond their connectivity to represent a more diverse class of economic entities. Links might have extended meaning where the random variables $\bar{\Omega}_\ell$ take vector values which represent different categories of exposures. We also recognize that even a very simple RFN is a complicated random variable of enormous dimension. Before proceeding to any analysis of network dynamics, the distributions of these collections of random variables must be fully specified. We will proceed stepwise, first focussing in the next chapter on characterizing possible models for the skeleton. In subsequent chapters we will consider how to specify the random structure of balance sheets and interbank exposures. Our optimistic view that something meaningful can be learned about systemic risk in the real world through the study of schematic or stylized RFNs is derived from our collective experience in other fields of complex disordered stochastic systems, rooted in the theory of statistical mechanics.

2.5 Bibliographic Notes

A seminal paper in 2000 by Allen and Gale [6] relates the structure of a stylized network of four banks to its resilience to default contagion, concluding that the complete network is the most resilient. The Eisenberg-Noe model [33] was originally intended to describe clearing mechanisms in payment systems or listed exchanges, but has since become a paradigm for modelling general financial systemic risk. The original paper dealt only with the case $\bar{\mathrm{Y}}_v \geq \bar{\mathrm{D}}_v$, and some of its main results were specific to this case. In two papers, [36] and [37], Elsinger et al applied this framework to the Austrian and UK networks and concluded that in the early 2000s these systems were rather resilient. Elsinger [35], Gourieroux et al [45] and Elliot et al [34] have studied model properties including uniqueness of clearing vectors in a more general setting that includes interbank equity crossholdings and multiple debt seniority layers. Rogers and Veraart [74] have explored the E-N model to understand when banks should cooperate to rescue a failing bank. Other authors have worked with models similar to the E-N framework aiming to understand the dependence between network connectivity and systemic resilience. In a simulation based study, Nier et al [72] showed that the resiliency of finite size networks can depend non-monotonically on key parameters. Glasserman and Young [43] and Acemoglu et al [1] also show that while for small initial shocks, connectivity improves stability, this relation can be reversed for large shocks. Upper, [80] gives an in-depth survey of 15 different papers that use simulation techniques to determine the possibility of contagion in interbank markets prior to 2007, concluding that none of them foresaw any indication of the upcoming crisis. Under the zero recovery assumption as in the G-K model, Amini et al [7] were able to prove asymptotic results on the cascade equilibrium in large random networks.

A fundamental treatment of the connection between funding liquidity and market liquidity by Brunnermeier and Pedersen [20] proposes a picture of how the funding constraints on a bank impact and are impacted by the liquidity of its market portfolio, that is its external assets. Funding liquidity cascades are a contagious, interbank version of the classic problem of bank runs that was studied by Diamond and Dybvig [31]. Despite a widespread opinion that funding liquidity cascades may be more important in real world crises, they have been less studied than the classic default cascade. Krishnamurthy [60] has investigated the systemic feedback due to both funding liquidity hoarding triggered by uncertainty, and asset fire sales. A paper by Minca and Amini [64] describes a number of network models for different channels for contagion, including funding liquidity cascades similar to those described in Section 2.2.

The justification for studying financial contagion on large random financial networks has been discussed by Gai and Kapadia [42], Amini, Cont and Minca [7],[64] and others.

Some recent papers have attempted to construct cascade mechanisms that effectively combine two or more contagion channels. The double cascade model for funding illiquidity and insolvency proposed by Hurd et al [53] unifies the assumptions of the Gai-Kapadia default model and the Gai-Kapadia liquidity model. Bookstaber [17] has developed an agent-based cascade modelling framework that incorporates versions of most of the contagion channels discussed in this chapter.

Chapter 3
Random Graph Models

Ye cannot live for yourselves; a thousand fibres connect you with your fellow-men, and along those fibres, as along sympathetic threads, run your actions as causes, and return to you as effects. [1]

Abstract The network of interbank counterparty relationships, or *skeleton*, is the random graph that acts as the medium through which financial contagion is propagated. The basic properties are developed for several promising families of random graph constructions including configuration graphs and inhomogeneous random graphs. A new extension, called the assortative configuration model, is proposed. The main results of this chapter are theorems describing the large graph asymptotics of this new assortative configuration model, including a proof of the locally tree-like property. Finally, measures of network structure are surveyed.

Keywords: Skeleton, counterparty network, configuration graph, inhomogeneous random graph, assortativity, random graph simulation, large graph asymptotics, network topology, locally tree-like.

Like people, banks are intensely social beings, and the "sympathetic threads" that connect banks transmit causes and effects just as do human connections. Our goal in this chapter is to explore models for the *skeleton* of a random financial network or RFN. At any moment in time, the skeleton is the random graph whose links indicate which pairs of nodes, representing banks, are considered to have a significant counterparty relationship. Depending on the context, links may be undirected lines or *directed* arrows that point from debtor to creditor.

Random graph theory is a general framework that can capture the most salient and basic attributes of macroscopic ensembles of partly connected similar entities that have been come to be called *complex adaptive systems*. Random graphs lie at the lowest structural layer of such systems and general categories of networks can be built by adding further layers of structure on top. The double adjective "complex adaptive" has taken on an additional higher meaning in recent years to describe the

[1] Rev. Henry Melvill, *Golden Lectures*, (London, 1855), p. 454

nature of a system whose totality is "more than the sum of its parts" and whose key responses and behaviours emerge in a way that cannot be anticipated from the microscopic interactions. The apt analogy is the way consciousness is presumed to emerge from the collective interaction of the brain's myriad neurons. The list of categories of complex adaptive systems in the real world is ever-growing, and questions about such systems reach beyond our depths of understanding. One might say that the theme of the present book is to use mathematics to explore the profound "complex adaptive" nature of the world's financial systems, using the tools of network science.

3.1 Definitions and Basic Results

In this section, we provide some standard graph theoretic definitions and develop an efficient notation for what will follow. Since in this book we are most interested in directed graphs rather than undirected graphs, our definitions are in that setting and the term "graph" will have that meaning. Undirected graphs fit in easily as a subcategory of the directed case.

Definition 6. 1. $g \in \mathscr{G}(N)$, a *graph* on N nodes, is a pair $(\mathscr{N}, \mathscr{E})$ where the set of *nodes* $\mathscr{N}$ is numbered by integers, $\mathscr{N} = \{1, \ldots, N\} := [N]$ and the set of *edges* is a subset $\mathscr{E} \subset \mathscr{N} \times \mathscr{N}$. Each edge or *link* $\ell \in \mathscr{E}$ is an ordered pair $\ell = (v, w)$, often labelled by integers $\ell \in \{1, \ldots, E\} := [E]$ where $E = |\mathscr{E}|$. Normally, "self-edges" with $v = w$ are excluded from $\mathscr{E}$, that is, $\mathscr{E} \subset \mathscr{N} \times \mathscr{N} \setminus \text{diag}$. For any $N \geq 1$, the collection of *directed graphs* on N nodes is denoted $\mathscr{G}(N)$.

2. A given graph $g = (\mathscr{N}, \mathscr{E}) \in \mathscr{G}(N)$ can be represented by its $N \times N$ *adjacency matrix* $M(g)$ with components

$$M_{vw}(g) = \mathbf{1}((v,w) \in g) = \begin{cases} 1 \text{ if } (v,w) \in g \\ 0 \text{ if } (v,w) \in \mathscr{N} \times \mathscr{N} \setminus g \end{cases} .$$

3. The *in-degree* $\deg^-(v)$ and *out-degree* $\deg^+(v)$ of a node v are

$$\deg^-(v) = \sum_w M_{wv}(g), \quad \deg^+(v) = \sum_w M_{vw}(g) .$$

A node $v \in \mathscr{N}$ has *node type* (j,k) when its in-degree and out-degree are $\deg^-(v) = j, \deg^+(v) = k$; the node set partitions into node types, $\mathscr{N} = \cup_{jk} \mathscr{N}_{jk}$. We shall write $k_v = k, j_v = j$ for any $v \in \mathscr{N}_{jk}$.

4. An edge $\ell = (v,w) \in \mathscr{E} = \cup_{kj} \mathscr{E}_{kj}$ has *edge type* (k, j) with in-degree j and out-degree k if it is an out-edge of a node v with out-degree $k_v = k$ and an in-edge of a node w with in-degree $j_w = j$. We shall write $\deg^+(\ell) = k_\ell = k$ and $\deg^-(\ell) = j_\ell = j$ whenever $\ell \in \mathscr{E}_{kj}$.

5. For completeness, an undirected graph is defined to be any directed graph g for which $M(g)$ is symmetric.

Remark 2. We use the term "type" for nodes and edges to denote characteristics that potentially extend beyond degree characteristics to include other characteristics such as the size, kind and location of a bank, or type of security. In this book, we narrow the definition of edge-type of $\ell = (v,w)$ to the pair (k_v, j_w) rather than the quadruple $(j_v, k_v; j_w, k_w)$ which may seem more natural: this is a choice that can be relaxed without difficulty and is made purely to reduce model complexity.

The standard visualization of a directed graph g on N nodes is to plot nodes as "dots" with labels $v \in \mathscr{N}$, and any edge (v,w) as an arrow pointing "downstream" from node v to node w. In our systemic risk application, such an arrow signifies that bank v is a debtor of bank w and the in-degree $\deg^-(w)$ is the number of banks in debt to w. In other words the existence of the edge (v,w) means "v owes w". Figure 3.1 illustrates the labelling of types of nodes and edges.

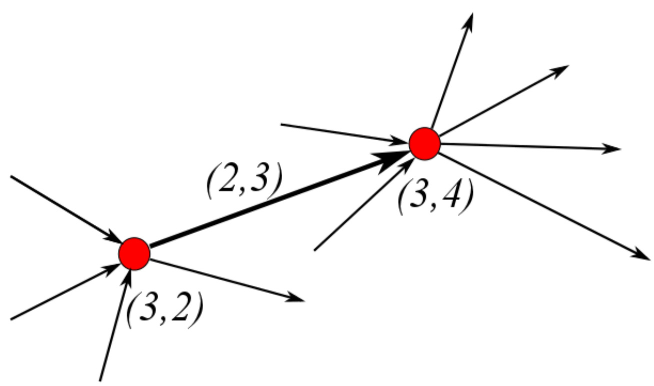


Fig. 3.1 A type $(3,2)$ debtor bank that owes to a type $(3,4)$ creditor bank through a type $(2,3)$ link.

There are constraints on the collections of node type $(j_v, k_v)_{v\in\mathscr{N}}$ and edge type $(k_\ell, j_\ell)_{\ell\in\mathscr{E}}$ if they derive from a graph. If we compute the total number of edges $E = |\mathscr{E}|$, the number of edges with $k_\ell = k$ and the number of edges with $j_\ell = j$ we find three conditions:

$$\begin{aligned} E &:= |\mathscr{E}| = \sum_v k_v = \sum_v j_v \\ e_k^+ &:= |\mathscr{E} \cap \{k_\ell = k\}| = \sum_\ell \mathbf{1}(k_\ell = k) = \sum_v k\mathbf{1}(k_v = k) \\ e_j^- &:= |\mathscr{E} \cap \{j_\ell = j\}| = \sum_\ell \mathbf{1}(j_\ell = j) = \sum_v j\mathbf{1}(j_v = j)\ . \end{aligned} \tag{3.1}$$

It is useful to define further graph theoretic objects and notation in terms of the adjacency matrix $M(g)$:

1. The *in-neighbourhood* of a node v is the set $\mathscr{N}_v^- := \{w \in \mathscr{N} | M_{wv}(g) = 1\}$ and the *out-neighbourhood* of v is the set $\mathscr{N}_v^+ := \{w \in \mathscr{N} | M_{vw}(g) = 1\}$.

2. $\mathscr{E}_v^+$ (or $\mathscr{E}_v^-$) denotes the set of out-edges (respectively, in-edges) of a given node v and v_ℓ^+ (or v_ℓ^-) denotes the node for which ℓ is an out-edge (respectively, in-edge).
3. Similarly, we have second-order neighbourhoods $\mathscr{N}_v^{--}, \mathscr{N}_v^{-+}, \mathscr{N}_v^{+-}, \mathscr{N}_v^{++}$ with the obvious definitions. Second and higher order neighbours can be determined directly from the powers of M and $M^\top$. For example, $w \in \mathscr{N}_v^{-+}$ whenever $(M^\top M)_{wv} \geq 1$.
4. We usually write j, j', j'', j_1, etc. to refer to in-degrees while k, k', k'', k_1, etc. refer to out-degrees.

Our financial network models typically have a sparse adjacency matrix $M(g)$ when N is large, meaning that the number of edges is a small fraction of the $N(N-1)$ potential edges. This reflects the fact that bank counterparty relationships are expensive to build and maintain, and thus $\mathscr{N}_v^+$ and $\mathscr{N}_v^-$ typically contain relatively few nodes even in a very large network.

3.1.1 Random Graphs

Random graphs are simply probability distributions on the sets $\mathscr{G}(N)$:

Definition 7. 1. A random graph of size N is a probability distribution $\mathbb{P}$ on the finite set $\mathscr{G}(N)$. When the size N is itself random, the probability distribution $\mathbb{P}$ is on the countably infinite set $\mathscr{G} := \cup_N \mathscr{G}(N)$. Normally, we also suppose that $\mathbb{P}$ is invariant under permutations of the N node labels.
2. For random graphs, we define the *node-type distribution* to have probabilities $P_{jk} = \mathbb{P}[v \in \mathscr{N}_{jk}]$ and the *edge-type distribution* to have probabilities $Q_{kj} = \mathbb{P}[\ell \in \mathscr{E}_{kj}]$.

P and Q can be viewed as bivariate distributions on the natural numbers, with marginals $P_k^+ = \sum_j P_{jk}, P_j^- = \sum_k P_{jk}$ and $Q_k^+ = \sum_j Q_{kj}, Q_j^- = \sum_k Q_{kj}$. Because they derive from actual graphs, edge and node type distributions must satisfy the following consistency conditions:

$$z := \sum_{jk} jP_{jk} = \sum_{jk} kP_{jk}; \quad Q_k^+ = kP_k^+/z, \quad Q_j^- = jP_j^-/z \quad \forall k, j \,. \tag{3.2}$$

Note that the mean in-degree and mean out-degree are both equal to z.

A number of random graph construction algorithms have been proposed in the literature, motivated by the need to create tractable families of graphs that match the types and measures of network topology observed in nature and society. The remainder of this chapter reviews the properties of the random graph constructions that are most closely related to the types of networks observed in financial systems. The textbook “Random Graphs and Complex Networks” by van der Hofstad [81] provides a complete and up-to-date review of the entire subject.

In what follows, asymptotic results for sequences of random graphs labelled by their size N are typically expressed in terms of convergence of random variables in probability, defined as:

Definition 8. A sequence $\{X_n\}_{n\geq 1}$ of random variables is said to *converge in probability* to a random variable X, written $\lim_{n\to\infty} X_n \stackrel{P}{=} X$ or $X_n \stackrel{P}{\longrightarrow} X$, if for any $\varepsilon > 0$

$$\mathbb{P}[|X_n - X| > \varepsilon] \to 0 \ .$$

We also recall further standard notation for asymptotics of sequences of real numbers $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ and random variables $\{X_n\}_{n\geq 1}$:

1. Landau's "little oh": $x_n = o(1)$ means $x_n \to 0$; $x_n = o(y_n)$ means $x_n/y_n = o(1)$;
2. Landau's "big oh": $x_n = O(y_n)$ means there is $N > 0$ such that x_n/y_n is bounded for $n \geq N$;
3. $x_n \sim y_n$ means $x_n/y_n \to 1$;
4. $X_n = o(y_n)$ means $X_n/y_n \stackrel{P}{\longrightarrow} 0$.
5. "w.h.p" means "with high probability, converging to 1".

3.2 Configuration Random Graphs

In their classic paper [38], Erdös and Renyi introduced the undirected model $G(N,M)$ that consists of N nodes and a random subset of exactly M edges chosen uniformly from the collection of $\binom{N}{M}$ possible such edge subsets. This model can be regarded as the Mth step of a random graph process that starts with N nodes and no edges, and adds edges one at a time selected uniformly randomly from the set of available undirected edges. Gilbert's random graph model $G(N,p)$, which takes N nodes and selects each possible edge independently with probability $p = z/(N-1)$, has mean degree z and similar large N asymptotics provided $M = zN/2$. In fact, it was proved by [15] and [67] that the undirected Erdös-Renyi graph $G(N,zN/2)$ and $G(N,p_N)$ with probability $p_N = z/(N-1)$ both converge in probability to the same model as $N \to \infty$ for all $z \in \mathbb{R}_+$. Because of their popularity, the two models $G(N,p) \sim G(N,zN/2)$ have come to be known as "the" random graph or the *Poisson graph* model since the degree distribution of $G(N,p)$ is $\mathrm{Bin}(N-1,p) \sim_{N\to\infty} \mathrm{Pois}(z)$.

Both the above constructions have obvious directed graph analogues: henceforth we use the notation $G(N,M)$ and $G(N,p)$ to denote the directed graph models. In the directed Gilbert $G(N,p)$ model, each possible directed edge selection is an independent Bernoulli trial, and thus the adjacency matrix $M(g)$ is easy to simulate in Matlab:

```
M=( rand(N,N) < p );
diag(M)=0;
```

Another undirected graph construction of interest is the random $r-$regular model with $r \geq 3$, that draws uniformly from the set of r-regular graphs on N nodes, that is, graphs for which each node has degree exactly r. This model is a particular case of a general class called "undirected configuration graphs" which takes as data an arbitrary degree distribution $P = \{P_k\}$; similarly "directed configuration graphs" take as data an arbitrary bi-degree distribution $P = \{P_{jk}\}$. When $P \sim \text{Pois}(z) \times \text{Pois}(z)$ the configuration graph turns out to be asymptotic for large N to both the directed Erdös-Renyi and Gilbert models.

The well known directed configuration multigraph model introduced by Bollobas [14] with general degree distribution $P = \{P_{jk}\}_{j,k=0,1,...}$ and size N is constructed by the following random algorithm:

1. Draw a sequence of N node-degree pairs $(j_1,k_1),\ldots,(j_N,k_N)$ independently from P, and accept the draw if and only if it is feasible, i.e. $\sum_{n\in[N]}(j_n - k_n) = 0$. Label the nth node with k_n *out-stubs* (picture this as a half-edge with an out-arrow) and j_n *in-stubs*.
2. While there remain available unpaired stubs, select (according to any rule, whether random or deterministic) any unpaired out-stub and pair it with an in-stub selected uniformly amongst unpaired in-stubs. Each resulting pair of stubs is a directed edge of the multigraph.

The algorithm leads to objects with self-loops and multiple edges which are usually called multigraphs rather than graphs. Only "simple" multigraphs, those that are free of self-loops and multiple edges, are considered to be graphs. For the most part, we do not care over much about the distinction, because the density of self-loops and multiple edges goes to zero as $N \to \infty$. In fact, Janson [55] has proved in the undirected case that the probability for a multigraph to be simple is bounded away from zero for well-behaved sequences $(g_N)_{N>0}$ of size N graphs with given P.

Exact simulation of the adjacency matrix in the general configuration model is problematic because the feasibility condition met in the first step above occurs only with vanishingly small asymptotic frequency $\sim \frac{\sigma}{\sqrt{2\pi N}}$. For this reason, practical Monte Carlo implementations use some type of rewiring or *clipping* to adjust each infeasible draw of node-degree pairs. We shall return to address this issue in Section 3.3.3.

3.3 Assortative Configuration Graphs

Because of the uniformity of the matching in step 2 of the configuration graph construction, the edge-type distribution of the resultant random graph is

$$Q_{kj} = \frac{jkP_k^+P_j^-}{z^2} = Q_k^+Q_j^- \tag{3.3}$$

which we call the *independent edge condition*. It has been observed that (3.3) is not true in financial networks. Rather, as noted in [78], [11] and [27], small banks have a tendency to choose large banks as counterparties, a property known as *disassortativity*, or as we prefer, *negative assortativity*. Therefore we wish to construct a class of *assortative configuration graphs* (ACG), which encompasses all reasonable type distributions (P,Q) and has special properties that make it suitable for exact analytical results.

3.3.1 The ACG Construction

The assortative configuration (multi-)graph of size N parametrized by the node-edge degree distribution pair (P,Q) that satisfy the consistency conditions (3.2) is defined by the following random algorithm:

1. Draw a sequence of N node-type pairs $X = ((j_1,k_1),\dots,(j_N,k_N))$ independently from P, and accept the draw if and only if it is feasible, i.e. $\sum_{v\in[N]} j_v = \sum_{v\in[N]} k_v$, and this defines the number of edges E that will result. Label the vth node with k_v *out-stubs* (each out-stub is a half-edge with an out-arrow, labelled by its degree k_v) and j_v *in-stubs*, labelled by their degree j_v. Define the partial sums $u^-_j = \sum_v \mathbf{1}(j_v = j), u^+_k = \sum_v \mathbf{1}(k_v = k), u_{jk} = \sum_v \mathbf{1}(j_v = j, k_v = k)$, the number $e^+_k = ku^+_k$ of *k-stubs* (out-stubs of degree k) and the number of *j-stubs* (in-stubs of degree j), $e^-_j = ju^-_j$.
2. Conditioned on X, the result of Step 1, choose an arbitrary ordering ℓ^- and ℓ^+ of the E in-stubs and E out-stubs. The matching sequence, or "wiring", W of edges is selected by choosing a pair of permutations $\sigma,\tilde{\sigma} \in S(E)$ of the set $[E]$. This determines the edge sequence $\ell = (\ell^- = \sigma(\ell), \ell^+ = \tilde{\sigma}(\ell))$ labelled by $\ell \in [E]$, to which is attached a probability weighting factor
$$\prod_{\ell\in[E]} Q_{k_{\sigma(\ell)} j_{\tilde{\sigma}(\ell)}} \ . \tag{3.4}$$

Given the wiring W determined in Step 2, the number of type (k,j) edges is
$$e_{kj} = e_{kj}(W) = \sum_{\ell\in[E]} \mathbf{1}(k_{\tilde{\sigma}(\ell)} = k,\ j_{\sigma(\ell)} = j) \ . \tag{3.5}$$

These numbers are constrained by the e^+_k, e^-_j determined by Step 1:
$$e^+_k = \sum_j e_{kj}, \quad e^-_j = \sum_k e_{kj}, \quad E = \sum_{kj} e_{kj} \ . \tag{3.6}$$

Intuitively, since Step 1 leads to a product probability measure subject to a single linear constraint that is true in expectation, one expects that it will lead to the independence of node degrees for large N, with the probability P. Similar logic suggests that since the matching weights in Step 2 define a product probability measure conditional on a set of linear constraints that are true in expectation, it should lead to edge degree independence in the large N limit, with the limiting probabilities given by Q. This logic turns out to be true, as proved in a recent paper [49]. In this section, we explore the highlights of the ACG results proved in that paper, but omit the proofs given there.

First, the paper analyzes certain combinatorial properties of the wiring algorithm of Step 2, conditioned on the node-type sequence X resulting from Step 1 for a finite N. To see how this goes, define two sequences $e^-_j(m), e^+_k(m)$ for $0 \le m \le E$ to be the number of available j-stubs and k-stubs available after m wiring steps.

Proposition 1 (Propositions 1 and 2, [49]). *Consider Step 2 of the assortative configuration graph construction for finite N with probabilities P, Q conditioned on the* $X = (j_i, k_i), i \in [N]$.

1. *The conditional probability of any wiring sequence* $W = (\ell \in [E])$ *is:*
$$\mathbb{P}[W|X] = C^{-1} \prod_{kj} (Q_{kj})^{e_{kj}(W)} , \tag{3.7}$$
$$C = C(e^-, e^+) = E! \sum_e \prod_{kj} \frac{(Q_{kj})^{e_{kj}}}{e_{kj}!} \prod_j \left(e^-_j !\right) \prod_k \left(e^+_k !\right) , \tag{3.8}$$
where the sum in (3.8) *is over collections* $e = (e_{kj})$ *satisfying the constraints* (3.6).
2. *The conditional probability of the first edge of the wiring sequence* $W = (\ell \in [E])$ *having type* k_1, j_1 *is*
$$\mathbb{P}[(k_1, j_1)|X] = \mathbb{E}[e_{k_1 j_1}|X]/E . \tag{3.9}$$
3. *The conditional probability of the first M edges of the wiring sequence* $W = (\ell \in [E])$ *having types* $(k_i, j_i)_{i\in[M]}$ *is*
$$\mathbb{P}[(k_i, j_i)_{i\in[M]}|X] = \frac{(E-M)!}{E!} \prod_{i\in[M]} \mathbb{E}[e_{k_i j_i}|e^-(i-1), e^+(i-1)] . \tag{3.10}$$

One can see from part 3 of this Proposition that the probability distribution of the first M edge types will be given asymptotically by $\prod_{i\in[M]} Q_{k_i j_i}$ provided our intuition is correct that $\mathbb{E}[E^{-1} e_{kj}] \stackrel{\mathrm{P}}{=} Q_{kj}(1 + o(1))$ asymptotically for large N. To keep the discussion as clear as possible, we confine the analysis to the case the distributions P and Q have support on the finite set $(j,k) \in \{0, 1, \ldots, K\}^2$. Then it turns out one can rigorously apply the Laplace asymptotic method to the cumulant generating function for the empirical edge-type random variables e_{kj}, conditioned on any feasible collection of (e^+_k, e^-_j) with total number $E = \sum_k e^+_k = \sum_j e^-_j$:

$$F(v; e^-, e^+) := \log \mathbb{E}[e^{\sum_{kj} v_{kj} e_{kj}}|e^-, e^+], \quad \forall v = (v_{kj}) \tag{3.11}$$
$$= \log \frac{\sum_e \prod_{kj} \frac{(Q_{kj} e^{v_{kj}})^{e_{kj}}}{e_{kj}!} \prod_j \left(e^-_j !\right) \prod_k \left(e^+_k !\right)}{\sum_e \prod_{kj} \frac{(Q_{kj})^{e_{kj}}}{e_{kj}!} \prod_j \left(e^-_j !\right) \prod_k \left(e^+_k !\right)} , \tag{3.12}$$

The constraints on (e_{kj}) can be introduced by auxiliary integrations over variables u^-_j, u^+_k of the form

$$2\pi\mathbf{1}(\sum_j e_{kj} = e_k^+) = \int_0^{2\pi} du_k^+ \, e^{i\tilde{u}_k(\Sigma_j e_{kj} - e_k^+)} \, .$$

This substitution leads to closed formulas for the sums over e_{kj}, and the expression for e^F:

$$\frac{\int_I d^{2K}u \exp[H(v,-iu;e)]}{\int_I d^{2K}u \ \exp[H(0,-iu;e)]} \tag{3.13}$$

where

$$H(v,\alpha;e) = \sum_{kj} e^{(\alpha_j^- + \alpha_k^+)} e^{v_{kj}} Q_{kj} - (\sum_j \alpha_j^- e_j^- + \sum_k \alpha_k^+ e_k^+) \, . \tag{3.14}$$

The integration in (3.13) is over the set $I := [0,2\pi]^{2K}$.

The Laplace asymptotic method (or saddlepoint method), reviewed for example in [48], involves shifting the u integration in (3.13) into the complex by an imaginary vector. The Cauchy Theorem, combined with the periodicity of the integrand in u, ensures the value of the integral is unchanged under the shift. The desired shift is determined by the e-dependent critical points $\alpha^* = (\alpha_j^-, \alpha_k^+)_{j,k\le K} \in \mathbb{R}^{2K}$ of H which are solutions of the system of equations

$$\sum_k e^{(\alpha_j^- + \alpha_k^+)} Q_{kj} = e_j^- \quad \forall j \tag{3.15}$$

$$\sum_j e^{(\alpha_j^- + \alpha_k^+)} Q_{kj} = e_k^+ \quad \forall k \tag{3.16}$$

The main theorem of [49] confirms that the large N asymptotics of the empirical node- and edge-type distributions agree with the target (P,Q) distributions.

Theorem 3 (Corollary 3.3, [49]). *Consider the ACG model with (P,Q) supported on $\{0,1,\ldots,K\}^2$.*

1. *Conditioned on X,*

$$E^{-1}e_{kj} \stackrel{\mathrm{P}}{=} [Q_{kj}e^{1-H(0,\alpha^*(x);x)-(\alpha_j^-+\alpha_k^+)}[1+O(E^{-1/2})]$$

   where $x = E^{-1}e$ and $e = (e^-(X), e^+(X))$.

2. *Unconditionally,*

$$N^{-1}u_{jk} \stackrel{\mathrm{P}}{=} P_{jk}[1+O(N^{-1/2})]$$
$$E^{-1}e_{kj} \stackrel{\mathrm{P}}{=} Q_{kj}[1+O(N^{-1/2})] \, .$$

3.3.2 Locally Tree-like Property

To understand percolation theory on random graphs, or to derive a rigorous treatment of cascade mappings on random financial networks, it turns out to be important that the underlying random graph model have a property sometimes called "locally tree-like". In this section, the local tree-like property of the ACG model will be characterized as a particular large N property of the probability distributions associated with *configurations*, that is, finite connected subgraphs g of the skeleton labelled by their degree types.

First consider what it means in the (P,Q) ACG model with size N to draw a random configuration g consisting of a pair of vertices v_1, v_2 joined by a link, that is, $v_2 \in \mathcal{N}^-_{v_1}$. In view of the permutation symmetry of the ACG algorithm, the random link can without loss of generality be taken to be the first link $W(1)$ of the wiring sequence W. Following the ACG algorithm, Step 1 constructs a feasible node degree sequence $X = (j_v, k_v), v \in [N]$, and conditioned on X, Step 2 constructs a random Q-wiring sequence $W = \big(\ell = (v^+_\ell, v^-_\ell)\big)_{\ell\in[E]}$ with $E = \sum_v k_v = \sum_v j_v$ edges. By an abuse of notation, we label their edge degrees by $k_\ell = k_{v^+_\ell}, j_\ell = j_{v^-_\ell}$ for $\ell \in [E]$. The configuration event in question, namely that the first link in the wiring sequence W attaches to nodes of the required degrees $(j_1,k_1),(j_2,k_2)$, has conditional probability $p = \mathbb{P}[v_i \in \mathcal{N}_{j_i,k_i}, i = 1,2|v_2 \in \mathcal{N}^-_{v_1}, X]$. To compute this, note that the fraction $j_1 u_{j_1k_1}/e^-_{j_1}$ of available j_1-stubs come from a j_1k_1 node and the fraction $k_2 u_{j_2k_2}/e^+_{k_2}$ available k_2-stubs come from a j_2k_2 node. Combining this fact with Part 2 of Proposition 1, equation (3.9) implies the *exact* configuration probability conditioned on X is

$$p = j_1 u_{j_1k_1} k_2 u_{j_2k_2} \, \frac{\mathbb{E}[e_{k_2 j_1}|e^-, e^+]}{E e^+_{k_2} e^-_{j_1}} \, . \tag{3.17}$$

By Theorem 3,

$$p \overset{\mathrm{P}}{=} \frac{j_1 k_2 P_{j_1k_1} P_{j_2k_2} Q_{k_2 j_1}}{z^2 Q^+_{k_2} Q^-_{j_1}} [1 + O(N^{-1/2})] \, . \tag{3.18}$$

This argument justifies the following informal computation of the correct asymptotic expression for p by successive conditioning:

$$\begin{aligned} p &= \mathbb{P}[v_i \in \mathcal{N}_{j_ik_i}, i = 1,2 \,\big|\, v_2 \in \mathcal{N}^-_{v_1}] & (3.19)\\ &= \mathbb{P}[v_1 \in \mathcal{N}_{j_1k_1} \,\big|\, v_2 \in \mathcal{N}^-_{v_1} \cap \mathcal{N}_{j_2k_2}]\, \mathbb{P}[v_2 \in \mathcal{N}_{j_2k_2} \,\big|\, v_2 \in \mathcal{N}^-_{v_1}] & (3.20)\\ &= (P_{k_1|j_1} Q_{j_1|k_2})(P_{j_2|k_2} Q^+_{k_2}) = \frac{P_{j_1k_1} P_{j_2k_2} Q_{k_2 j_1}}{P^+_{k_2} P^-_{j_1}} & (3.21) \end{aligned}$$

where we introduce conditional degree probabilities $P_{k|j} = P_{jk}/P^-_j$ etc.

Occasionally in the above matching algorithm, the first edge forms a self-loop, i.e. $v_1 = v_2$. The probability of this event, jointly with fixing the degree of v_1, can be computed exactly for finite N as follows:

$$\tilde{p} := \mathbb{E}[v_1 = v_2, v_1 \in \mathcal{N}_{jk} | v_2 \in \mathcal{N}^-_{v_1}, X] = \left(\frac{jk u_{jk}}{e^-_j e^+_k}\right) \frac{\mathbb{E}[e_{kj}|X]}{E} \, .$$

Again by Theorem 3, as $N \to \infty$ this goes to zero, while $N\tilde{p}$ approaches a finite value:

$$N\tilde{p} \xrightarrow{\mathrm{P}} \frac{jkP_{jk}Q_{kj}}{z^2 Q_k^+ Q_j^-} \tag{3.22}$$

which says that the relative fraction of edges being self loops is the asymptotically small $\sum_{jk} \frac{jkP_{jk}Q_{kj}}{Nz^2 Q_k^+ Q_j^-}$. In fact, following results of [55] and others on the undirected configuration model, one expects that the total number of self loops in the multi-graph converges in probability to a Poisson random variable with finite parameter

$$\lambda = \sum_{jk} \frac{jkP_{jk}Q_{kj}}{z^2 Q_k^+ Q_j^-} \ . \tag{3.23}$$

A general *configuration* is a connected subgraph h of an ACG graph $(\mathscr{N}, \mathscr{E})$ with L ordered edges and each node labelled by its degree type. It results from a growth process that starts from a fixed node w_0 called the root and at step $\ell \le L$ adds one edge ℓ that connects a node w_ℓ to a specific existing node w'_ℓ. The following is a precise definition:

Definition 9. A configuration rooted to a node w_0 with degree $(j,k) := (j_0, k_0)$ is a connected subgraph h consisting of a sequence of L edges that connect nodes $(w_\ell)_{\ell \in [L]}$ of types (j_ℓ, k_ℓ), subject to the following condition: For each $\ell \ge 1$, w_ℓ is connected by the edge labelled with ℓ to a node $w'_\ell \in \{w_j\}_{j \in \{0\} \cup [\ell-1]}$ by either an in-edge (that points into w'_ℓ) (w_ℓ, w'_ℓ) or an out-edge (w'_ℓ, w_ℓ).

A random realization of the configuration results when the construction of the size N ACG graph $(\mathscr{N}, \mathscr{E})$ is conditioned on X arising from Step 1 and the first L edges of the wiring sequence of Step 2. The problem is to compute the probability of the node degree sequence $(j_\ell, k_\ell)_{\ell \in [L]}$ conditioned on X, the graph h and the root degree (j,k), that is

$$p = \mathbb{P}[w_\ell \in \mathscr{N}_{j_\ell, k_\ell}, \ell \in [L] | w_0 \in \mathscr{N}_{jk}, h, X] \ . \tag{3.24}$$

Note that there is no condition that the node w_ℓ at step ℓ is distinct from the earlier nodes $w_{\ell'}, \ell' \in \{0\} \cup [\ell-1]$. With high probability each w_ℓ will be new, and the resultant subgraph h will be a tree with L distinct added nodes (not including the root) and L edges. With small probability one or more of the w_ℓ will be preexisting, i.e. equal to $w_{\ell'}$ for some $\ell' \in \{0\} \cup [\ell-1]$: in this case the subgraph h will have $M < L$ added nodes, will have cycles and not be a tree.

The following sequences of numbers are determined given X and h:

- $e_{j,k}(\ell)$ is the number of j-stubs connected to (j,k) nodes available after ℓ wiring steps;
- $e_{k,j}(\ell)$ is the number of k-stubs connected to (j,k) nodes available after ℓ wiring steps.
- $e_j^-(\ell) := \sum_k e_{j,k}(\ell)$ and $e_k^+(\ell) := \sum_j e_{k,j}(\ell)$ are the number of j-stubs and k-stubs respectively available after ℓ wiring steps.

Note that $e_{j,k}(0) = ju_{jk}$ and $e_{k,j}(0) = ku_{jk}$, and both decrease by at most 1 at each step. The analysis of configuration probabilities that follows is inductive on the step ℓ.

Theorem 4 (Theorem 4.1, [49]). *Consider the ACG sequence with (P,Q) supported on $\{0,1,\ldots,K\}^2$. Let h be any fixed finite configuration rooted to $w_0 \in \mathcal{N}_{jk}$, with M added nodes and $L \geq M$ edges, labelled by the node-type sequence $(j_m,k_m)_{m\in[M]}$. Then, as $N \to \infty$, the joint conditional probability $p = \mathbb{P}[w_m \in \mathcal{N}_{j_m k_m}, m \in [M] | w_0 \in \mathcal{N}_{jk}, h, X]$ is given by*

$$p \stackrel{\mathrm{P}}{=} \prod_{m\in[M],\ \textit{out-edge}} P_{k_m|j_m} Q_{j_m|k_{m'}} \prod_{m\in[M],\ \textit{in-edge}} P_{j_m|k_m} Q_{k_m|j_{m'}} \left[1 + O(N^{-1/2})\right] \tag{3.25}$$

if h is a tree and

$$O(N^{M-L}) \ . \tag{3.26}$$

if h has cycles. For trees, the ℓth edge has $m = \ell$, and $m' \in \{0\} \cup [\ell - 1]$ numbers the node to which w_ℓ attaches.

Remarks 1.

1. Formula (3.26) shows clearly what is meant by saying that configuration graphs are *locally tree-like* as $N \to \infty$. It means the number of occurrences of any fixed finite size graph h with cycles embedded within a configuration graph of size N remains bounded with high probability as $N \to \infty$.
2. Even more interesting is that (3.25) shows that large configuration graphs exhibit a strict type of conditional independence. Selection of any root node w_0 of the tree graph h splits it into two (possibly empty) trees h_1, h_2 with node-types $(j_m,k_m), m \in [M_1]$ and $(j_m,k_m), m \in [M_1 + M_2] \setminus [M_1]$ where $M = M_1 + M_2$. When we condition on the node-type of w_0, (3.25) shows that the remaining node-types form independent families:

$$\begin{aligned} &\mathbb{P}[w_m \in \mathcal{N}_{j_m k_m}, m \in [M], h \big| X, w_0 \in \mathcal{N}_{jk}] \\ &\quad = \mathbb{P}[w_m \in \mathcal{N}_{j_m k_m}, m \in [M_1], h_1 \big| X, w_0 \in \mathcal{N}_{jk}] \\ &\quad \times \mathbb{P}[w_m \in \mathcal{N}_{j_m k_m}, m \in [M_1 + M_2] \setminus [M_1], h_2 \big| X, w_0 \in \mathcal{N}_{jk}] \ . \end{aligned} \tag{3.27}$$

   We call this deep property of the general configuration graph the *locally tree-like independence property* (LTI property). In [50], the LTI property provides the key to unravelling cascade dynamics in large configuration graphs.

Proof of Theorem 4: First, suppose Step 1 generates the node-type sequence X. Conditioned on X, now suppose the first step generates an in-edge (w_1, w_0). Then, a refinement of the argument leading to Part 2 of Proposition 1 shows that the conditional probability $\mathbb{P}[w_1 \in \mathcal{N}_{j_1 k_1} | w_0 \in \mathcal{N}_{jk}, h, X]$ can be written

$$\frac{\mathbb{P}[w_1 \in \mathcal{N}_{j_1 k_1}, w_0 \in \mathcal{N}_{jk} | h, X]}{\mathbb{P}[w_0 \in \mathcal{N}_{jk} | h, X]} = \left(\frac{k_1 u_{k_1, j_1}}{k_1 u^+_{k_1}}\right) \left(\frac{\mathbb{E}[e_{k_1 j} | e^-(0), e^+(0)]}{e^-_j(0)}\right) \ . \tag{3.28}$$

Now, for $N \to \infty$, Part 2 of Theorem 3 applies and shows that for the case of an in-edge on the first step, with high probability, X is such that:

$$\mathbb{P}[w_1 \in \mathscr{N}_{j_1 k_1} | w_0 \in \mathscr{N}_{jk}, h, X] \stackrel{\mathrm{P}}{=} P_{j_1|k_1}\ Q_{k_1|j}\left[1 + O(N^{-1/2})\right] .$$

The case of an out-edge is similar.

Now we continue conditionally on X from Step 1, assume inductively that (3.25) is true for $M-1$, and prove it for M. Suppose the final node w_M is in-connected to the node $w_{M'}$ for some $M' \le M$. The ratio $\mathbb{P}[w_m \in \mathscr{N}_{j_m k_m}, m \in [M] | v \in \mathscr{N}_{jk}, h, X]/\mathbb{P}[w_m \in \mathscr{N}_{j_m k_m}, m \in [M-1] | w_0 \in \mathscr{N}_{jk}, h, X]$ can be treated just as in the previous step and shown to be

$$\left(\frac{e_{k_M, j_M}(M-1)}{e^+_{k_M}(M-1)}\right)\left(\frac{\mathbb{E}[e_{k_M j_{M'}} | e^-(M-1), e^+(M-1)]}{e^-_{j_{M'}}(M-1)}\right)$$

which with high probability equals $P_{j_M|k_M}\ Q_{k_M|j_{M'}}\left[1 + O(N^{-1/2})\right]$. The case w_M is out-connected to the node $w_{M'}$ is similar.

The first step m that a cycle is formed can be treated by imposing a condition that $w_m = w_{m''}$ for some fixed $m'' < m$. One finds that the conditional probability of this is

$$\begin{aligned}
&\mathbb{P}[w_m = w_{m''}, w_\ell \in \mathscr{N}_{j_\ell k_\ell}, \ell \in [m-1] | w_0 \in \mathscr{N}_{jk}, h, X] \\
&\qquad = \frac{k_{m''}}{e^+_{k_{m''}}(m-1)} \times \mathbb{P}[w_\ell \in \mathscr{N}_{j_\ell k_\ell}, \ell \in [m-1] | w_0 \in \mathscr{N}_{jk}, h, X] .
\end{aligned}$$

The first factor is $O(N^{-1})$ as $N \to \infty$, which proves the desired statement (3.26) for cycles.

Finally, since (3.26) is true for cycles, with high probability all finite configurations are trees. Therefore their asymptotic probability laws are given by (3.25), as required. □

3.3.3 Approximate ACG Simulation

It was observed in Section 3.2 that Step 1 of the configuration graph construction draws a sequence $(j_v, k_v)_{v\in[N]}$ of node types that is iid with the correct distribution P, but is only feasible, $\sum_v (k_v - j_v) = 0$, with small probability. Step 2 of the exact ACG algorithm in Section 3.3.1 requires is even less feasible in practice. Practical simulation algorithms address the first problem by "clipping" the drawn node bidegree sequence when the discrepancy $D = D_N := \sum_v (k_v - j_v)$ is not too large, meaning it is adjusted by a small amount to make it feasible, without making a large change in the joint distribution. Step 1 of the following simulation algorithm generalizes slightly the method introduced by [23] who verify that the effect of clipping vanishes with high probability as $N \to \infty$. The difficulty with Step 2 of the ACG construction is overcome by an approximate sequential wiring algorithm.

The *approximate assortative configuration simulation algorithm* for multigraphs of size N, parametrized by the node-edge degree distribution pair (P,Q) that have support on the finite set $(j,k) \in \{0,1,\dots,K\}^2$, involves choosing a suitable threshold $T = T(N)$ and modifying the steps identified in Section 3.3.1:

1. Draw a sequence of N node-type pairs $X = (j_v,k_v), v \in [N]$ independently from P, and accept the draw if and only if $0 < |D| \le T(N)$. When the sequence X is accepted, it is adjusted by adding a few stubs, either in- or out- as needed. First draw a random subset $\sigma \subset \mathcal{N}$ of size $|D|$ with uniform probability $\binom{N}{|D|}^{-1}$, and then define the feasible sequence $\tilde{X} = (\tilde{j}_v,\tilde{k}_v), v \in [N]$ by adjusting the degree types for $v \in \sigma$ as follows:

$$\tilde{j}_v = j_v + \xi_v^-; \quad \xi_v^- = \mathbf{1}(v \in \sigma, D > 0) \tag{3.29}$$

$$\tilde{k}_v = k_v + \xi_v^+; \quad \xi_v^+ = \mathbf{1}(v \in \sigma, D < 0)\ . \tag{3.30}$$

2. Conditioned on $\tilde{X}$, the result of Step 1, randomly wire together available in and out stubs *sequentially*, with suitable weights, to produce the sequence of edges W. At each $\ell = 1,2,\dots,E$, match from available in-stubs and out-stubs weighted according to their degrees j,k by

$$C^{-1}(\ell)\frac{Q_{kj}}{Q_k^+Q_j^-}\ . \tag{3.31}$$

In terms of the bivariate random process $(e_j^-(\ell-1), e_k^+(\ell-1))$ with initial values $(e_j^-(0), e_k^+(0)) = (e_j^-, e_k^+)$ that at each ℓ counts the number of available degree j in-stubs and degree k out-stubs, the ℓ dependent normalization factor $C(\ell)$ is given by:

$$C(\ell) = \sum_{jk} e_j^-(\ell-1)e_k^+(\ell-1)\frac{Q_{kj}}{Q_k^+Q_j^-}\ . \tag{3.32}$$

Remark 3. An alternative simulation algorithm for the ACG model has been proposed and studied in [30].

Chen and Olvera-Cravioto, [23], addresses the clipping in Step 1 and shows that the discrepancy of the approximation is negligible as $N \to \infty$:

Theorem 5. *Fix $\delta \in (0,1/2)$, and for each N let the threshold be $T(N) = N^{1/2+\delta}$. Then:*

1. *The acceptance probability $\mathbb{P}[|D_N| \le T(N)] \to 1$ as $N \to \infty$;*
2. *For any fixed finite M, Λ, and bounded function $f : (\mathbb{Z}_+ \times \mathbb{Z}_+)^M \to [-\Lambda,\Lambda]$*

$$\left|\mathbb{E}[f\left((\tilde{j}_i,\tilde{k}_i)_{i=1,\dots,M}\right)] - \mathbb{E}[f\left((\hat{j}_i,\hat{k}_i)_{i=1,\dots,M}\right)]\right| \to 0\ ; \tag{3.33}$$

3. *The following limits in probability hold:*

$$\frac{1}{N}\tilde{u}_{jk} \xrightarrow{\mathrm{P}} P_{jk}, \qquad \frac{1}{N}\tilde{u}^+_k \xrightarrow{\mathrm{P}} P^+_k, \qquad \frac{1}{N}\tilde{u}^-_j \xrightarrow{\mathrm{P}} P^-_j \ . \tag{3.34}$$

Similarly it is intuitively clear that the discrepancy of the approximation in Step 2 is negligible as $N \to \infty$. As long as $e^-_j(\ell-1), e^+_k(\ell-1)$ are good approximations of $(E-\ell)Q^-_j, (E-\ell)Q^+_k$, (3.31) shows that the probability that edge ℓ has type (k,j) will be approximately Q_{kj}. Since the detailed analysis of this problem is not yet complete, we state the desired properties as a conjecture:

Conjecture 1. In the approximate assortative configuration simulation algorithm with probabilities P,Q, the following convergence properties hold as $N \to \infty$.

1. The fraction of type (k,j) edges in the matching sequence $(k_l, j_\ell)_{\ell\in[E]}$ concentrates with high probability around the nominal edge distribution Q_{kj}:

$$\frac{e_{kj}}{E} = Q_{kj} + o(1) \ . \tag{3.35}$$

2. For any fixed finite number L, the first L edges $\ell, \ell \in [L]$ have degree sequence $(k_l, j_\ell)_{\ell\in[L]}$ that converges in distribution to $(\hat{k}_l, \hat{j}_\ell)_{\ell\in[L]}$, an independent sequence of identical Q distributed random variables.

Although the conjecture is not yet completely proven, extensive simulations have verified the consistency of the approximate configuration simulation algorithm with the theoretical large N probabilities.

We will soon find that random financial networks with ACG skeletons are useful for modelling default and liquidity cascades in simple settings where banks properties can be assumed to be determined by their node degrees. Even if such an assumption is too simple to be very realistic, such models are able to explain very clearly how system resilience is related to the connectivity of the network. However, a more complicated cascade mechanism such as the many asset fire sale mechanism of Section 2.3.2 cannot be placed on an ACG skeleton. The next class of random graph models shows how to extend the notion of node-type to support complex cascades such as this.

3.4 Inhomogeneous Random Graphs

Generalizations of the Erdös-Renyi random graph known as *inhomogeneous random graphs* (IRG) or *generalized random graphs* (GRG) provide an alternative to the ACG framework for modelling large scale financial networks. Now nodes have *types* other than degree: we can have commercial banks, investment banks, hedge funds, and so on, each with additional continuous characteristics such as geographical location or size. This class originates in papers [24] and [18] and has been studied in generality in [16]. Although this book deals mostly with directed graphs, for simplicity, we present here the discussion for undirected graphs. For further details about this class, please see the textbook [81].

Recall that for the ER graph (or rather the Gilbert graph) one selects each potential edge (v,w) independently with a fixed probability $p = z/(N-1)$. In the IRG, we select conditionally independently, with probabilities $p_{vw} = p_{wv}$ that depend on the random type of the nodes v and w. We suppose that the collection $u = (u_v)_{v\in[N]}$ of node types are identical independent non-negative random variables with cumulative distribution function $F : \mathbb{R}_+ \to [0,1]$. Then, conditioned on the node types u_v, u_w the probability of an edge between v and w is defined to be

$$p_{vw} = \frac{\kappa(u_v,u_w)}{1+\kappa(u_v,u_w)}$$

where $\kappa : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is symmetric and nondecreasing in both arguments. For example, when $\kappa(u,u') = uu'$, we obtain a variant of the Chung-Lu model [24]. Taking κ, F subject to the above conditions, but otherwise arbitrary, gives the most general IRG.

One example of an IRG, a bipartite undirected Erdös-Renyi graph described by two parameters, is the simplest possible setting for the asset fire sale model of Section 2.3.2. We take $u_v \in \{0,1\}$ with $\mathbb{P}[u_v = 0] = p$ where type 0 nodes are banks that connect only to the asset classes they own, which are the type 1 nodes. κ is a two-by-two symmetric matrix

$$\kappa = \begin{pmatrix} 0 & \kappa_{01} \\ \kappa_{01} & 0 \end{pmatrix} ,$$

with $\kappa_{01} > 0$.

A big advantage of the IRG model is that it is almost as easy to simulate as the Gilbert model. To generate the adjacency matrix M of a graph with N nodes, one can follow these steps:

1. Simulate the iid random variables $u_v,\ v \in [N]$ from the distribution F;
2. Compute the N by N upper-triangular matrix A with the v,w entry given by $\frac{\kappa(u_v,u_w)}{1+\kappa(u_v,u_w)}$;
3. Generate an upper-triangular matrix B with each entry iid uniform on $[0,1]$;
4. Let the ones of the adjacency matrix M be put where $B \le A$.

The following discussion aims to understand the large N asymptotics of both the node and edge degree distributions of the IRG $g = (\mathcal{N}, \mathcal{E})$ with given κ, F. Conditioned on the values of $u = (u_v)_{v\in[N]}$, the independent Bernoulli variables $X_{vw} = \mathbf{1}((v,w) \in g), 1 \le v < w \le N$ have values $x_{vw} \in \{0,1\}$ with joint probability

$$\mathbb{P}[X_{vw} = x_{vw}, 1 \le v < w \le N | (u_v), v \in [N]] = \prod_{1\le v<w\le N} \frac{\kappa(u_v,u_w)^{x_{vw}}}{1+\kappa(u_v,u_w)} . \qquad (3.36)$$

The degree of node v in a given configuration is

$$d_v(x) = \sum_{w<v} x_{vw} + \sum_{w>v} x_{vw} , \qquad (3.37)$$

where we define $x_{vw} = x_{wv}$ for $w < v$.

Lemma 2. *The conditional joint probability generating function is given by*

$$\Psi\Big((t_v), v \in [N] \,\Big|(u_v), v \in [N]]\Big) := \mathbb{E}\Big[\prod_{v\in[N]}(t_v)^{d_v(X)}\Big|(u_v), v \in [N]\Big]$$
$$= \prod_{1\le v<w\le N} \frac{1+t_v t_w \kappa(u_v,u_w)}{1+\kappa(u_v,u_w)} \,. \qquad (3.38)$$

Proof of Lemma 2: Since the total probability is 1, (3.36) implies an identity

$$\sum_x \prod_{1\le v<w\le N} \kappa(u_v,u_w)^{x_{vw}} = \prod_{1\le v<w\le N}(1+\kappa(u_v,u_w)) \qquad (3.39)$$

where the sum is over all possible configurations $x = (x_{vw}), x_{vw} \in \{0,1\}$. From (3.37) we deduce a second identity

$$\prod_{v=1}^{N}(t_v)^{d_v(x)} = \prod_v \left[\prod_{1\le w<v} t_v^{x_{vw}} \times \prod_{v<w\le N} t_v^{x_{vw}}\right]$$
$$= \left[\prod_{1\le w<v\le N} t_v^{x_{vw}}\right] \times \left[\prod_{1\le v<w\le N} t_v^{x_{vw}}\right]$$
$$= \left[\prod_{1\le v<w\le N} t_w^{x_{vw}}\right] \times \left[\prod_{1\le v<w\le N} t_v^{x_{vw}}\right] = \prod_{1\le v<w\le N}(t_v t_w)^{x_{vw}} \,.$$

These two formulas lead to

$$\Psi\Big((t_v), v \in [N] \,\Big|(u_v), v \in [N]]\Big) = \mathbb{E}\left[\prod_{1\le v<w\le N}(t_v t_w)^{X_{vw}}\Big|(u_v), v \in [N]\right]$$
$$= \frac{\sum_x \prod_{1\le v<w\le N}(t_v t_w \kappa(u_v,u_w))^{x_{vw}}}{\prod_{1\le v<w\le N}(1+\kappa(u_v,u_w))} = \prod_{1\le v<w\le N} \frac{1+t_v t_w \kappa(u_v,u_w)}{1+\kappa(u_v,u_w)} \,. \qquad (3.40)$$

□

To obtain interesting asymptotic behaviour of the node degree distribution as $N \to \infty$, the next theorem assumes that $\kappa := \kappa^{(N)}$ scales with N.

Theorem 6. *In the IRG sequence with cumulative distribution function* $F : \mathbb{R}_+ \to [0,1]$, *assume for all N that*

$$\kappa^{(N)}(u,u') = (N-1)^{-1}\kappa(u,u') \qquad (3.41)$$

where for some $\alpha > 0$

$$||\kappa||_{1+\alpha,F} := \int_{\mathbb{R}_+^2} |\kappa(u,u')|^{1+\alpha} dF(u)dF(u') < \infty \,. \qquad (3.42)$$

1. As $N \to \infty$, the univariate generating function is given by

$$\Psi^{(N)}(t_1) \;=\; \mathbb{E}[e^{(t_1-1)G^{-1}(U)}]\,(1+o(1)) \;, \tag{3.43}$$

$$G^{-1}(u) := \int_0^\infty \kappa(u,u')dF(u') \;. \tag{3.44}$$

where U is F distributed.

2. For any fixed integer $M > 1$, the joint degrees $d_v, v \in [M]$ converge in distribution to an independent collection of identical random variables.

Part (1) tells us that the degree of a randomly selected node converges in distribution as $N \to \infty$ to a random variable whose probability distribution is given by $(P_k)_{k\geq 0}$ where

$$P_k = \int_{\mathbb{R}_+} \frac{e^{-\lambda}\lambda^k}{k!} dF(G(\lambda)) \;.$$

In other words, the node degrees all converge to a mixture of Poisson random variables with random parameter λ having the mixing cumulative distribution function $F(G(\lambda))$. The mean node-degree is thus $z = \int_{\mathbb{R}_+} \lambda \; dF(G(\lambda))$. Part (2) ensures the asymptotic independence of any finite collection of degrees.

We can see from the theorem that the model parametrization by an arbitrary pair (κ, F) contains redundant information. By defining

$$\tilde\kappa(v,v') = \kappa(G(v),G(v')), \quad \tilde F(v) = F(G(v)) \tag{3.45}$$

one finds that the pair $(\tilde\kappa, \tilde F)$ leads to the same model:

$$\Psi^{(N)}(t_1) = \mathbb{E}[e^{(t_1-1)V}]\,(1+o(1))$$

where V is $\tilde F$ distributed, and

$$v = \int_0^\infty \tilde\kappa(v,v')d\tilde F(v') \;. \tag{3.46}$$

Without loss of generality therefore, one can take a pair (κ, F) such that (3.44) holds with G the identity mapping, under which condition the Poisson mixing distribution turns out to be F.

A rule of thumb says that a mixed Poisson distribution with an unbounded distribution of the mixing variable inherits the tail distribution of the mixing variable. Thus, if F has a Pareto tail of order $\tau - 1$ for some $\tau > 2$, Theorem 6 applies with $\alpha < \tau - 2$, and leads to a fat-tailed degree distribution with the same order. Note further that many potential integer degree distributions are not Poisson mixtures, for example any distribution with finite support.

Proof of Theorem 6: Part (1). For fixed N we compute Ψ by an intermediate conditioning on the random variables u_v and use (3.38) to write

$$\Psi((t_v), v \in [N]) = \mathbb{E}\left[\prod_{1\le v<w\le N} \frac{1+t_v t_w \kappa^{(N)}(u_v,u_w)}{1+\kappa^{(N)}(u_v,u_w)}\right] .$$

Putting $t_v = 1$ for $v > 1$ leads to cancellations of numerator and denominator factors:

$$\Psi(t_1) = \mathbb{E}^{(N)}\left[\prod_{2\le v\le N} \frac{1+t_1\kappa^{(N)}(u_1,u_v)}{1+\kappa^{(N)}(u_1,u_v)}\right] = \mathbb{E}\left[(\psi^{(N)}(t_1,u_1))^{N-1}\right] \qquad (3.47)$$

where

$$\psi^{(N)}(t,u) = \int_{\mathbb{R}_+} \frac{1+t\kappa^{(N)}(u,u')}{1+\kappa^{(N)}(u,u')}\, dF(u') = \int_{\mathbb{R}_+} \frac{1+t(N-1)^{-1}\kappa(u,u')}{1+(N-1)^{-1}\kappa(u,u')}\, dF(u') .$$

Now, for any $\alpha > 0$ and every $x \ge 0$, there is $C(\alpha)$ such that

$$\frac{1+tx}{1+x} = 1+(t-1)x+R(x)$$

where the remainder is bounded $|R(x)| \le |t-1|C(\alpha)x^{1+\alpha}$. Using this bound and the definition of G^{-1} we find

$$\begin{aligned}\psi^{(N)}(t,u) &= \int_{\mathbb{R}_+} \left(1+(t-1)(N-1)^{-1}\kappa(u,u')\right)\, dF(u') + \tilde{R} \\ &= 1+(t-1)(N-1)^{-1}G^{-1}(u) + \tilde{R} .\end{aligned}$$

By the bound (3.42) on κ, the remainder $\tilde{R}$ is bounded by $(N-1)^{-1-\alpha}$ times a function with bounded $L^{1+\alpha}$-norm. Now, by a standard limiting argument, as $N \to \infty$,

$$\left[1+(t-1)(N-1)^{-1}G^{-1}(u)+\tilde{R}\right]^{N-1} = e^{(t-1)G^{-1}(u)}(1+O(N^{-\alpha}))$$

which leads to the desired result.

The proof of part (2) is similar, and left as an exercise. □

We can go further and investigate the *shifted* bivariate distribution of edge degrees $(k_\ell - 1, k'_\ell - 1)$ by computing the expectation $e = \mathbb{E}^{(N)}[t_1^{d_1-1} t_2^{d_2-1} | (1,2) \in g]$ under the parametrization with G equal to the identity mapping. Following exactly the same steps as above, we find the expression

$$\begin{aligned} e = &\left(\mathbb{P}^{(N)}[(1,2)\in g]\right)^{-1} \\ &\times \mathbb{E}^{(N)}\left[\frac{\kappa^{(N)}(u_1,u_2)}{1+\kappa^{(N)}(u_1,u_2)} \prod_{v\ge 3}\left(\frac{1+t_1\kappa^{(N)}(u_1,u_v)}{1+\kappa^{(N)}(u_1,u_v)}\right)\left(\frac{1+t_2\kappa^{(N)}(u_2,u_v)}{1+\kappa^{(N)}(u_2,u_v)}\right)\right] .\end{aligned}$$

The same logic that leads to Theorem 6 leads to

$$e = (\mathbb{E}[\kappa(u_1,u_2)])^{-1}\ \ \mathbb{E}\left[\kappa(u_1,u_2)e^{(t_1-1)u_1}e^{(t_2-1)u_2}\right](1+o(1))) . \qquad (3.48)$$

In the Chung-Lu class of models where $\kappa(u_1,u_2) = u_1 u_2$, (3.48) implies that asymptotically, the edge degree distribution is the independent case

$$Q_{kk'} = \frac{kk' P_k P_{k'}}{z^2} \ .$$

In general there is correlation between the edge degrees, i.e. the graph is assortative, where the edge-type distribution Q equals a bivariate mixture of independent Poisson random variables, shifted up by the vector $(1,1)$. To verify this statement one needs to check that Q defined by

$$Q_{kk'} = \int_0^\infty \int_0^\infty \mathrm{Pois}(v, k-1)\ \mathrm{Pois}(v', k'-1)\ dG(v,v'), \quad \forall k,k' \geq 1 \qquad (3.49)$$

with

$$dG(v,v') := z^{-1} \kappa(v,v')\ dF(v) dF(v') \qquad (3.50)$$

is consistent with

$$e = \sum_{kk'} t_1^{k-1}\ t_2^{k'-1} Q_{kk'} \ .$$

As a useful variation of the IRG framework, one can introduce a more abstract *node type* space $\mathscr{A}$ with probability measure $\mathrm{d}f$ and a mapping $u : \mathscr{A} \to \mathbb{R}_+$. We define the probability of a type a and type b node to wire together to be

$$\frac{\kappa(u(a),u(b))}{1+\kappa(u(a),u(b))}$$

where κ and $\mathrm{d}f$ are consistent:

$$u(a) = \int_{\mathscr{A}} \kappa(u(a),u(b))\ \mathrm{d}f(b), \quad \forall a \in \mathscr{A} \ .$$

Then $u(a)$ will be the average degree of a type a node. The next example illustrates this kind of network construction.

Example 1. (**Three Bank Types**) Consider a financial network with small, medium and large banks whose network fractions are $f = (f_a)_{a=1,2,3} = [0.80, 0.15, 0.05]$ and whose conditional average degrees are $u = (u_a)_{a=1,2,3} = [2, 31/3, 51/2]$. The mean degree is $z = f \cdot u = 4.35$. One connectivity kernel that satisfies the consistency condition $\sum_{b=1}^3 \kappa_{ab} f_b = z^{-1} u_a$ is:

$$\kappa^{(1)} = \begin{bmatrix} 0 & \frac{1}{15} & \frac{1}{5} \\ \frac{1}{15} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{7}{10} \end{bmatrix} \ .$$

It is easy to verify that this choice leads to the consistent negatively assortative pair of degree distributions

$$Q^{(1)}_{kk'} = z^{-1} \sum_{a,b=1}^{3} \kappa^{(1)}_{ab} \,\text{Pois}(u_a, k-1)\,\text{Pois}(u_b, k'-1)\, f_a\, f_b \tag{3.51}$$

$$P_k = \sum_{a=1}^{3} \text{Pois}(u_a, k)\, f_a \,. \tag{3.52}$$

The independent edge type distribution $Q^{(2)}$ arises by taking $\kappa^{(2)}_{ab} = z^{-1}\, u_a\, u_b$. To simulate a random network with N banks that has such large N asymptotics, one draws N bank types $a_1, \dots, a_N$ from the f distribution. Then for each pair of banks $v < w \in [N]$ one creates an undirected edge between them independently with probability $\frac{\kappa_{a_v,a_w}}{N-1+\kappa_{a_v,a_w}}$.

The undirected IRG construction parametrized by pairs (κ, F) that satisfy (3.44) with G equal to the identity, defines a rich class that generalizes the Erdös-Renyi graph. When the skeleton of a random financial network is modelled by an IRG, nodes can have a continuum of types, giving a flexibility not possible within the class of ACG models. IRG models have nice properties: They are straightforward to simulate on a computer, and their asymptotic node type and edge type distributions can be fully characterized as mixtures of Poisson random variables. The natural question is what the IRG class has to do with the configuration graph model. It has been proven in [18] that the subclass of Chung-Lu models is asymptotic to a subclass of non-assortative *simple* configuration graphs. Although it appears not to have been proven yet, it seems that IRG models, like the *ACG* model, always have the desirable *locally tree-like property* and are therefore a promising foundation for implementing cascade mechanisms on financial networks.

3.5 Measures of Network Topology

The term *network topology* is used to refer to a wide variety of characteristics observed in the random graphs underlying large scale networks, both synthetic and in the real world. A *measure of network topology* is a summary statistic that tells us something important about the way nodes connect or about the relative importance of different node and edge types.

3.5.1 Connectivity

Overall *connectivity* of the network can be measured by the fraction of the number of actual directed links to the number of potential links. Thus, in a finite network with N nodes and E directed edges, the connectivity is given by $C = \frac{E}{N(N-1)}$. However, since our networks are typically sparse enough that $C \to 0$ for large N, we normally focus instead on the mean degree $z = E/N$.

3.5.2 Measures from the Degree Distributions

In addition to measuring moments of the degree distribution P_{jk}, such as the mean (in- and out-) degree $z = \sum_{j,k} jP_{jk} = \sum_{j,k} kP_{jk}$, network practitioners also focus on the tail properties. *Tail exponents* are defined by large graph limits:

$$\alpha^{\pm} = -\limsup_{j\to\infty} \frac{\log P^{\pm}_j}{\log j} \ .$$

Finite tail exponents are indicators of what are called *Pareto tails*, and signal the existence of non-negligible numbers of *hubs*, or highly connected nodes that can be significant focal points for systemic risk. Log-log plots capture the characteristic tail exponents as the negative of the slope of the best fit line, above a certain cutoff level. Clauset et al [26] provides the definitive statistical inference method for determining Pareto tail exponents for random samples from a distribution with power law tail.

3.5.3 Centrality Measures

Centrality measures require the full adjacency matrix $M(g)$, and aim to decide the relative importance of nodes. At their heart they rest on the fact that the kth power of the adjacency matrix provides the number of (directed) k-step paths between nodes. Different centrality measures typically formalize the idea that important nodes are those that have important neighbours, by summing over these paths with different weights. For directed graphs where we need to distinguish forward paths from backward paths, centrality measures come in several versions.

1. *Degree centrality*: For undirected graphs, this is simply the degree of the node, while for directed graphs, one refers to the in-degree and out-degree centralities.
2. *Eigenvalue centrality*: By the Perron-Frobenius theorem for non-negative matrices, there exists a non-negative right-eigenvector of M, and this eigenvector is associated with the maximal eigenvalue λ (which is necessarily positive). If M is irreducible, or equivalently if the directed graph g is strongly connected, then there is a unique right-eigenvector, call it $\mathbf{x}^+ = [x^+_1,\dots,x^+_N]$, normalized so that $\sum_i x^+_i = 1$. The component for node i is positive, $x^+_i > 0$, and is called its forward eigenvalue centrality measure. Conversely, the backward eigenvalue centrality measures x^-_i derive from the maximal left-eigenvector of M. Both measures can be easily computed by power iteration using the formulas:

$$x^+_i = \lambda^{-1}\sum_j M_{ij}x^+_j \ ; \qquad x^-_i = \lambda^{-1}\sum_j M_{ji}x^-_i \ .$$

   In more generality, one can apply the same measure to a weighted adjacency matrix, where the link weights are arbitrary non-negative values.

3. *Katz centrality*: This is a parametric family of measures that generalizes both degree and eigenvalue centrality by penalizing long paths with an attenuation factor. For any $\alpha \in (0, \lambda^{-1})$, we define the forward and backward Katz-centrality indices by

$$x_i^{\alpha,+} = \sum_{k=1}^{\infty} \sum_j \alpha^k (M^k)_{ij} \; ; \qquad x_i^{\alpha,-} = \sum_{k=1}^{\infty} \sum_j \alpha^k (M^k)_{ji} \; .$$

We can see that $x_i^{\alpha,+} = \sum_j \alpha M_{ij}(x_j^{\alpha,+} + 1)$ from which it can be shown that the eigenvalue centrality x_i^+ is proportional to the limit of $x_i^{\alpha,+}$ as α approaches λ^{-1} from below.
4. *Betweenness centrality*: This measure differs from the others in considering only *shortest paths* between nodes. For two nodes $v \neq v'$, we define $\sigma_{vv'}$ to be the number of shortest directed paths from v to v'; for any third node $w \neq v, v'$, $\sigma_{vv'}(w)$ is the number of shortest directed paths from v to v' that go through w. Then the betweenness centrality of node w is defined by the formula

$$b_w = \sum_{v,v' \neq w} \frac{\sigma_{vv'}(w)}{\sigma_{vv'}} \; .$$

3.5.4 Clustering Coefficients

Clustering in social networks refers to the propensity of the friends of our friends to be our friends. In a general setting it means the likelihood that a connected triple of nodes forms a triangle. It is usually measured by the *clustering coefficient*, the ratio of the number of triangles to connected triples,

$$C(g) = \frac{3 \times (\text{number of triangles})}{(\text{ number of connected triples})} \; ,$$

where the factor of 3, corresponding to the number of connected triples in a triangle, ensures that $C \in [0, 1]$.

In directed networks, an even more basic notion of clustering is *reflexivity*, that refers to the fraction of the number of node pairs that have a *reflexive* pair of edges (i.e. an edge pointing in both directions) to the total number of directed edges. For triples of nodes, accounting for the direction of edges means there are two different kinds of triangles in directed graphs, and three different connected triples. One can use the results of Section 3.3.1 to compute asymptotic values for all these ratios in the ACG model.

3.5.5 Connectivity and Connected Components

Contagion can only propagate across connected components in a network, and thus the sizes of connected subgraphs of a network are additional measures of its susceptibility to large scale contagion. A *strongly connected component* (SCC) of a network is a subgraph each of whose nodes are connected to any other node by a path of downstream edges, and which is maximal in the sense that it is not properly contained in a larger SCC. For any SCC, one defines its *in-component* to be the maximal subgraph of nodes that are downstream connected to the SCC. Similarly, its *out-component* is the maximal subgraph of nodes that are upstream-connected to the SCC. A *weakly connected component* (WCC) of a network is a maximal subgraph each of whose nodes are connected to any other nodes by a path consisting of undirected edges. A WCC may contain a multitude of side branches that are not downstream or upstream connected to the SCC: pieces called *tendrils* and *tubes* form subgraphs that are not connected to the SCC but are either upstream connected to the in-component or downstream connected to the out-component, or both. In undirected graphs, there is no distinction between strong and weak connectivity, and thus any WCC is an SCC.

In random graph models, the probability distribution of sizes of connected components is a topic of interest. When we consider infinite random graphs, a critical question is whether there is strongly connected component that is infinite or even a positive fraction of the entire network. When this happens in a random graph model, the infinite SCC is typically unique, and we call it the *giant strongly connected component* (GSCC). Its associated giant in- and out-components are called G-IN and G-OUT, and GWCC denotes the *giant weakly connected component*. Clearly we have the inclusions:

$$\text{GSCC} = \text{G-IN} \cap \text{G-OUT} \subset \text{G-IN} \cup \text{G-OUT} \subset \text{GWCC}$$

The complement of the GWCC falls into a disjoint collection of finite sized weakly-connected pieces. Figure 3.2 shows a schematic “bow-tie” rendering of the various connected components of a typical directed network, in this instance the World Wide Web as it appeared in 1999.

It has been found that the existence of a giant connected component in a large network is the single most relevant and important distinguishing characteristic that determines its susceptibility to domino-like cascades. The next chapter develops this intuitive observation into a theory called *percolation on random graphs*, and explores its relation to cascade modelling.

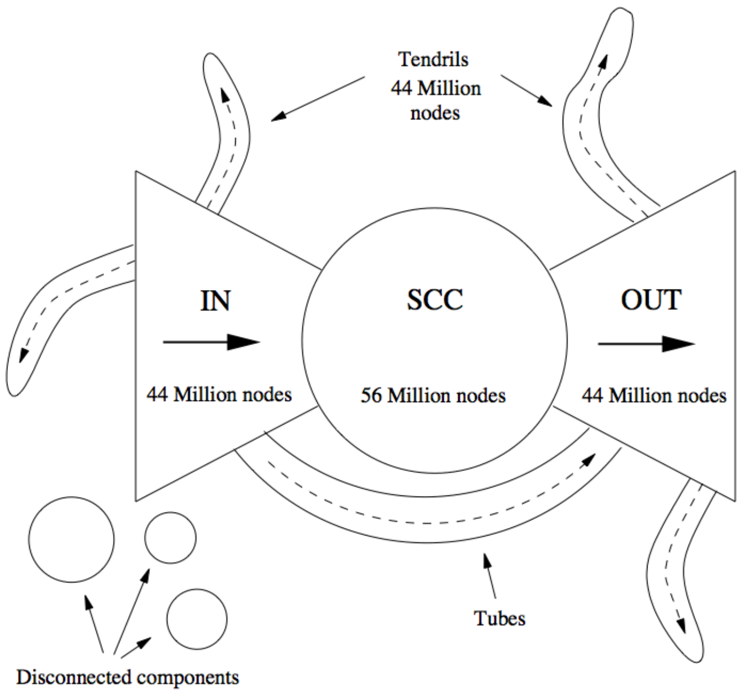


Fig. 3.2 The connected components of the World Wide Web in 1999. (Source: [19].)

Chapter 4
Percolation and Cascades

Abstract The right kind of connectivity turns out to be both necessary and sufficient for large scale cascades to propagate in a network. After outlining percolation theory on random graphs, we develop an idea known as "bootstrap percolation" that proves to be the precise concept needed for unravelling and understanding the growth of simple network cascades. These principles are illustrated by the famous Watts model of information cascades.

Keywords: Graph connectivity, branching process, bond and site percolation, bootstrap percolation, vulnerable edge, vulnerable cluster, Watts' cascade model, without regarding property, local tree-like independence.

As a warmup to understanding cascading shocks on random financial networks, we can answer simpler questions about whether or not the network is highly connected. Obviously cascades cannot propagate if the network isn't sufficiently connected, but what is not so obvious is that the right kind of connectivity is also a sufficient condition for the possibility of large scale cascading to occur. Fortunately, there is a rich and beautiful theory on the connectivity of networks known as *percolation*, and as it will turn out, this idea of percolation actually permeates our problems of cascades. The connected components, called "clusters" , of a given undirected graph $g = (\mathscr{N}, \mathscr{E})$ can be ordered in decreasing size from largest to smallest. Percolation theory answers the question of whether the largest cluster, denoted by $\mathscr{C}$, of the infinite graph, its itself infinite. When this occurs, we say the graph has a *giant cluster* or *giant component*. More precisely, given a well-behaved sequence of a finite size random graphs, percolation theory says finite graphs will have, with high probability, a large cluster exceeding a certain size whenever there exists a giant cluster in the infinite graph limit.

Our primary aim in this chapter is to understand, and in some cases, to prove, conditions under which a simple model of *undirected* networks due to Duncan Watts is susceptible to large scale cascades. It will be revealed as we proceed that sometimes our understanding extends beyond what we are able to prove. For this reason, we shall label most of our results as "formal propositions", for which a "formal", but not rigorous, proof is given. The shortfall in rigour is always of the same nature: while we can prove the existence of an asymptotic cascade mapping, we cannot easily prove the consistency of the fixed point problem. In one simple case, however, we provide rigorous proof of a weakened version of such a formal proposition.

The clearest results on percolation on random graphs apply to the family of configuration graphs, because as Section 3.3.2 shows, when a configuration graph is large enough and the density of edges is not too great, it has the "locally treelike" property that with high probability the graph has few short cycles or closed loops. The "locally treelike" property provides the key to obtaining exact analytical formulas for percolation. Some intuition about percolation on configuration graphs stems because when we start from a random node, and consider growing its connected component one neighbourhood at a time, the process looks like a growing tree or pure branching process, of the type known as a Galton-Watson (GW) process. The potential for this graph component to become large is measured by the potential for the associated GW process to be unbounded. This relation to Galton-Watson processes can then be established rigorously in the asymptotic regime as the number of nodes goes to infinity. Therefore, before considering percolation, we first review the essential properties of GW branching processes.

4.1 Branching Processes

A *Galton-Watson process*, the simplest class of *branching process*, describes a population evolving in time, starting with a single individual in the 0th generation. In the nth (non-overlapping) generation, let there be Z_n individuals. Each individual i of the nth generation, $n \geq 0$, is assumed to produce a random number $X_{n,i}$ of children or offspring, each drawn independently from an identical distribution X on the non-negative integers.

The central question to answer about a GW process is whether the population will ultimately survive or go extinct. The answer is best expressed in terms of the *probability generating function* of the X distribution:

$$g(s) := \mathbb{E}s^X = \sum_{k\geq 0} P_k\, s^k, \quad P_k = \mathbb{P}[X=k]\ .$$

We also need the generating functions $H_n(s) := \mathbb{E}s^{Z_n}$ for $n \geq 0$, and the fundamental dynamic identity:

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{i,n}\ . \tag{4.1}$$

Using iterated conditioning, one computes

$$H_n(s) = \mathbb{E}s^{Z_n} = \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{Z_{n-1}} s^{X_{n,i}} \middle| Z_{n-1}\right]\right] = \mathbb{E}\left[\prod_{i=1}^{Z_{n-1}} \mathbb{E}\left[s^{X_{n,i}} \middle| Z_{n-1}\right]\right] \quad (4.2)$$

$$= \mathbb{E}\left[(g(s))^{Z_{n-1}}\right] = (H_{n-1}\circ g)(s) \quad (4.3)$$

where $(H_{n-1}\circ g)(s) := H_{n-1}(g(s))$. Here, the third equality follows from the mutual conditional independence property of the two index collection $X_{n,i}$. This composition identity, when iterated, leads to the key formula:

$$H_n = \underbrace{g\circ g\cdots\circ g}_{n \text{ factors}} = g\circ H_{n-1}\ . \quad (4.4)$$

The extinction probability is $\eta := \mathbb{P}[\ \exists\ n : Z_n = 0]$ and for each n, define $\eta_n = \mathbb{P}[Z_n = 0] = H_n(0)$. Since $\{Z_{n-1} = 0\} \subset \{Z_n = 0\}$, the sequence η_n is increasing, and converges to η. Since $\eta_n = H_n(0)$,

$$\eta_n = g(H_{n-1}(0)) = g(\eta_{n-1})\ .$$

Like any generating function, g has $g(1) = 1$ and is continuous, increasing and convex on $[0,1]$. Therefore, by continuity,

$$\eta = \lim_{n\to\infty}\ \eta_n = \lim_{n\to\infty} g(\eta_{n-1}) = g\left(\lim_{n\to\infty} \eta_{n-1}\right) = g(\eta)\ ,$$

so $\eta \in [0,1]$ is a fixed point of g. By the convexity of g, there can be at most 2 fixed points. By induction, one can easily verify that if ψ is any fixed point of g, then $\eta_n \le \psi$ for all n, and hence $\eta \le \psi$: Note that $\eta_0 = P_0 \le \psi$, and if $\eta_{n-1} \le \psi$ then $\eta_n = g(\eta_{n-1}) \le g(\psi) = \psi$. Thus η is the smallest fixed point of g on $[0,1]$. This argument proves the following:

Theorem 7. *The extinction probability $\eta \in [0,1]$ is the smallest fixed point of g.*

1. *If $\mathbb{E}X > 1$, then $\eta < 1$, which says that with positive probability $1-\eta$ the population will survive forever.*
2. *If $\mathbb{E}X \le 1$, then apart from a trivial exception, $\eta = 1$ and the population becomes extinct almost surely. The single trivial exception is if $\mathbb{E}X = 1$ and $g"(1) = 0$, which implies $\mathbb{P}[X = 1] = 1$ and that the population remains 1 for all time.*

Note that this theorem applies even if one or both of $g'(1) = \mathbb{E}X, g"(1) = \mathbb{E}X(X-1)$ are infinite. Case (1), when survival is possible, is called the *supercritical* case. The case of almost sure extinction subdivides: case $\mathbb{E}X < 1$ is called *subcritical*, and the case $\mathbb{E}X = 1$ and $g"(1) > 0$ is called *critical*.

From our comments about the relation between percolation and GW processes, it should not be a surprise that in the next section we will find that the possibility of a giant cluster boils down to conditions similar to those given in this theorem.

4.2 Percolation on Configuration Graphs

The paper of Janson [54], based on work by Molloy and Reed [68], considers percolation on undirected configuration (multi)graphs for each finite N based on a general degree sequence $\mathbf{d} = (d_v)_{v=1}^N$ with an even number of stubs $\sum_v d_v = 2E$ (note that usually we neglect to show explicit N dependence). In what follows, we consider asymptotics of the sequence of random degree sequences as $N \to \infty$ and continue to use the asymptotic notation summarized in Section 3.1. The finite random configuration multigraph model is denoted by $G^*(N,\mathbf{d})$ and $G(N,\mathbf{d})$ denotes $G^*(N,\mathbf{d})$ under the condition that the multigraph is simple, that is that the multigraph is in fact a graph.

Assumption 1. The sequence of undirected multigraphs $G^*(N,\mathbf{d})$ is *well-behaved* in the sense that there exists a probability distribution $(P_k)_{k=0,1,\ldots}$ over nonnegative integers such that:

- The empirical degree density converges in distribution:

$$N^{-1}\sum_v \mathbf{1}(d_v = k) \xrightarrow{\mathrm{P}} P_k \ .$$

- The mean degree is finite and positive: $N^{-1}\sum_v \sum_k k\mathbf{1}(d_v = k) \xrightarrow{\mathrm{P}} z$ where $z := \sum_{k=1}^{\infty} kP_k < \infty$.

For undirected random graphs, the "size-biased" distribution of d_v denotes the probabilities Q_k that either of the nodes attached to an arbitrary edge $\ell = (v,w)$ has $k-1$ remaining edges, and one can show that

$$Q_k := \mathbb{P}[k_w = k | w \in \mathcal{N}_v] = \frac{kP_k}{z} \ . \tag{4.5}$$

This fact is relevant to understanding the growth of successive neighbourhoods of a randomly selected node. The degree of v, i.e. the number of neighbours of v, has PMF P. However, neglecting the possibility of cycles, we can see that each neighbour of v has k new neighbours with probability $P^*_k := Q_{k+1}$. Therefore, the growth of neighbourhoods of a random node v, that is the growth of the cluster containing v, approximately follows a GW process whose zeroth generation has offspring probability governed by P while for each successive generation, the offspring distribution is given by P^*.

If we let $g(x) = \sum_k P_k x^k$ be the generating function of P then $g^*(x) = g'(x)/z$ is the generating function of P^*. Now, from Theorem 7 for GW processes, the extinction probability ξ of any node other than the root node is determined by the condition $\xi = g^*(\xi)$. If there is a fixed point $\xi < 1$ then the GW process is supercritical and non-extinction occurs with positive probability $1-\xi$, otherwise $\xi = 1$ is the unique fixed point and extinction occurs almost surely. The key insight is that the same condition determines whether the random graph is supercritical or not. The following is a refinement of the main theorem of [68] proved rigorously in [56], which asserts the precise conditions under which the random graph has a giant cluster (i.e. is supercritical). We will present here only a more intuitive "formal" argument.

Proposition 2 (Proposition 3.1, [54]). *Consider the random multigraph sequence $G^*(N,\mathbf{d})$ satisfying Assumption 1. Let $g(x)$ be its generating function and let $\mathscr{C}$ be the largest cluster. Then the empirical probabilities of a random node or edge being in $\mathscr{C}$ are governed by the following asymptotic properties:*

1. *If $\sum_k k(k-2)\, P_k > 0$, then there is a unique $\xi \in (0,1)$ such that $g^*(\xi) = \xi$ and*

$$\mathbb{P}[v \in \mathscr{C}] \xrightarrow{\mathrm{P}} 1 - g(\xi) > 0\,, \tag{4.6}$$

$$\mathbb{P}[v \in \mathscr{C} \cap \mathscr{N}_k] \xrightarrow{\mathrm{P}} P_k(1-\xi^k),\ \textit{for every}\ k \ge 0\,, \tag{4.7}$$

$$\mathbb{P}[\ell \in \mathscr{C}] \xrightarrow{\mathrm{P}} (1-\xi^2) > 0\,. \tag{4.8}$$

2. *If $\sum_k k(k-2)\, P_k \le 0$, then unless $P_2 = 1$, $\mathbb{P}[v \in \mathscr{C}] \xrightarrow{\mathrm{P}} 0$ and $\mathbb{P}[\ell \in \mathscr{C}] \xrightarrow{\mathrm{P}} 0$.*
3. *In the trivial special case when $P_2 = 1$, then $\sum_k k(k-2)\, P_k = 0$ and $\mathbb{P}[v \in \mathscr{C}] \xrightarrow{\mathrm{P}} 1$ and $\mathbb{P}[\ell \in \mathscr{C}] \xrightarrow{\mathrm{P}} 1$.*

This proposition can be phrased in w.h.p. terms: for example, Part 1 means that with high probability each $G^*(N,\mathbf{d})$ has a giant cluster if $\sum_k k(k-2)\, P_k > 0$. The formal proof of this result is based on the asymptotic locally tree-like property of large configuration graphs expressed in equation (3.27) that says the outgoing edges of any node connect to random subgraphs that can be treated as independent. First we introduce a useful definition that will enable us to interpret ξ as a probability of a specific event.

Definition 10. Suppose a node property $\mathscr{P}$ is *local*, meaning the condition $w \in \mathscr{P}$ is determined by conditions on the nearest neighbours $v \in \mathscr{N}_w$. Then, for any directed edge (v,w), we say that w satisfies the local property $\mathscr{P}$ *without regarding* v, and write "$w \in \mathscr{P}$ WOR v", if the property is determined by these conditions on all nearest neighbours $v' \in \mathscr{N}_w$ excluding v.

In our problem, connectedness is a local node property and so for any directed edge (v,w),

$$\{w \in \mathscr{C}^c \text{ WOR } v\} = \{(\mathscr{N}_w \setminus v) \cap \mathscr{C} = \emptyset\}$$

whereas

$$\{w \in \mathscr{C}^c\} = \{\mathscr{N}_w \cap \mathscr{C} = \emptyset\}\,.$$

Formal Proof of Proposition 2: We can analyze the probability $a = \mathbb{P}[v \in \mathscr{C}^c]$ in terms of WOR probabilities. If v has degree k, then $v \in \mathscr{C}^c$ is a local property equivalent to $w \in \mathscr{C}^c$ WOR v for each of the k neighbours $w_i \in \mathscr{N}_v, i \le k$. Moreover, by the LT property, these k events form a mutually independent collection. Thus, if we define

$$\xi = \mathbb{P}[w \in \mathscr{C}^c \text{ WOR } v | w \in \mathscr{N}_v] \tag{4.9}$$

then

$$\mathbb{P}[v \in \mathscr{C}^c] = \sum_k \mathbb{P}[k_v = k]\mathbb{P}[w_i \in \mathscr{C}^c \text{ WOR } v, i \le k | \mathscr{N}_v = \{w_i\}_{i \le k}] = \sum_k P_k \xi^k := g(\xi) \; . \tag{4.10}$$

If w has degree k and $w \in \mathscr{N}_v$ then $w \in \mathscr{C}^c$ WOR v is equivalent to $w' \in \mathscr{C}^c$ WOR w for each of the remaining $k-1$ neighbours $w' \in \mathscr{N}_w \setminus v$. Thus, just as before,

$$\xi = \sum_k \mathbb{P}[k_w = k | w \in \mathscr{N}_v] \xi^{k-1} = \sum_k Q_k \xi^{k-1} = g^*(\xi) \; . \tag{4.11}$$

Since g^* is itself the generating function of the P^* distribution, the discussion leading to Theorem 7 applies here, and we find the three cases for percolation, supercritical, critical and subcritical, correspond to the cases $g^{*'}(1) > 1$, $g^{*'}(1) = 1$, and $g^{*'}(1) < 1$. We also have

$$\mathbb{P}[v \in \mathscr{C}^c, v \in \mathscr{N}_k] = \mathbb{P}[v \in \mathscr{C}^c | k_v = k] P_k = P_k \xi^k$$

which verifies (4.7). Finally, for $\ell = (v, w)$, one can easily see that $\ell \in \mathscr{C}^c$ means both $w \in \mathscr{C}^c$ WOR v and $v \in \mathscr{C}^c$ WOR w and since these two events are independent and have probability ξ, (4.8) follows. We also see that the fixed point ξ of the equation $g^*(\xi) = \xi$ can be interpreted as a "without regarding" probability. □

To show why this formal proof cannot easily be made rigorous, but nonetheless it indicates a fundamental fact about *finite N* configuration graphs, we now provide a rigorous proof of an analogous, but very much weaker result. For any sequence of size N undirected random graphs $g = (\mathscr{N}, \mathscr{E})$, define the following WOR probabilities recursively:

$$\hat{\eta}_k^{(n,N)} = \mathbb{E}^N \Big[E^{-1} \sum_{\ell = (v,w) \in \mathscr{E}} \mathbf{1}(k_w = k, \mathscr{N}_w^n = \emptyset \text{ WOR } v) \Big] \tag{4.12}$$

where $\mathscr{N}_w^1 = \emptyset$ WOR v means $\mathscr{N}_w^1 = \{v\}$ and for $n > 1$, $\mathscr{N}_w^n = \emptyset$ WOR v means $\mathscr{N}_{w'}^{n-1} = \emptyset$ WOR w for all $w' \in \mathscr{N}_w \setminus v$.

Proposition 3. *Consider the undirected configuration graph sequence with P supported on the finite set* $\{0, 1, \ldots, K\}$. *Then for any finite* $n \ge 1$ *and* $k \le K$, *as* $N \to \infty$

$$|\hat{\eta}_k^{(n,N)} - Q_k (\xi_{n-1})^{k-1}| \stackrel{\mathrm{P}}{=} O(N^{-1/2}) \tag{4.13}$$

where $\xi_{n-1} = (\underbrace{g^* \circ g^* \cdots \circ g^*}_{n \text{ factors}})(0)$, $g^*(x) = \sum_{k>0} Q_k \, x^{k-1}$, *and* $Q_k := kP_k / z$.

Proof of Proposition 3: The proof depends on having a rigorous proof of the LT property for the underlying undirected configuration graph construction. This can be proved by a much simpler version of the proof of Theorem 4 for the directed ACG model. Fortunately, we do not need to spell out these details here, but we will use its basic properties.

Note that by edge permutation symmetry of Step 2 of the configuration graph construction, and iterated expectations with the intermediate conditioning on the result X of Step 1,

$$\hat{\eta}_k^{(n,N)} = \mathbb{E}^N\Big[\mathbb{E}^N[\mathbf{1}(k_{w_1}=k, \mathcal{N}_{w_1}^n = \emptyset \text{ WOR } w_0)|X]\Big] \tag{4.14}$$

where $\ell = (v,w) = (w_0, w_1)$ has been taken as the first link of the wiring sequence W. Now introduce a recursive labelling of the nodes in the increasing sequence of neighbourhoods $\mathcal{N}_{w_1}^0 \subset \mathcal{N}_{w_1}^1 \cdots \subset \mathcal{N}_{w_1}^{n-1}$. Label $w_1 := \mathcal{N}_{w_1}^0$ by $\lambda(w_1) = 1$ and let its degree be $k(\lambda(w_1)) = k_1$. Label any $w \in \mathcal{N}_{w_1}^i \setminus \mathcal{N}_{w_1}^{i-1}, i > 0$ by the $i+1$ component vector $(\lambda(s(w)), j)$ for some index $j \le k(\lambda(s(w))$ where $s(w) \in \mathcal{N}_{w_1}^{i-1}$ denotes the node to which w attaches . Assign this node w a positive degree $k(\lambda(w))$. Finally, note that for $\mathcal{N}_{w_1}^n = \emptyset$ WOR $w_0)$ it is necessary and sufficient for $k(\lambda(w)) = 1$ for all $w \in \mathcal{N}_{w_1}^{n-1} \setminus \mathcal{N}_{w_1}^{n-2}$.

For fixed n, the set of all possible degree labellings is finite, and has a natural lexicographic order which we follow in growing the skeleton graph link-by-link. The inner expectation of (4.14) can be written

$$\sum_{\mathbf{k}_1} \cdots \sum_{\mathbf{k}_{n-1}} \mathbb{E}^N\Big[\mathbf{1}(k_{w_1}=k) \prod_{w \in \mathcal{N}_{w_1}^{n-1}} \mathbf{1}(k_w = k(\lambda(w)))\Big| h, X\Big] \ . \tag{4.15}$$

Here each sum over $\mathbf{k}_i$ is a sum over the possible degrees of the nodes in $\mathcal{N}_{w_1}^i \setminus \mathcal{N}_{w_1}^{i-1}$. Each term in the overall sum corresponds to one possible neighbourhood $h = \mathcal{N}_{w_1}^{n-1}$, which is a fully labelled "configuration" h rooted to w_0, as it was defined in Section 3.3.2. By an undirected, non-assortative version of Theorem 4, we can conclude that the finite sum in (4.15) equals its $N \to \infty$ limit $\hat{\eta}_k^{(n,\infty)}$ up to an $1 + O(N^{-1/2})$ factor.

Since by Theorem 4, configurations h in (4.15) with cycles go to zero in the limit, (4.15) is asymptotic to a sum over tree configurations h, which has the value

$$\hat{\eta}_k^{(n,\infty)} = Q_k \sum_{\mathbf{k}_1} \cdots \sum_{\mathbf{k}_{n-1}} \prod_{w \in \mathcal{N}_{w_1}^{n-1}} Q_{k_w}$$

which is an exact version of the GW process with generating function g^*. We can therefore conclude that $\hat{\eta}_k^{(n,N)} \stackrel{\mathrm{P}}{=} Q_k(\xi_{n-1})^{k-1} + O(N^{-1/2})$ as required. □

Remarks 2.

1. Obviously, this proposition gives little indication of the accuracy of the $N = \infty$ approximation to $\hat{\eta}_k^{(n,N)}$. Nor does it allow us to interchange the $N \to \infty$ and $n \to \infty$ limits to deduce anything about $\hat{\eta}_k^{(\infty,N)}$.
2. On the other hand, the logic of the proof is completely robust, and can be applied whenever an LT property similar to Theorem 4 is true. It proves that the first n steps of the limiting branching process with generating functions g, g^* approx-

imates the probability that a random connected cluster has diameters less than n.

3. We can argue that this type of rigorous result is sufficient to justify the validity of the formal propositions discussed in this chapter.

4.3 Site Percolation

Site percolation on a random graph asks about the connected clusters of subgraphs created by the deletion of random nodes and their edges, and it has been found that Proposition 2 extends beautifully to this more general setting. If we delete nodes v (and their incident edges) of a configuration multigraph independently with probabilities $1-\delta_k$ determined by their degree k, it can be shown that the resultant subgraph is also a configuration multigraph with a new degree distribution P'. The following theorem due to [54] gives the rigorous statement:

Theorem 8 (Theorem 3.5, [54]). *Consider site percolation with deletion probabilities $1-\delta_k \in [0,1]$, on the random configuration multigraph sequence $G^*(N,\mathbf{d})$ satisfying Assumption 1. Then the subgraph has a giant cluster $\mathscr{C}$ with high probability if and only if*

$$\sum_k k(k-1)\ \delta_k\ P_k > z := \sum_k k\ P_k\ . \tag{4.16}$$

1. If (4.16) *holds then there is a unique $\xi \in (0,1)$ such that*

$$\sum_k k\ \delta_k\ P_k(1-\xi^{k-1}) = z(1-\xi) \tag{4.17}$$

and then

$$\mathbb{P}[v \in \mathscr{C}] \xrightarrow{\mathrm{P}} \sum_{k\geq 1} \delta_k\ P_k(1-\xi^k) > 0\ , \tag{4.18}$$

$$\mathbb{P}[\ell \in \mathscr{C}] \xrightarrow{\mathrm{P}} \frac{2}{z}\left[(1-\xi)\sum_k k\ \delta_k\ P_k - \frac{1}{2}(1-\xi)^2 \sum_k k\ P_k\right]\ . \tag{4.19}$$

2. If (4.16) *does not hold, $\mathbb{P}[v \in \mathscr{C}] \xrightarrow{\mathrm{P}} 0$ and $\mathbb{P}[\ell \in \mathscr{C}] \xrightarrow{\mathrm{P}} 0$.*

Why is this apparently special result relevant to our problem of financial cascades? The detailed answer will be revealed shortly, but in the meantime, we can explain that the key to understanding cascades in a simple financial network is to focus on banks that are "vulnerable" in the sense that they default if only one debtor bank defaults, and to delete all other banks (or edges). The resultant network of vulnerable banks has a giant in-cluster $\mathrm{G}-\mathrm{IN}_V$ if and only if the analogue of (4.16) for directed graphs holds. In this case, if any bank owing money to a bank in this giant vulnerable in-cluster were to default, then all of the giant strongly connected vulnerable cluster GSCC_V eventually defaults. Moreover, the giant vulnerable out-cluster

$G - OUT_V$, must also eventually default. The result will be a *global cascade*. The size of the global cascade will clearly be at least as big as $G - OUT_V$. However, it may be much larger since some banks that are not vulnerable to a single debtor's default may be vulnerable to two or more debtor defaults. Such secondary defaults may seem unlikely in a locally-tree-like network since they are impossible on a tree. However, they are likely to happen in a large LTI network if $G - OUT_V$ is itself a large fraction of the network. This fact illustrates a subtlety in the term "locally tree-like": Even if there are few short cycles in a large LTI network, there are many large cycles that connect the network on a large scale, and allow secondary defaults to occur starting from a single seed default.

To understand this better, we now follow the idea that each bank has a threshold number of defaulted neighbours that when exceeded implies its own default, which is reminiscent of a concept that goes by the name of *bootstrap percolation*.

4.4 Bootstrap Percolation

We have just seen that the theory of site percolation can give an indication of the conditions under which a global cascade can occur in a random network. When any node adjacent to the giant vulnerable in-cluster defaults, then the entire giant vulnerable out-cluster will default. However, the resultant global cascade may be much larger in extent than this, because of the possibility that less vulnerable nodes may eventually succumb if they are susceptible when more than one neighbour is triggered. *Bootstrap percolation* considers the general problem of dynamical cascades on random networks where each *susceptible* (or *inactive*) node has a *threshold* r and will *succumb* or become *activated* when the number of its succumbed (activated) neighbours exceeds r.

In general, given a set of nodes $v \in \mathscr{N}$ (which may be a graph or a regular lattice) and a *threshold function* $r : v \in \mathscr{N} \to \{0, 1, \dots\}$, then bootstrap percolation is an increasing process of subgraphs defined by setting $\mathscr{A}_0 = \{v \in \mathscr{N} | r(v) = 0\}$ and for $t \geq 1$,

$$\mathscr{A}_t = \{v \in \mathscr{N} \mid r(v) \ \leq \ |\mathscr{N}_v \cap \mathscr{A}_{t-1}|\} \ .$$

One can say that the threshold model *percolates* if the closure $\mathscr{A} = \cup_{t \geq 0} \mathscr{A}_t$ grows linearly with $\mathscr{N}$ as $|\mathscr{N}| \to \infty$.

The name *bootstrap percolation* was introduced in a paper [22] in a statistical physics context to denote this type of dynamic percolation theory, and the subject has had a rich development since then. The recent paper [10] focusses on a version of bootstrap percolation on the random regular graph, and contains results and references that are quite relevant to financial cascades.

Our investigations will now follow a similar type of process as we next consider the simplest cascade model, that is essentially bootstrap percolation on the undirected Poisson random graph. The Watts Cascade Model introduced in [82] has been the inspiration for much subsequent work on financial cascades, and provides a prototype for the analytical techniques we shall develop in the remainder of the book.

4.5 Watt's 2002 Model of Global Cascades

This classic paper [82] at the heart of network science considers the problem of cascades on a generic undirected random graph. A nice illustrative context is of a social network, with nodes representing people linked to their friends. Each individual is assigned a random threshold and is assumed to adopt a specific new technology, say the latest iPhone, as soon as the number of their adopting friends exceeds this threshold. The model addresses the question "how can one understand how social contacts influence people to adopt the new technology?". The mathematical analysis of the model focuses on the transmission of "adoption" shocks over the friendship links of the network. It determines conditions under which these shocks accumulate and create a large scale *adoption cascade*, explaining why the product may eventually gain a large share of the market. This simple cascade model will serve as a template for studying financial cascades such as the propagation of defaults and liquidity shocks.

4.5.1 The Framework

The framework of the Watts model consists of two layers: the *skeleton* graph of individuals linked to their friends, and an assignment of random values to people that represents the threshold number of friends having adopted the technology that will trigger that person to adopt. Amongst these nodes are the *early adopters* or *seed nodes* defined to be those whose thresholds are zero. In comparison to our notion of random financial network, we don't need a third layer for exposures since friendship links are all trivially set to have weights equal to 1. The full specification of the model is:

1. The skeleton graph is the random undirected Gilbert graph model $G(N,p)$ of Section 3.2 with edge probability p and mean degree $z=(N-1)p$. Here $N\in\{1,2,\dots\}$ denotes the finite number of people and we introduce the node-degree distribution $P=(P_k)_{k=0,1,\dots}$ with Binomial probabilities $P_k=\mathbb{P}[v\in\mathcal{N}_k]=\mathrm{Bin}(N-1,p,k)$.[1] Recall that $\mathrm{Bin}(N-1,z/(N-1))\xrightarrow{\mathrm{D}}\mathrm{Pois}(z)$ as $N\to\infty$.
2. The thresholds are integer random variables $\bar{\Delta}_v\geq 0$ meaning v will adopt when at least $\bar{\Delta}_v$ of her friends have adopted. Conditioned on the skeleton, the collection $\bar{\Delta}_v$ is assumed to be independent, and distributed depending on the degree k_v, that

[1] For any $0\leq k\leq N$ and $p\in[0,1]$, the Binomial probability $\mathrm{Bin}(N,p,k)=\binom{N}{k}p^k(1-p)^{N-k}$ is the probability of exactly k successes in N independent Bernoulli trials each with success probability p.

is,

$$\mathbb{P}[\bar{\Delta}_v = m | v \in \mathscr{N}_k] = d_{k,m}, \quad k,m \in \mathbb{Z}_+$$

for discrete probability distributions $d_{k,\cdot}$ parametrized by k. Let the cumulative probability distributions be given by

$$D_{k,m} := \mathbb{P}[\bar{\Delta}_v \leq m | v \in \mathscr{N}_k] = \sum_{y=0}^{m} d_{k,m}, \quad k,m \in \mathbb{Z}_+ \ .$$

3. For each $n = 0,1,\dots$, we let $\mathscr{D}_n$ denote the set of nodes that have adopted after n steps of the adoption cascade. These sets are defined inductively by $\mathscr{D}_0 = \{v \in \mathscr{N} : \bar{\Delta}_v = 0\}$ and
$$\mathscr{D}_n = \{v \in \mathscr{N} : \bar{\Delta}_v \leq |\mathscr{N}_v \cap \mathscr{D}_{n-1}|\} \tag{4.20}$$
for $n \geq 1$. Their conditional probabilities are defined to be
$$p_k^{(n)} := \mathbb{P}[v \in \mathscr{D}_n | v \in \mathscr{N}_k] \ . \tag{4.21}$$

A randomly selected degree k node will be an early adopter or seed with probability $p_k^{(0)} := d_{k,0}$. Also, observe that the sequence $\{\mathscr{D}_n\}_{n\geq 0}$ is increasing and converges in at most N steps to $\mathscr{D}_\infty$, the set of nodes that eventually adopt.

In [82], the benchmark threshold specification was $d_{k,m} = \rho\mathbf{1}(m=0) + (1-\rho)\mathbf{1}(m-1 < 0.18k \leq m)$ which means early adopters are selected independently with uniform probability $p_k^{(0)} = \rho$ and the remaining nodes adopt when at least a fraction $\phi = 0.18$ of their friends have adopted.

Exact large N results about the Watts model derive from a property based on the *without regarding* concept introduced in Definition 10 in Section 4.2. This property implies that a source of feedback that can be expected in general cascade models is not present in the Watts model. Let D_v^n be the indicator function for the node set $\mathscr{D}_n$, and $\tilde{D}_{v,w}^n$ be the indicator for the set of directed edges (v,w) such that $v \in \mathscr{D}_n$ WOR w. That is, with initial conditions $D_v^{-1} = 0$, it holds that for $n \geq 0$:

$$D_v^n = \mathbf{1}(v \in \mathscr{D}_n) = \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w' \in \mathscr{N}_v} D_{w'}^{n-1}\right) \tag{4.22}$$

$$\tilde{D}_{v,w}^n = \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w' \in \mathscr{N}_v} D_{w'}^{n-1}\mathbf{1}(w' \neq w)\right) \ . \tag{4.23}$$

Although (4.23) seems an intuitive definition for the condition $v \in \mathscr{D}_n$ WOR w, the next proposition shows that it is in fact natural to replace it by a self-consistent, not equivalent, definition: Let $D_{v,w}^{-1} = 0$ and for $n \geq 0$,

$$D_{v,w}^n = \mathbf{1}(v \in \mathscr{D}_n \text{ WOR } w) = \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w' \in \mathscr{N}_v} D_{w',v}^{n-1}\mathbf{1}(w' \neq w)\right) \ . \tag{4.24}$$

Proposition 4 (The WOR property of the Watts model). *Let the Watts model be specified by* $(\mathscr{N},\mathscr{E},\{\bar{\Delta}_v\})$ *and the sequences* $\{D^n_v,\tilde{D}^n_{v,w},D^n_{v,w}\}_{n=-1,0,1,...}$ *defined by the recursive equations* (4.22), (4.23), (4.24) *with the initial conditions* $D^{-1}_v,D^{-1}_{v,w},\tilde{D}^{-1}_{v,w}=0$. *Then for all* $n\geq 0$ *and* $(v,w)\in\mathscr{E}$

$$D^n_v = \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w'\in\mathscr{N}_v} D^{n-1}_{w',v}\right) \tag{4.25}$$

$$= \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w'\in\mathscr{N}_v} \tilde{D}^{n-1}_{w',v}\right) . \tag{4.26}$$

In general, $D^n_{v,w}\leq\tilde{D}^n_{v,w}$*, with strict inequality only occurring if* $D^{n-1}_w=1$.

The last part of the theorem says that adoption shocks $D^n_{v,w}$ and $\tilde{D}^n_{v,w}$ transmitted to w can only differ when their impact is inconsequential because w has already adopted.

Proof of Proposition 4: To prove (4.25) we introduce additional variables $\tilde{D}^n_v$ defined recursively for $n\geq -1$ by $\tilde{D}^{-1}_v=0$ and

$$\tilde{D}^n_v = \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w'\in\mathscr{N}_v} D^{n-1}_{w',v}\right) . \tag{4.27}$$

We show by induction on n that

$$\tilde{D}^n_v = D^n_v \tag{4.28}$$

for all n,v. The proof is based on the monotonicity in n of these recursive mappings, that is $D^{n-1}_v\leq D^n_v$ etcetera. First, note that (4.28) is true for $n=-1,0$. Then note that by monotonicity $\tilde{D}^n_v\leq D^n_v$ for all n,v.

Now assume there is a minimal $n\geq 1$ and v such that $0=\tilde{D}^n_v<D^n_v=1$. Parsing the defining conditions leads to the implication that $\sum_{w\in\mathscr{N}_v}D^{n-1}_{w,v}<\sum_{w\in\mathscr{N}_v}D^{n-1}_w$. Since n is minimal, this in turn implies $D^{n-1}_{w,v}<D^{n-1}_w=\tilde{D}^{n-1}_w$ for some $w\in\mathscr{N}_v$. Thus $D^{n-2}_{v,w}=1$. This means $D^{n-2}_v\geq D^{n-2}_{v,w}=1$ while $\tilde{D}^{n-2}_v\leq\tilde{D}^n_v=0$. But by the minimality of n, we must have that $D^{n-2}_v=\tilde{D}^{n-2}_v$, which is a contradiction. We conclude the non-existence of a minimal $n\geq 2$ and v such that $0=\tilde{D}^n_v<D^n_v=1$. Thus (4.25) follows.

Since $D^n_{v,w}\leq\tilde{D}^n_{v,w}\leq\tilde{D}^n_v=D^n_v$ it must be the case that

$$\begin{aligned} D^n_v = \tilde{D}^n_v &= \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w'\in\mathscr{N}_v} D^{n-1}_{w',v}\right) \\ &\leq \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w'\in\mathscr{N}_v} \tilde{D}^{n-1}_{w',v}\right) \leq \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w'\in\mathscr{N}_v} D^{n-1}_{w'}\right) = D^n_v \end{aligned}$$

which proves (4.26). Finally, to prove the last part of the theorem, suppose $D^n_{v,w} = 0, \tilde{D}^n_{v,w} = 1$. In this case $\tilde{D}^n_v = D^n_v \geq \tilde{D}^n_{v,w} = 1$ as well. Hence it must be that $D^{n-1}_{w,v} = 1$ and thus $D^{n-1}_w = 1$. $\square$

4.5.2 The Main Result

To state the core result in the Watts model, we introduce conditional and unconditional *without regarding* adoption probabilities:

$$\hat{p}^{(n)}_k := \mathbb{P}[w \in \mathscr{D}_n \text{ WOR } v | w \in \mathscr{N}_k \cap \mathscr{N}_v] \tag{4.29}$$

$$\hat{\pi}^{(n)} := \mathbb{P}[w \in \mathscr{D}_n \text{ WOR } v | w \in \mathscr{N}_v] \tag{4.30}$$

and make use of (4.24) and (4.25) to derive inductive formulas for the collection of probabilities $p^{(n)}_k := \mathbb{P}[v \in \mathscr{D}_n | v \in \mathscr{N}_k]$, $\hat{p}^{(n)}_k, \hat{\pi}^{(n)}$ for $n \geq 1$ in terms of the WOR-probabilities $\hat{\pi}^{(n-1)}$, with the initial conditions $\hat{p}^{(0)}_k := d_{k,0}, \hat{\pi}^{(0)} = \sum_k d_{k,0} Q_k$. These formulas are valid asymptotically in the limit as the network size N goes to infinity, while keeping the probability data $(P_k, d_{k,0})$ fixed.

Formal Proposition 9 (Watts Cascade Theorem)**.** *Consider the Watts model in the limit as $N \to \infty$, with fixed mean degree $z > 0$ and with adoption threshold distribution functions $d_{k,\cdot}, D_{k,\cdot}$ for $k \geq 0$. Then:*

1. *The initial adoption probabilities are $p^{(0)}_k = \hat{p}^{(0)}_k = d_{k,0}, \hat{\pi}^{(0)} = \sum_k d_{k,0} Q_k$.*
2. *The collections $p^{(n)}_k$, $\hat{p}^{(n)}_k, \hat{\pi}^{(n)}$ for $n \geq 1$ are given by the recursion formulas*

$$p^{(n)}_k = G_k(\hat{p}^{(n-1)}) := \sum_{m=0}^{k} D_{k,m} \,\text{Bin}(k, \hat{\pi}^{(n-1)}, m) \tag{4.31}$$

$$\hat{p}^{(n)}_k = \hat{G}_k(\hat{p}^{(n-1)}) := \sum_{m=0}^{k-1} D_{k,m} \,\text{Bin}(k-1, \hat{\pi}^{(n-1)}, m) \tag{4.32}$$

$$\hat{\pi}^{(n)} = \sum_k \hat{p}^{(n)}_k \, Q_k \; . \tag{4.33}$$

3. *The probability $\hat{\pi}^{(n)}$ is given by a scalar mapping $\hat{\pi}^{(n)} = G(\hat{\pi}^{(n-1)})$ where*

$$G(\pi) = \sum_k \sum_{m=0}^{k-1} \frac{kP_k}{z} \, D_{k,m} \,\text{Bin}(k-1, \pi, m) \; . \tag{4.34}$$

   The mapping $G : [0,1] \to [0,1]$ is continuous, monotonically increasing and has $G(0) = \hat{\pi}^{(0)}$. Therefore the sequence $\hat{\pi}^{(n)}$ converges to the least fixed point $\pi \in [0,1]$ with $\pi = G(\pi)$.

Formal Proof of Proposition 9: Part 1 is trivial. The proof of Part 2 is based on two properties of the model. The first is the LT property of the skeleton as long as N is sufficiently large, and the second is the conditional independence of the thresholds $\bar{\Delta}$, conditioned on the skeleton. Therefore, adoption shocks coming into a node v along different links are always asymptotically independent as $N \to \infty$, and this collection of shocks is independent of the threshold $\bar{\Delta}_v$.

By the original definition of the set $\mathscr{D}_n$,

$$p_k^{(n)} = \mathbb{P}\Big[\bar{\Delta}_w \le \sum_{w' \in \mathscr{N}_w} \mathbf{1}(w' \in \mathscr{D}_{n-1}) | w \in \mathscr{N}_k\Big] \tag{4.35}$$

One might try to argue that conditioned on $w \in \mathscr{N}_k$, the events $\{w' \in \mathscr{D}_{n-1}\}$ over nodes $w' \in \mathscr{N}_w$ are mutually independent in the limit $N \to \infty$ because of the locally tree-like property that becomes exact as $N \to \infty$, and independent of $\bar{\Delta}_v$ because of the further independence assumption. But this is erroneous: because the links are bi-directional, each $\{w' \in \mathscr{D}_{n-1}\}$ is in fact dependent on $\{w \in \mathscr{D}_{n-2}\}$ and hence on $\bar{\Delta}_w$. However, by (4.24) each $\{w' \in \mathscr{D}_{n-1} \text{ WOR } w\}$ is conditionally independent of the state of $\bar{\Delta}_w$. Using (4.25), (4.35) can be rewritten

$$p_k^{(n)} = \mathbb{P}\Big[\bar{\Delta}_w \le \sum_{w' \in \mathscr{N}_w} \mathbf{1}(w' \in \mathscr{D}_{n-1} \text{ WOR } w) | w \in \mathscr{N}_k\Big] \tag{4.36}$$

where the terms in the sum are k independent Bernoulli random variables. Furthermore, because of the independent edge condition, the Bernoulli probabilities are independent of k_w, and thus are each $\hat{\pi}^{(n-1)}$. This leads to equation (4.31). Similarly, using (4.24) we can compute that for random links (v, w)

$$\begin{aligned}\hat{p}_k^{(n)} &:= \mathbb{P}[w \in \mathscr{D}_n \text{ WOR } v | w \in \mathscr{N}_v \cap \mathscr{N}_k] \\ &= \mathbb{P}\Big[\bar{\Delta}_w \le \sum_{w' \in \mathscr{N}_w \backslash v} \mathbf{1}(w' \in \mathscr{D}_{n-1} \text{ WOR } w) | w \in \mathscr{N}_v \cap \mathscr{N}_k\Big]\ .\end{aligned}$$

where now there are $k-1$ independent Bernoulli random variables in the sum, leading to (4.32). We also note that

$$\hat{\pi}^{(n)} = \sum_k \mathbb{P}[w \in \mathscr{D}_n \text{ WOR } v | w \in \mathscr{N}_v \cap \mathscr{N}_k]\ \mathbb{P}[k_w = k | w \in \mathscr{N}_v] \tag{4.37}$$

and since $\mathbb{P}[k_w = k | w \in \mathscr{N}_v] = Q_k = \frac{kP_k}{z}$ from (4.5), we find

$$\hat{\pi}^{(n)} = \sum_k \mathbb{P}[w \in \mathscr{D}_n \text{ WOR } v | w \in \mathscr{N}_v \cap \mathscr{N}_k]\ Q_k \tag{4.38}$$

which leads to (4.38). This verifies Part 2 of the proposition.

For Part 3, we note that the argument why $\lim_{n\to\infty} \hat{\pi}^{(n)} = \pi$ is the smallest fixed point of G is precisely the same as the argument given for the extinction probability η for percolation in Section 4.2. $\square$

Reviewing for a moment Section 4.4, we can see that the main Proposition of the Watts model and its formal proof are a template for what we can expect in variations of bootstrap percolation dynamics.

4.5.3 The Cascade Condition

In this version of the Watts model, we notice that two initial seed probability distributions $(p_k^{(0)})$ giving the same scalar value $\hat{\pi}^{(0)}$ lead to the same sequence $(p_k^{(n)})$ for $n \geq 1$. So we can consider equally weighted schemes where initial seeds are uniformly random with $p_k^{(0)} = \hat{\pi}^{(0)}$. Then, essentially, the cascade is the iterated scalar mapping G that converges to the fixed point π, and the probability that a degree k node eventually defaults is

$$p_k^{(\infty)} = G_k(\pi)$$

Let us consider initial seed probabilities $p_k^{(0)} = \hat{\pi}^{(0)} = \varepsilon$ and small $\varepsilon > 0$. We can write $D_{k,m} = \varepsilon + (1-\varepsilon)\tilde{D}_{k,m}$ where $\tilde{D}_{k,m} = \mathbb{P}[\bar{\Delta}_v \leq m | v \in \mathscr{N}_k, \bar{\Delta}_v \neq 0]$ is the threshold CDF conditioned on not being an early adopter and consider the fixed point of $G(\cdot;\varepsilon,\tilde{D})$ as a function of ε for fixed $\tilde{D}$. The most important question to ask is whether the fixed point $\pi(\varepsilon)$ is of order ε or of order 1 as $\varepsilon \to 0$. In other words, what is the *cascade condition* that determines if an infinitesimally small seed fraction will grow to a large-scale cascade? It turns out this depends on the derivative $\partial G(0;\varepsilon,\tilde{D})/\partial\pi$ at $\varepsilon = 0$, which is easy to calculate:

$$\frac{\partial G(0;0,\tilde{D})}{\partial \pi} = \sum_k \sum_{m=0}^{k-1} \frac{kP_k}{z} \tilde{D}_{k,m} \binom{k-1}{m} [\mathbf{1}(m=1) - (k-1)\mathbf{1}(m=0)] = \sum_k \frac{k(k-1)\, P_k\, d_{k,1}}{z} . \tag{4.39}$$

Formal Proposition 10. *Consider the Watts model in the limit as $N \to \infty$, with fixed mean degree $z > 0$.*

1. *If the cascade condition $\partial G(0;0,\tilde{D})/\partial\pi > 1$ is true, then there is $\bar{\varepsilon} > 0$ such that $|\pi(\varepsilon)| > \bar{\varepsilon}$ for all $\varepsilon > 0$. That is, under this condition, an initial seed with a positive fraction of nodes will almost surely trigger a cascade fraction larger than $\bar{\varepsilon}$.*
2. *If $\partial G(0;0,\tilde{D})/\partial\pi < 1$ then there is $\bar{\varepsilon} > 0$ and C such that for all $0 < \varepsilon < \bar{\varepsilon}$, $|\pi(\varepsilon)| \leq C\varepsilon$. That is, this network will almost surely not exhibit large scale cascades for any initial seed with fractional size less than $\bar{\varepsilon}$.*

We can interpret this condition by comparing (4.39) to the main result Proposition 8 for site percolation. We see that the above cascade condition is identical to the site percolation condition of (4.16), if we look at the connectivity of the subgraph obtained by deleting all sites except those with $\bar{\Delta} \leq 1$. This connection to site percolation becomes even clearer in the next subsection.

4.5.4 The Frequency of Global Cascades

A necessary condition for a single seed node to trigger a very large cascade is that some of its neighbours are *vulnerable* in the sense that they are susceptible to adopt given that only one of their neighbours adopts. If there are few cycles in the graph, then any potential large scale cascade must first grow through vulnerable nodes, and only when the cascade is sufficiently developed will less vulnerable nodes begin to adopt.

In the infinite network with a single seed, $d_{k,0} = 0$ for all k and the set of vulnerable nodes $\mathscr{V} \subset \mathscr{N}$ is defined by the condition $\bar{\Delta}_v \leq 1$, which has probability $d_{k,1}$ if $k_v = k$. We consider whether or not $\mathscr{V}$ has a giant cluster $\mathscr{C}_V$. Supposing the cluster $\mathscr{C}_V$ is giant, then the condition $v \in \mathscr{C}_V^c$ means either $v \notin \mathscr{V}$ or $v \in \mathscr{V}$ and all neighbours $w \in \mathscr{N}_v$ are not in $\mathscr{C}$ without regarding v. If $\xi = \mathbb{P}[w \in \mathscr{C}_V^c \text{ WOR } v | w \in \mathscr{N}_v]$, the logic outlined in Section 4.5.2 implies

$$\mathbb{P}[v \in \mathscr{C}_V^c] = \sum_k P_k \left[(1 - d_{k,1}) + d_{k,1} \xi^k \right] . \tag{4.40}$$

Moreover, following the same logic,

$$\xi := \mathbb{P}[w \in \mathscr{C}_V^c \text{ WOR } v | w \in \mathscr{N}_v] = \sum_k Q_k \left[(1 - d_{k,1}) + d_{k,1} \xi^{k-1} \right] := f(\xi) . \tag{4.41}$$

Now the function f maps $[0,1]$ to itself, is increasing and convex, and has $f(0) > 0$ and $f(1) = 1$. As illustrated by Figure 4.1, for (4.41) to have a non-trivial fixed point $\xi < 1$, it is necessary and sufficient that $f'(1) > 1$, that is,

$$\sum_k k(k-1) P_k d_{k,1} > z$$

which we recognize as the cascade condition from the previous subsection. When $\sum_k k(k-1) P_k d_{k,1} < z$, the argument can be reversed and one finds that there can be no giant vulnerable cluster.

For a global cascade to arise from a single seed, it is sufficient and almost necessary for the random seed to have at least one neighbour in the giant vulnerable cluster $\mathscr{C}_V$, which occurs with frequency

$$f = \sum_k P_k \left(1 - \xi^k\right) . \tag{4.42}$$

In this case, a cascade of at least the size of $\mathscr{C}_V$ will result. However, since the $\mathscr{C}_V$ is a positive fraction of the entire network, a positive fraction of adoptions by less vulnerable nodes is possible, and therefore the extent of the global cascade that results may be substantially larger than $\mathscr{C}_V$.

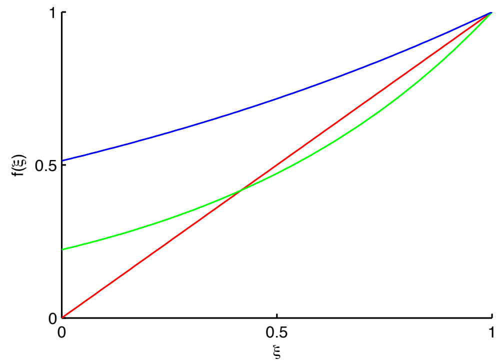


Fig. 4.1 Two possible fixed point configurations are illustrated. The supercritical case is found for the green curve showing the function $f(\xi) = e^{-3(\xi-1)/2}$, which has a non-trivial fixed point $\xi < 1$. The blue curve showing $f(\xi) = e^{-2(\xi-1)/3}$ has only the trivial fixed point at $\xi = 1$ and corresponds to a sub-critical random graph.

4.6 Numerical Experiments on the Watts Model

We illustrate the Watts Cascade model by reproducing some of the numerics of the original specification:

- The skeleton is an undirected $G(N,p)$ (Gilbert) random graph with $N = 10000$ and edge probability $p = z/(N-1)$, chosen with the mean node degree $z \in [0,10] \ll N$ so that the graph is sparse.
- There is an initial seed of adopters chosen uniformly with probability $\mathbb{P}[v \in \mathscr{D}_0] := p^{(0)} \ll 1$ and the remaining nodes have thresholds

$$\bar{\Delta}_v = \min\{i : i \geq \phi^* k_v\} \mathbf{1}(v \notin \mathscr{D}_0)$$

  for the default value $\phi^* = 0.18$.
- Link weights are equal to one: $\bar{\Omega}_{wv} = 1$.

Since $N = 10000$ is large, and the $G(N,p)$ model is a configuration graph, we expect Proposition 9 to yield the probability of eventual default $p_k^{(\infty)} = G_k(\pi)$ with only small finite size error effects. To make a proper comparison between the finite network computed by Monte Carlo simulation and the infinite network computed analytically, it is important to understand how to generate the initial seed. In the infinite network, any positive initial seed probability generates an infinite set of early adopters. When the density of adopters $p^{(0)}$ is small, each seed generates a "mini-cascade" far from the others, and a large scale cascade will be generated if at least one seed succeeds in generating a large cascade. This will happen almost surely if each single seed has a positive probability of generating a large cascade, that is when the cascade condition of Proposition 10 is true. Moreover, in this case percolation theory suggests that the fractional size of the large scale cascade, that is, the expected default probability, will exceed the fractional size of the giant vulnerable cluster

$\mathscr{C}_V$ (recall that on undirected networks, the various giant clusters coincide: $\mathscr{C}_V = \mathrm{G-IN}_V = \mathrm{GSCC}_V = \mathrm{G-OUT}_V$). On the other hand, the probability of a "mini-cascade" growing to a global cascade is zero when there is no giant vulnerable cluster: This defines the non-cascading region of parameter space, where almost surely no global cascade will arise.

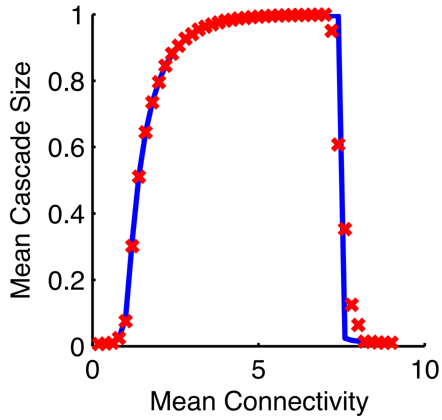
In a Monte Carlo realization of the finite network, we can assert that with high probability a single random adoption seed will grow to a large fraction of the network if and only if it lands on or beside a giant vulnerable cluster $\mathscr{C}_V$. This will happen with probability f given approximately by (4.42). On the other hand, taking an intermediate seed fraction, say $100/10000$, will generate 100 roughly independent "mini-cascades", each of which has a probability f of becoming a large scale cascade taking up a significant fraction of the network. Supposing the probability f of hitting $\mathscr{C}_V$ is not too small, say $f \gtrsim 0.01$, then the probability of a large scale cascade will be about $1-(1-f)^{100}$, which will be close to one.

Experiment 1: We generated Nsim $= 50$ realizations of the network, each with 50 randomly generated seed nodes and compared the results to the analytic formula for the eventual adoption probability $p^{(\infty)} = \sum_k P_k p_k^{(\infty)}$ from Theorem 9 with initial seed probability $\pi^{(0)} = 0.005$. Figure 4.2(a) shows the results.

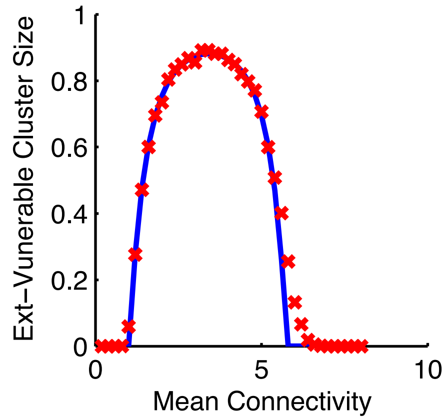
In addition to the close agreement between the Monte Carlo and analytical results, we observe in this graph the well known "contagion window" for z in the approximate range $[1,7]$. The upper and lower thresholds of this windowcorrespond to the values of the mean degree for which the connected vulnerable out-component $\mathscr{C}_V$ becomes "giant". The contagion window determined by the analytical formula becomes exact as $\pi^{(0)} \to 0$. The transition near the upper threshold shown in the Monte Carlo result corresponds to the region where the default frequency $f \ll 0.01$ and thus in most realizations none of the initial seeds hits $\mathscr{C}_V$. $\mathscr{C}_V$ is small in this parameter region because when $\phi^* = 0.18$ only links pointing to nodes with degree 5 or less are vulnerable.

Experiment 2: To make the correct comparison between the cascade frequency f computed analytically by (4.42), and the Monte Carlo simulations, we should generate only a single seed in our Monte Carlo simulations, and count the number of simulations that lead to a cascade that can be considered "large" or "global". In a 10000 node network, we consider that a cascade of more than 50 adoptions is a global cascade. Figure 4.2(b) shows the frequency at which single seeds trigger a global cascade of at least 50 nodes in the finite network, and the infinite network frequency given by (4.42).

Taken together, the two parts of Figure 4.2 verify the vulnerable percolation picture that global cascades can arise whenever $\mathscr{C}_V$ is a significant fraction of the entire network (say any positive fraction if $N = \infty$, or 0.5% if $N = 10000$). When this happens, a single isolated seed that happens to have at least one neighbour in $\mathscr{C}_V$ will trigger all of $\mathscr{C}_V$ and more, a possibility that happens with probability f given by the frequency formula (4.42).



(a)



(b)

Fig. 4.2 These two graphs show (a) the mean fractional cascade size, and (b), the fractional extended vulnerable cluster size, in the benchmark Watts model with $\phi^* = 0.18$, as a function of mean degree z, as computed using the large N analytics of Theorem 9 (blue curve) and by Monte Carlo simulation (red crosses). In Figure (a), the Monte Carlo computation involved Nsim $= 50$ realizations of the $N = 10000$ node graph, each with an initial seed generated by selecting nodes independently with probability 0.5%. In Figure (b), the simulation involved Nsim $= 2500$ realizations of the $N = 10000$ node graph and a single random initial seed node.

4.7 Extensions of the Watts Model

The basic construction and main results of the previous section have been worked out in many different guises. We outline here some of the main lines that have been developed.

1. **General degree distributions:** The Poisson random graph model fails to capture most of the structural features observed in real world social networks. In particular, it has a thin tailed degree distribution that is incompatible with the type of fat-tails and power laws observed in real world networks. It turns out to be straightforward to analyze the Watts model on general configuration graphs, obtaining the same result as given by Proposition 9, for an arbitrary degree distribution P_k.
2. **Mixtures of directed and undirected edges:** In [13], percolation results are proved in random networks with both directed and undirected edges and arbitrary two-point correlations. Analysis of the Watts model in this network seems to be a straightforward next step.
3. **Assortative graphs:** In Section 3.3, the usual directed configuration graph structure was generalized to allow for *assortativity*, that is, non-independent edge-type distributions Q_{kj}. This generalization is important because observed financial networks are typically disassortative, and this fact is expected to strongly influence a network's resilience to default. It was demonstrated in [51] that for

the Gai-Kapadia 2010 default cascade model on assortative directed configuration graphs, similar arguments lead to a characterization of the limiting default probabilities in terms of the fixed points of a vector valued cascade mapping $G: \mathbb{R}^{\mathbb{Z}_+} \to \mathbb{R}^{\mathbb{Z}_+}$.

4. **Random edge weights:** The Watts model allows only equal link weights, whereas in reality, one might expect the strength of friendship links to be stochastic. One might also expect that the statistical properties of friendship strength will be related to the degree of connectivity of social contacts, i.e to the edge-type. As first shown in [52], even capturing the adoption cascade with purely deterministic dependence between link-strength and edge-degree in the Watts model turns out to require an analysis of random link weights. In this more general setting, that paper finds that limiting adoption probabilities are characterized in terms of the fixed points of another vector-valued cascade mapping $G: \mathbb{R}^{\mathbb{Z}_+} \to \mathbb{R}^{\mathbb{Z}_+}$.

In the next chapter, we will explore such extensions, no longer in the Watts model but in the context of systemic risk, by studying variations of the Gai-Kapadia default cascade model. First however, we pay a bit more attention to the mathematical structure at its core that leads to such elegant large graph asymptotic results.

4.8 Dependence in Random Financial Networks

A random financial network, as defined in Section 2.4, consists of a skeleton graph, decorated by additional random structures, namely balance sheets for each node and exposures for each directed link. We have not yet considered how these nearly infinite-dimensional random variables might be provided with multivariate distributions that reflect the nature of banks and their balance sheets, as well as the often high uncertainty in our observations and knowledge of the system at an arbitrary moment in time. Let us consider what reasonable constraints can be made on the possible dependence relating different balance sheet entries and exposures.

A reasonable hypothesis is that banks control their balance sheets while subject to constraints imposed by the financial regulator. Important constraints are usually expressed in the form of capital ratios, such as the minimum core tier one capital or “capital adequacy ratio” (CAR), the liquidity coverage ratio (LCR) and net stable funding ratio (NSFR). However, banks are monitored only intermittently by the regulators, and therefore banks have non-negligible probability of being non-compliant at a random moment in time. Indeed, a financial crisis, almost by definition, typically occurs at a moment when one or more banks have been hit by shocks that have made them non-compliant.

A second reasonable hypothesis is that individual banks control their balance sheets without knowledge of the balance sheets of other banks in the system. Interbank exposure size is the result of a bargaining game between two counterparties whose outcome depends on their connectivity and balance sheets. Notice the intrinsic conceptual difficulty in imagining the results of the myriad of overlapping games being played within the network: to move forward will require overarching assumptions that capture some salient features of the observed interbank exposure sizes. Moreover, each interbank link leads to some dependence between the interbank components of balance sheets of counterparty banks, through the consistency relations (2.37).

Pushing harder, we might make a third reasonable hypothesis that banks' balance sheets depend on the "type" of the bank, but when conditioned on the type of bank, can be treated as independent of other banks. In the simplest RFN settings where we equate bank type to node-degree, imagining that bank size (measured by total assets) is strongly correlated with bank connectivity, implies that balance sheet dependency arises only through the skeleton graph.

As a final consideration, we note that real financial network data is often highly aggregated across counterparties and it is a problem to identify bank specific random variables consistent with these observations. Treating such aggregated measurements as constraints leads to system-wide dependence amongst banks' balance sheets that is impractical to model. Instead, one can impose such aggregate observations in expectation, providing linear constraints on the means of the more basic (and independent) bank-specific random variables. This type of inconsistency is reduced by the law of large numbers that assures that such aggregated random variables become equal to their means in the large network limit.

We are about to commit to making a conceptual leap that will tame balance sheet and exposure dependence by tying it directly to the dependence built into the skeleton. Before doing so, one should imagine the alternative as an arbitrarily high-dimensional random describing a large, partly measured financial network. Such an object is extremely complex. Even the marginal distributions of essential quantities necessarily have high uncertainty, while specifying dependence in such a setting is orders of complexity beyond specifying marginals. The dependence assumptions we now make are intended to be reasonable: they enable both Monte Carlo and analytical experiments to run side by side, and so computational tools become available to explore the consequences of the assumptions. We do not claim that our dependence assumptions rest on fundamental economic principles.

We consider a RFN adapted to a cascade mechanism such as those discussed in Chapter 2, parametrized by the reduced quadruple $(\mathscr{N},\mathscr{E},\bar{\Delta},\bar{\Omega})$. The key to taming dependence is to allow the skeleton $(\mathscr{N},\mathscr{E})$ to determine the overall dependence in the network. Conceptually, we first draw a random skeleton, finite or infinite. Then, conditioned on the realization of the skeleton, we draw remaining random variables *independently*. Moreover, the dependence of these random variables on the skeleton will be *local*. That is, the marginal distributions of node variables (buffers) depend only on the type of the node itself and not on the type of its neighbours. Similarly, the marginal distribution of each edge variable (exposure) depends only on the type of the edge.

It is helpful to express this definition in terms of general probability theory as developed for example in Chapters One and Two in Volume Two of the popular book on stochastic calculus by Shreve [77]. The probability framework is expressed in terms of the triple $(\Omega',\mathscr{F},\mathbb{P})$, where Ω' is the sample space, $\mathbb{P}$ is the probability measure, and $\mathscr{F}$ is the information set, or sigma-algebra. $\mathscr{F}$ can be reduced to a union of sub-sigma-algebras

$$\mathscr{F} = \mathscr{G} \vee \mathscr{F}_\Delta \vee \mathscr{F}_\Omega \ .$$

where: $\mathscr{G}$ denotes the information contained in the directed skeleton $(\mathscr{N}, \mathscr{E})$; $\mathscr{F}_\Delta$ denotes the information of the collection of buffers $\bar{\Delta}_v$; $\mathscr{F}_\Omega$ denotes the information of the collection of exposures $\bar{\Omega}_\ell$. Then the dependence of $\bar{\Delta}_v$ only on the type (j_v, k_v) of the node v can be expressed in terms of expectations conditioned on knowing all information apart from the information of $\bar{\Delta}_v$, measured by the sigma algebra $\sigma(\bar{\Delta}_v$ it generates: it holds that for any bounded Borel function f and a randomly selected node $v \in \mathscr{N}$, there is a bounded Borel function $g : \mathbb{Z}_+^2 \to \mathbb{R}$ such that

$$\mathbb{E}[f(\bar{\Delta}_v) | \mathscr{F} \setminus \sigma(\bar{\Delta}_v)] = g(j_v, k_v) \ .$$

This implies that the conditional CDF of $\bar{\Delta}_v$ is a function that depends only on its type: there are Borel functions D_{jk} such that for all $x \geq 0$

$$\mathbb{E}[\bar{\Delta}_v \leq x | \mathscr{F} \setminus \sigma(\bar{\Delta}_v), v \in \mathscr{N}_{jk}] = D_{jk}(x) \ . \tag{4.43}$$

In the same way, for the exposure on a type (k, j) edge, it holds that there are Borel functions W_{kj} such that for all $x \geq 0$

$$\mathbb{E}[\bar{\Omega}_\ell \leq x | \mathscr{F} \setminus \sigma(\bar{\Omega}_\ell), \ell \in \mathscr{E}_{kj}] = W_{kj}(x) \ . \tag{4.44}$$

While the dynamics of cascades is determined by the specification of the reduced RFN, questions of interpretation, and in particular, the impact of the cascade on the financial system and the larger economy also require specifying the remaining balance sheet random variables. Of course, the interbank assets and liabilities $\bar{Z}_v, \bar{X}_v$ are determined endogenously by the exposures $\bar{\Omega}$, and the buffer variables $\bar{\Delta}$ typically amount to additional linear constraints on external balance sheet entries. In absence of more detailed information than this, it makes sense to model such additional random variables as combinations of a minimal collection of further independent random variables.

The proposal to build RFNs this way is common in the literature on random financial networks. For example, Nier et al in [72], start with a random directed Poisson skeleton with 25 nodes, and conditioned on the skeleton, specify random variables for buffers and exposures that depend on node and edge types, but are otherwise independent.

4.8.1 The LTI Property

In the subsequent sections of the book, our analysis will focus on random ACG skeletons, which in the large N limit have the locally tree-like property. In the ACG setting, our dependence hypothesis becomes the following definition.

Definition 11. A random financial network (RFN) $(\mathcal{N},\mathcal{E},\bar{\Delta},\bar{\Omega})$ is said to have the *locally tree-like independence (LTI) property* when it satisfies the following conditions:

1. The skeleton graph is an infinite (directed, indirected or mixed) ACG $(\mathcal{N},\mathcal{E})$, with arbitrary node and edge type distributions $\{P,Q\}$. In this general setting, each node v is assigned its degree type τ_v, that is jk or k in the case of directed and undirected graphs and more general for mixed graphs. Similarly, each edge ℓ is assigned its type τ_ℓ, which may be kj or kk' or more general. Here, the indices j,k,k' run over the collection of possible integer degrees.
2. Conditioned on the realization of the skeleton graph $(\mathcal{N},\mathcal{E})$, the buffer random variables $\bar{\Delta}_v,\ v\in\mathcal{N}$ and exposure random variables $\bar{\Omega}_\ell,\ \ell\in\mathcal{E}$ form a mutually independent collection. Moreover, the buffer distribution of $\bar{\Delta}_v$ depends only on the type τ_v of v and the exposure distribution of $\bar{\Omega}_\ell$ depends only on the type τ_ℓ of ℓ.

4.8.2 Ramifications of the LTI Property

As we shall see in the remainder of this book, the LTI property is designed to enable a cascade analysis in complex RFNs that follows the computational template laid out in our simple version of the Watts cascade model. This works essentially because under LTI, the shocks transmitted through different edges entering a node never have a chance to become dependent. The mathematical advantages that stem from LTI will become clear as we proceed to other models.

To reiterate a second point, an essentially arbitrary dependence structure for the balance sheet random variables $\bar{\Delta}_v,\bar{\Omega}_\ell$ is dramatically reduced under LTI to the dependence encoded into the skeleton graph. Thus LTI reins in the amount of information needed to completely specify an RFN model. If one feels a given dependence structure induced by the skeleton is overly restrictive, yet wishes to retain the LTI property, one can explore the further option to add complexity to the RFN at the skeleton level, for example, by extending the concept of node and edge types using the IRG construction described in Section 3.4.

The question to ask of our proposed conceptual framework is thus pragmatic: why or when can we hope that computations based on the LTI assumption will have some predictive power when transferred to a real world network model where the assumption is apparently far from true. Pushing further, is it possible to weaken the LTI assumption, and to extend its exact consequences to a broader setting? Some measure of optimism that LTI computations are predictive even when the skeleton graph has high clustering or is otherwise far from locally-treelike can be gained by extensive investigations of the question in the random network literature. For example, [63] makes a strong case for the "unreasonable effectiveness" of tree-based theory for networks with clustering, and gives some guidelines to predict the circumstances under which tree-based theory can provide useful approximations.

Chapter 5
Zero Recovery Default Cascades

Abstract This chapter realizes the central aim of the book, which is to understand a simple class of cascades on financial networks as a generalization of percolation theory. The main results apply to random financial networks with locally tree-like independence and characterize zero-recovery default cascade equilibria as fixed points of certain cascade mappings. The proofs of the main results follow a new and distinctive template presented here for the first time, that has the important virtue that its logic extends to LTI financial networks of arbitrary complexity. Numerical computations, both large network analytics and finite Monte Carlo simulations, verify that essential characteristics such as cascade extent and cascade frequency can be derived from the properties of the cascade fixed points.

Keywords: Directed configuration graph, random buffers, random edge weights, cascade mapping theorem, numerical implementation.

In Chapter 3 we have understood something about possible models for the skeleton of a financial network, and in Chapter 4 we showed how two concepts, bootstrap percolation and site percolation, reveal the nature of the cascades observed in the Watts model. It is time to return our attention to financial networks and to investigate how well the various systemic cascade mechanisms described in Chapter 2 can be adapted to the setting of random financial networks, using the Watts model as a template. In this chapter, we confine the discussion to simple default cascades, under the condition of zero recovery as assumed by [42]: during a crisis banks recover none of their assets loaned to a defaulted bank.

At a theoretical level, perhaps the most important new contribution of this chapter is the proof of Theorem 12 which provides the explicit large N asymptotic limit of the cascade mapping for the extended G-K default cascade model. This proof is very different in spirit and detail than the well-known proof given in a similar setting by Amini, Cont and Minca [7].

We begin again with a network of N banks whose schematic balance sheets are as shown in Table 2.1, consisting of the collection of *book values* $(\bar{\mathrm{Y}},\bar{\mathrm{Z}},\bar{\mathrm{D}},\bar{\mathrm{X}},\bar{\Omega})$ with interbank exposures $\bar{\Omega}_\ell = \bar{\Omega}_{wv}$ (what w owes v) that are consistent with the interbank assets and liabilities

$$\bar{Z}_v = \sum_w \bar{\Omega}_{wv}, \quad \bar{X}_v = \sum_w \bar{\Omega}_{vw} .$$

The initial capital buffer of bank v is defined by $\bar{\Delta}_v = \bar{Y}_v + \bar{Z}_v - \bar{D}_v - \bar{X}_v$. At the onset of our schematic financial crisis, certain banks are found to be initially insolvent, defined by the condition $\bar{\Delta}_v \le 0$. By the law of limited liability, insolvency is assumed to trigger immediate default. Under an assumption about the loss given default, losses will be transmitted to both external creditors, and, importantly for systemic risk, to creditor banks. This implies the possibility of a default cascade crisis.

We suppose that counterparty relations, i.e. links, are expensive to maintain, and so in a large network, the matrix of counterparties is sparse. These relationships are also changing in time, and not easily observable, and can therefore considered to be random: the skeleton graph is a large, sparse, directed random graph $(\mathcal{N}, \mathcal{E})$. The balance sheets likewise change rapidly, are not well observed, and can be thought of as random variables. Under such conditions, we have argued that it is appropriate to adopt the notion of random financial network (RFN) with the *locally tree-like independent* (LTI) assumption of Section 4.8.1 as the underlying dependence hypothesis.

In the present chapter we focus our attention on the simplest of default mechanisms, namely the zero recovery mechanism of Gai and Kapadia [42], governed by the cascade equation (2.14) we derived in Chapter 2 :

$$\Delta_v^{(n)} = \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}(1 - \tilde{h}(\Delta_w^{(n-1)})) = \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv} \mathbf{1}(\Delta_w^{(n-1)} \le 0)) \qquad (5.1)$$

with initial values $\Delta_v^{(0)} = \bar{\Delta}_v$. As we observed earlier, these equations, and therefore the entire cascade dynamics, depend only on the skeleton graph $(\mathcal{N}, \mathcal{E})$ and the reduced balance sheet variables $\bar{\Delta}, \bar{\Omega}$. Our central aim is to apply this cascade mechanism to a RFN that satisfies the LTI hypothesis, and to determine the large scale properties of the behaviour that results.

5.1 The Gai-Kapadia Model

The specification of the Gai-Kapadia model we first present generalizes the default cascade introduced by [42] in several respects, but retains a restrictive condition on exposures $\bar{\Omega}$ that will be removed later in the chapter. It also satisfies the LTI property of Section 4.8.1.

Assumptions 6.

1. As in the description of the model in Section 2.1.3, banks have limited liability and receive a zero recovery[1] of interbank liabilities from any defaulted debtor bank.
2. The skeleton graph is a directed ACG with probabilities (P,Q) that satisfy the consistency conditions (3.2) with mean degree z.
3. Conditionally on the skeleton, banks' capital buffers $\bar{\Delta}_v$ are a collection of independent non-negative random variables whose cumulative probability distributions depend on the node type (j_v,k_v) and are given by

$$\mathbb{P}[\bar{\Delta}_v \le x | v \in \mathcal{N}_{jk}] := D_{jk}(x),\ x \ge 0\ . \tag{5.2}$$

4. Interbank exposures $\bar{\Omega}_{wv}$ are deterministic constants that are equal across the debtors $w \in \mathcal{N}_v^-$ of each bank v, and depend on the degree type (j_v,k_v) of v.[2] This implies $\bar{\Omega}_{wv} = \bar{\Omega}_{jk}$ when $v \in \mathcal{N}_{jk}$ where $\bar{\Omega}_{jk} = \bar{Z}_{jk}/j$ for a collection of interbank asset parameters $\bar{Z}_{jk}$.
5. The remaining balance sheet quantities are freely specified.

Apart from the restrictive condition on exposures $\bar{\Omega}$, this is a natural setting for the G-K default cascade. The G-K cascade mechanism has a symmetry that allows the rescaling of every bank and its exposures, while leaving the sequence of defaults unchanged. As one checks easily, if $\lambda_v, v \in \mathcal{N}$ is any collection of positive numbers, specifications $\bar{\Delta}_v, \bar{\Omega}_{wv}$ and $\lambda_v\bar{\Delta}_v, \lambda_v\bar{\Omega}_{wv}$ lead to identical cascades. Therefore, our restriction on exposures is equivalent to saying that they can simultaneously be set to 1 under this symmetry, by taking $\lambda_v = \lambda_{jk} = j/\bar{Z}_{jk}$ for each $v \in \mathcal{N}_{jk}$. For example, taking $\lambda_{jk} = 5j$ changes the benchmark exposures chosen by Gai and Kapadia in their paper [42] to constant exposures $\bar{\Omega} = 1$.

5.1.1 Shocks and the Solvency Condition

Like the early adopters that initiate Watts' adoption cascades, our schematic financial crises are triggered by a set $\mathcal{D}_0$ of initially defaulted banks. Perhaps they default as the result of a collective shock to the system, leaving the remaining banks with depleted capital buffers. Or perhaps one bank defaults for idiosyncratic reasons. All such cases can be modelled by supposing these initially defaulted banks have $\bar{\Delta}_v = 0$ while the remaining banks have positive buffers. The set of initially defaulted banks $\mathcal{D}_0 = \{v : \bar{\Delta}_v = 0\}$ thus has conditional probability

$$p^{(0)}_{jk} := \mathbb{P}[v \in \mathcal{D}_0 | v \in \mathcal{N}_{jk}] = D_{jk}(0)\ . \tag{5.3}$$

[1] It is a trivial matter to extend the model slightly to allow for a constant fractional recovery with $R < 1$.

[2] Temporarily, we allow $\bar{\Omega}_{wv}$ to depend only on the node type of v rather than the edge type of wv. In the next section we will revert to the usual convention.

If we define $D_v^{-1} = 0$ for all v, then the indicator functions for the set $\mathscr{D}_n$ of defaulted banks at step $n \geq 0$ are defined recursively by the formula

$$D_v^n := \mathbf{1}(v \in \mathscr{D}_n) = \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w' \in \mathscr{N}_v} \bar{\Omega}_{w'v} D_{w'}^{n-1}\right) . \quad (5.4)$$

Note that the directed graph condition means that $\bar{\Omega}_{w'v} \neq 0$ only if $w' \in \mathscr{N}_v^-$, which implies the sum in (5.4) is actually over $\mathscr{N}_v^-$. We now present a *formal* derivation of a recursive formula for the conditional default probabilities $p_{jk}^{(n)} := \mathbb{P}[v \in \mathscr{D}_n | v \in \mathscr{N}_{jk}]$ after $n \geq 0$ cascade steps.

Consider first (5.4) for bank v at the $n = 1$ step, conditioned on the locally tree-like skeleton $(\mathscr{N}, \mathscr{E})$. By the LTI property, when $v \in \mathscr{N}_{jk}$, the debtor nodes $w_i \in \mathscr{N}_v^-, i \in \{1, \dots, j\} := [j]$ are distinct and $\{\bar{\Delta}_v, D_{w_i}^0\}$ are independent random variables. Therefore,

$$\mathbb{P}[v \in \mathscr{D}_1 | \mathscr{N}, \mathscr{E}, v \in \mathscr{N}_{jk}] = \mathbb{P}\left[\bar{\Delta}_v \leq \bar{\Omega}_{jk} \cdot \sum_{i \in [j]} D_{w_i}^0 \middle| \mathscr{N}, \mathscr{E}, v \in \mathscr{N}_{jk}\right] \quad (5.5)$$
$$= \sum_{m=0}^{j} \mathbb{P}\left[\bar{\Delta}_v \leq \bar{\Omega}_{jk} m \middle| v \in \mathscr{N}_{jk}\right] \mathbb{P}\left[\sum_{i \in [j]} D_{w_i}^0 = m \middle| \mathscr{N}, \mathscr{E}, v \in \mathscr{N}_{jk}\right]$$

where we sum over all possible values m for the number of defaulted neighbours. Knowing the skeleton determines the node types (j_i, k_i) of each w_i, and so summing over σ, denoting the possible size m subsets of the index set $[j]$, leads to

$$\mathbb{P}\left[\sum_{i \in [j]} D_{w_i}^0 = m \middle| \mathscr{N}, \mathscr{E}, v \in \mathscr{N}_{jk}\right]$$
$$= \sum_{\sigma \subset [j], |\sigma| = m} \left(\prod_{i \in \sigma} p_{j_i k_i}^{(0)}\right) \left(\prod_{i \notin \sigma} (1 - p_{j_i k_i}^{(0)})\right) . \quad (5.6)$$

Now, to take the expectation over the skeleton $(\mathscr{N}, \mathscr{E})$ we refer back to the details of the ACG construction of Section 3.3, in particular Theorem 4, which we apply to the tree graph g which consists of v joined to its debtor nodes $w_i, i \in [j]$. Using a conditional version of (3.25) for this tree graph leads to the asymptotic large N formula

$$\mathbb{E}\left[\left(\prod_{i \in \sigma} p_{j_i k_i}^{(0)}\right) \left(\prod_{i \notin \sigma} (1 - p_{j_i k_i}^{(0)})\right) \middle| v \in \mathscr{N}_{jk}\right]$$
$$= \sum_{j_1, k_1} \cdots \sum_{j_j, k_j} \prod_{i \in \sigma} p_{j_i k_i}^{(0)} \prod_{i \notin \sigma} (1 - p_{j_i k_i}^{(0)}) \prod_{i \in [j]} (P_{j_i | k_i} Q_{k_i | j}) .$$

By introducing $\tilde{\pi}_j^{(0)} := \sum_{j', k'} p_{j' k'}^{(0)} P_{j' | k'} Q_{k' | j}$ this expectation can be written

$$(\tilde{\pi}_j^{(0)})^m\ (1-\tilde{\pi}_j^{(0)})^{j-m}\ . \tag{5.7}$$

Finally, we put (5.6), (5.6), (5.7) together with the cumulative distribution function for $\bar{\Delta}_v$ to obtain the formula for the $n=1$ conditional default probability:

$$p_{jk}^{(1)} = \mathbb{P}[v \in \mathscr{D}_1 | v \in \mathscr{N}_{jk}] = \sum_{m=0}^{j} D_{jk}(m\bar{\Omega}_{jk})\ \mathrm{Bin}(j,\tilde{\pi}_{jk}^{(0)},m) \tag{5.8}$$

where $\bar{\Omega}_{jk} = \bar{Z}_{jk}/j$, $\mathrm{Bin}(\cdot,\cdot,\cdot)$ denotes a binomial probability just as in (4.34), and

$$\tilde{\pi}_{jk}^{(0)} = \mathbb{P}[w \in \mathscr{D}_0 | w \in \mathscr{N}_v^-, v \in \mathscr{N}_{jk}] = \sum_{j',k'} p_{j'k'}^{(0)} P_{j'|k'} Q_{k'|j}\ . \tag{5.9}$$

Note that $\tilde{\pi}_{jk}^{(0)} = \tilde{\pi}_j^{(0)}$ is independent of k.

The extension of this argument to all $n \geq 1$ rests on the special relation between the G-K mechanism and the LTI dependence structure of the model. We observe that conditioned on $(\mathscr{N},\mathscr{E})$, the event $v \in \mathscr{D}_1$ depends on $\bar{\Delta}_v, \bar{\Delta}_{w_i}, i \in [j_v]$ where the $w_i \in \mathscr{N}_v^-$ are nodes "up-stream" from v. We see that this upstream dependence extends to all n when $(\mathscr{N},\mathscr{E})$ is a tree graph, and will be asymptotically true as $N \to \infty$ on configuration graphs. Thus, under the LTI assumption, by induction on n, the event $v \in \mathscr{D}_n$ depends on the events $w_i \in \mathscr{D}_{n-1}$ for $w_i \in \mathscr{N}_v^-, i \in [j_v]$ which are conditionally independent and have identical default probabilities

$$\tilde{\pi}_j^{(n-1)} := \mathbb{P}[w \in \mathscr{D}_{n-1} | w \in \mathscr{N}_v^-,\ v \in \mathscr{N}_{jk}] \tag{5.10}$$

that do not depend on k. For reasons that will become clear in the next section, it is natural to reexpress (5.10) in terms of a further collection of conditional probabilities:

$$\tilde{\pi}_j^{(n-1)}\ =\ \sum_{k'} \pi_{k'}^{(n-1)} Q_{k'|j} \tag{5.11}$$

$$\pi_{k'}^{(n-1)} := \mathbb{P}[w \in \mathscr{D}_{n-1} | k_w = k'] = \sum_{j'} p_{j'k'}^{(n-1)}\ P_{j'|k'}\ . \tag{5.12}$$

Moreover, the conditional default probabilities $p_{jk}^{(n)} = \mathbb{P}[v \in \mathscr{D}_n | v \in \mathscr{N}_{jk}]$ will be given by a formula just like (5.8). This completes the formal justification for cascade mapping formulas for the basic G-K model:

Formal Proposition 11 (Gai-Kapadia Cascade)**.** *Consider the sequence of G-K financial networks $(N,P,Q,\bar{\Delta},\bar{\Omega})$ satisfying Assumptions 6. Let $p_{jk}^{(0)} = D_{jk}(0)$ and $\pi_{k'}^{(0)} = \mathbb{P}[w \in \mathscr{D}_0 | k_w = k']$. Then the following formulas hold in the limit as $N \to \infty$:*

1. *$\tilde{\pi}_j^{(0)} = \mathbb{P}[w \in \mathscr{D}_0 | w \in \mathscr{N}_v^-,\ v \in \mathscr{N}_{jk}] = \sum_{k'} \pi_{k'}^{(0)}\ Q_{k'|j}$, which is independent of k.*
2. *For any $n = 1,2,\ldots$, the quantities $\tilde{\pi}_j^{(n-1)}, p_{jk}^{(n)}, \pi_k^{(n)}$ satisfy the recursive formulas*

$$\tilde{\pi}_j^{(n-1)} = \sum_{k'} \pi_k^{(n-1)} Q_{k'|j} , \tag{5.13}$$

$$p_{jk}^{(n)} := = \mathbb{P}[v \in \mathscr{D}_n | v \in \mathscr{N}_{jk}] = \sum_{m=0}^{j} D_{jk}(m\bar{\Omega}_{jk}) \operatorname{Bin}(j, \tilde{\pi}_j^{(n-1)}, m) \tag{5.14}$$

$$\pi_k^{(n)} = \sum_{j'} p_{j'k}^{(n)} P_{j'|k} . \tag{5.15}$$

3. *The new probabilities $\pi^{(n)} = (\pi_k^{(n)})$ are a vector valued function $G(\pi^{(n-1)})$ which is explicit in terms of the specification $(N, P, Q, \bar{\Delta}, \bar{\Omega})$.*
4. *The cascade mapping G maps $[0,1]^{\mathbb{Z}_+}$ onto itself, and is monotonic, i.e. $G(\mathbf{a}) \leq G(\mathbf{b})$ whenever $\mathbf{a} \leq \mathbf{b}$, under the partial ordering relation defined by $\mathbf{a} \leq \mathbf{b}$ if and only if $a_k \leq b_k$ for all k. Since $\pi^{(0)} = G(0)$, the sequence $\pi^{(n)}$ converges to the least fixed point $\pi^* \in [0,1]^{\mathbb{Z}_+}$, that is*

$$\pi^* = G(\pi^*) . \tag{5.16}$$

Remark 4. When the skeleton is non-assortative, meaning $Q_{k'|j} = Q^+_{k'}$ is independent of j, then each $\tilde{\pi}_j^{(n)} = \tilde{\pi}^{(n)}$ is independent of j. Under this condition, the cascade mapping can be reduced to a scalar function $\tilde{G} : \tilde{\pi} \in [0,1] \to [0,1]$ and our theorem is subsumed in the scalar cascade mapping of Theorem 3.6 in [7].

As compelling as Proposition 11 is, its power is weakened by the restrictive assumption on the form of the interbank exposures. Let us therefore repair this deficiency in the next section, before exploring the broader implications of the G-K cascade mapping theorem.

5.2 The G-K Model with Random Link Weights

The primary motivation for allowing the link weights to become random variables is to correct the asymmetry in the way interbank exposures are specified in the previous section. According to Assumption 6.4, "interbank exposures are deterministic constants that are equal across the debtors of each bank", which means essentially that the creditor bank is choosing to lend equally to its counterparties. One can easily argue the opposite case that the debtor bank might be the counterparty that chooses to borrow equally. And that the reality is likely closer to a compromise, wherein exposure sizes depend on attributes of both the borrower and the lender.

A look at the argument leading to Proposition 11 will convince one that if the exposures $\bar{\Omega}_\ell$ for $\ell \in \mathscr{E}_v^-$ depend deterministically on k_ℓ rather than j_ℓ, then they will be effectively random variables when conditioned on the degree $j_\ell = j_v$. And the main Proposition 11 cannot handle this fact. As we will now find, dealing with the simple variation where the $\bar{\Omega}_\ell$ depend deterministically on k_ℓ rather than j_ℓ is just as difficult as to deal with the more general case where the exposures $\bar{\Omega}_\ell$ are arbitrary random variables depending on both k_ℓ, j_ℓ. It is in this more general specification that we find what is arguably the most natural setting for LTI default models with zero recovery.

Thus, in this section we analyze the final version of the model assumptions, namely Assumptions 6 with 6.4 replaced by 4.4′

Assumption (6.4’). Each interbank exposure $\bar{\Omega}_\ell$ depends randomly on its edge type (k_ℓ, j_ℓ). Conditionally on the skeleton, they form a collection of independent nonnegative random variables, independent as well from the default buffers $\bar{\Delta}_v$. Their cumulative distribution functions (CDFs) and probability distribution functions (PDFs) are given for $x \geq 0$ and any k, j by

$$\begin{aligned} W_{kj}(x) &= \mathbb{P}[\bar{\Omega}_\ell \leq x | \ell \in \mathscr{E}_{kj}], \\ w_{kj}(x) &= dW_{kj}(x)/dx\ . \end{aligned} \tag{5.17}$$

Note that these generalized assumptions, particularly the assumed independence built into Assumptions 6.3, 6.4′, are still consistent with the basic LTI structure identified in Section 4.8. But now that exposures $\bar{\Omega}_\ell$ are random variables, we must reconsider the previous analysis of the default cascade.

First we reconsider (5.5) which now takes the form

$$\mathbb{P}[v \in \mathscr{D}_1 | \mathscr{N}, \mathscr{E}, v \in \mathscr{N}_{jk}] = \mathbb{P}\left[\bar{\Delta}_v \leq \sum_{i \in [j]} \bar{\Omega}_{\ell_i} D^0_{w_i} \middle| \mathscr{N}, \mathscr{E}, v \in \mathscr{N}_{jk}\right]$$

where $\ell_i = (w_i v)$ for each i. Now, under this condition, $\bar{\Omega}_{\ell_i}, D^0_{w_i}, i \in [j]$ is a collection of independent random variables. We can calculate that

$$\widetilde{W}^{(0)}_j(x) \ := \ \mathbb{P}[\bar{\Omega}_{\ell_i} D^0_{w_i} \leq x | \mathscr{N}, \mathscr{E}, \ell_i \in \mathscr{E}^-_v, v \in \mathscr{N}_{jk}]$$

is independent of k and given by

$$\widetilde{W}^{(0)}_j(x) = \sum_{k'} Q_{k'|j} \left((1 - \pi^{(0)}_{k'}) \mathbf{1}(x \geq 0) + \pi^{(0)}_{k'} W_{k'j}(x) \right)\ . \tag{5.18}$$

where as before, $\pi^{(0)}_{k'} = \sum_{j'} p^{(0)}_{j'k'}\ P_{j'|k'}$. By the LTI property, this conditional independence of edge weights $\bar{\Omega}_{\ell_i}$ and indicators $D^{n-1}_{w_i}$ continues for higher n, and we find that (5.18) extends to

$$\widetilde{W}^{(n-1)}_j(x) = \sum_{k'} Q_{k'|j} \left((1 - \pi^{(n-1)}_{k'}) \mathbf{1}(x \geq 0) + \pi^{(n-1)}_{k'} W_{k'j}(x) \right) \tag{5.19}$$

for all $n \geq 1$. For this, we need to make the inductive verification using the LTI property that for $n \geq 0$ the collection of defaulted node events $w_i \in \mathscr{D}_{n-1}$ is mutually independent, and independent of all buffer and exposure random variables downstream to w_i. Then it follows that, conditioned on $(\mathscr{N}, \mathscr{E})$ and $v \in \mathscr{N}_{jk}$, the collection of random variables $\mathbf{1}(w_i \in \mathscr{D}_{n-1}), \bar{\Omega}_{\ell_i}$ and $\bar{\Delta}_v$ are mutually independent.

Bringing these pieces together, we find that for $v \in \mathcal{N}_{jk}$, the conditional default event $v \in \mathcal{D}_n = \{\bar{\Delta}_v \le \sum_{i\in[j]} \bar{\Omega}_{\ell_i} D^{n-1}_{w_i}\}$ has the form

$$X \le Z, \quad Z := \sum_{i=1}^{j} Y_i$$

where $X, Y_1, \ldots, Y_j$ are independent non-negative random variables. Moreover, their CDFs are $F_X(x) = D_{jk}(x)$ and $F_Y(x) = \widetilde{W}^{n-1}_j(x)$ with $Y_i \sim Y$, respectively. The probability in question is thus

$$\mathbb{P}[X \le Z] = \int_0^\infty \int_0^\infty \mathbf{1}(x \le z)\; f_X(x) f_Z(z) dx dz = \int_{\mathbb{R}} F_X(z) f_Z(z) dz = \langle F_X, f_Z \rangle \quad (5.20)$$

where $\langle f, g \rangle := \int_{\mathbb{R}} \bar{f}(x) g(x) dx$ is the usual (Hermitian) inner product over $\mathbb{R}$. To determine the PDF $f_Z(z) = F'_Z(z)$ of Z, let us stop for a moment to review the convolution of probability density functions.

The sum of independent random variables $Z = \sum_{i=1}^{j} Y_i$ with known PDFs $f_i(x)$ has a PDF $f_Z(x)$ given by convolution:

$$f_Z(x) = [f_1 \circledast f_2 \circledast \cdots \circledast f_j](x)$$

where the convolution product of functions is defined by

$$[f \circledast g](x) = \int_{\mathbb{R}} f(x-y) g(y) dy \; . \quad (5.21)$$

In the present context, this means f_Z is a convolution power, $f_Z = (f_Y)^{\circledast j}$ with

$$f_Y(x) = \tilde{w}^{(n-1)}_j(x) = \frac{d\widetilde{W}^{(n-1)}_j(x)}{dx} = \sum_{k'} Q_{k'|j} \left((1 - \pi^{(n-1)}_{k'}) \delta_0(x) + \pi^{(n-1)}_{k'} w_{k'j}(x) \right) \; . \quad (5.22)$$

Note that here we need to represent the point masses at $Y_i = 0$ and other potential point masses using delta functions $\delta_a(x)$.

We can now put together what is perhaps the main result of this book, which is the generalization of Proposition 11 to account for random link weights. Given its importance to the overall theory of financial cascades, we will also provide in the subsequent section a rigorous justification, similar to Proposition 3 in Section 4.2, for the formal $N = \infty$ arguments we have been contented with until now.

Formal Proposition 12 (Gai-Kapadia Cascade, RLW version)**.** *Consider the sequence of G-K financial networks $(N, P, Q, \bar{\Delta}, \bar{\Omega})$ satisfying Assumptions 6, with Assumption 6.4 replaced by the random link weight Assumption 6.4' above. Let $p^{(0)}_{jk} = D_{jk}(0)$ and $\pi^{(0)}_k = \mathbb{P}[w \in \mathcal{D}_0 | k_w = k]$ be initial default probabilities. Then the following formulas hold as $N \to \infty$:*

1. For any $n = 1, 2, \ldots$, the quantities $p^{(n)}_{jk}, \pi^{(n)}_k$ satisfy the recursive formulas

$$p_{jk}^{(n)} = \langle D_{jk}, (\widetilde{w}_j^{(n-1)})^{\circledast j} \rangle \ , \tag{5.23}$$

$$\pi_k^{(n)} = \sum_{j'} p_{j'k}^{(n)} P_{j'|k} \tag{5.24}$$

where the PDFs $\widetilde{w}_j^{(n-1)}(x)$ *are given by* (5.22)*.*

2. *The new probabilities* $\pi^{(n)}$ *are a vector valued function* $G(\pi^{(n-1)})$ *which is explicit in terms of the specification* $(N,P,Q,\bar{\Delta},\bar{\Omega})$*.*
3. *The cascade mapping G maps* $[0,1]^{\mathbb{Z}_+}$ *onto itself, and is monotonic. Since* $\pi^{(0)} = G(0)$*, the sequence* $\pi^{(n)}$ *converges to the least fixed point* $\pi^* \in [0,1]^{\mathbb{Z}_+}$*, that is*

$$\pi^* = G(\pi^*) \ . \tag{5.25}$$

Remarks 3.

1. We observe that the main conclusion of Proposition 11, namely the existence of a vector-valued monotonic cascade mapping $G : [0,1]^{\mathbb{Z}_+} \to [0,1]^{\mathbb{Z}_+}$ remains true with general random link weights.
2. The vector-valued fixed point equation seems to be the natural feature of this type of model. In order to get a scalar fixed point condition such as the one obtained in [7], we need to assume both non-assortatitivity, i.e. $Q_{kj} = Q_k^+ Q_j^-$, and that the probability distributions of $\bar{\Omega}_{wv}$ depend on the degree type of v but not w.
3. The numerical computation of this cascade mapping now makes intensive use of equation 5.23, which involves repeated high-dimensional integrations. Fortunately, as demonstrated in [52], Fourier Transform methods make this type of computation extremely efficient.

5.2.1 A Theorem on the G-K Cascade

The aim here is to support the formal argument for Proposition 12 with a rigorous proof of a weaker statement about the same model. Amini, Cont and Minca, in [7], have proved a scalar fixed point theorem for the cascade equilibrium in a narrower version of the G-K model by an argument that extends random graph methods developed by [68] and many others. However, it seems difficult to extend their method and ideas directly to more general settings. The method we now present provides an alternative, flexible and simple approach that is applicable to a broad range of LTI cascade models. We focus on computing the probability of events $v \in \mathscr{D}_n$:

$$\tilde{p}_{jk}^{(n,N)} := \mathbb{E}^N \Big[N^{-1} \sum_{v \in \mathscr{N}} \mathbf{1}(v \in \mathscr{D}_n \cap \mathscr{N}_{jk}) \Big] \ . \tag{5.26}$$

on G-K networks of finite size N.

Theorem 13. *Consider the sequence of G-K financial networks* $(N,P,Q,\bar{\Delta},\bar{\Omega})$ *satisfying Assumptions 6, with Assumption 6.4 replaced by the random link weight As-*

sumption 6.4' above. Suppose also P,Q are supported on the finite set $\{0,1,\ldots,K\}^2$. Then for any finite $n \geq 1$ and $j,k \leq K$, as $N \to \infty$

$$|\tilde{p}^{(n,N)}_{jk} - P_{jk} p^{(n)}_{jk}| \stackrel{\mathrm{P}}{=} O(N^{-1/2}) \tag{5.27}$$

where the quantities $p^{(n)}_{jk}$ are given by the recursive formulas (5.23), (5.24).

Proof of Theorem 13: Note that by the node permutation symmetry of the ACG construction, and iterated expectations with the intermediate conditioning on the skeleton $(\mathscr{N},\mathscr{E})$

$$\hat{p}^{(n,N)}_{j_0k_0} = \mathbb{E}^N\Big[\mathbb{E}^N[\mathbf{1}(w_0 \in \mathscr{D}_n \cap \mathscr{N}_{j_0k_0})|,(\mathscr{N},\mathscr{E})]\Big] \tag{5.28}$$

where $v = w_0$ is defined as the root of the wiring sequence W. The key observation is that the inner conditional expectation depends only on the n-th order in-neighbourhood $\mathscr{N}^{-,n}_{w_0}$ of w_0 in $(\mathscr{N},\mathscr{E})$ and is completely N independent. The N-dependent outer expectation can be analyzed similarly to the proof of Proposition 3.

Introduce a recursive labelling $\lambda(w)$ of the nodes $w \in \mathscr{N}^{-,n}_{w_0}$, together with their node types $\tau = (j,k)$, in the increasing sequence of in-neighbourhoods $\mathscr{N}^{-,i}_{w_0}$. We grow the neighbourhoods by sequentially adding in-edges following precisely the algorithm developed in Section 3.3.2. Label $w_0 := \mathscr{N}^{-,0}_{w_0}$ by $\lambda(w_0) = 0$ and note its type is $\tau(w_0) = (j_0,k_0)$. Recursively, label any $w \in \mathscr{N}^{-,i}_{w_0} \setminus \mathscr{N}^{-,i-1}_{w_0}, i > 0$ by the $i+1$ component vector $(\lambda(s(w)), j)$ for some index $j \leq j(s(w))$. Here $s(w) \in \mathscr{N}^{i-1}_{w_0}$ denotes the node to which w attaches. Assign node w a node-type $\tau(\lambda(w))$.

For fixed n, the set $\mathscr{L}$ of all possible node labelings λ is finite, and has a natural lexicographic order which we follow in growing the skeleton graph link-by-link. The expectation in (5.28) can be written

$$P_{j_0k_0} \sum_{\tau_1} \cdots \sum_{\tau_n} \mathbb{P}^N[\tau_1,\ldots,\tau_n | h, w_0 \in \mathscr{N}_{j_0k_0}]\ \mathbb{E}[\mathbf{1}(w_0 \in \mathscr{D}_n)|\tau_1,\ldots,\tau_n] \tag{5.29}$$

We emphasize that here the expectation is N independent. Each sum over τ_i is a sum over the possible types of the nodes in $\mathscr{N}^{-,i}_{w_0} \setminus \mathscr{N}^{-,i-1}_{w_0}$, and thus each term in the overall sum corresponds to one possible neighbourhood $h = \mathscr{N}^{-,n}_{w_0}$, which is a fully labelled "configuration" h rooted to w_0 as defined in Section 3.3.2. By Theorem 4 we can conclude that

$$\mathbb{P}^N[\tau_1,\ldots,\tau_n | h, w_0 \in \mathscr{N}_{j_0k_0}] = \prod_{w \in \mathscr{N}^{-,n}_{w_0}} (P_{j_w|k_w} Q_{k_w|j_{s(w)}})(1 + O(N^{-1/2}))$$

when h is a tree, and $O(N^{-1}$ when h has cycles. Therefore, up to an $1 + O(N^{-1/2})$ factor, the finite sum in (5.29) equals its $N \to \infty$ limit which is a sum over tree configurations h which by inspection has the value $p^{(n)}_{j_0k_0}$ given by Proposition 12.

We conclude that
$\hat{p}^{(n,N)}_{j_0k_0} \stackrel{\mathrm{P}}{=} P_{j_0k_0} p^{(n)}_{j_0k_0} + O(N^{-1/2})$ as required. □

The structure of this proof is indeed not sensitive to the specific nature of the cascade dynamics and random financial network, but is highly dependent on the LTI property. Therefore, we have confidence that large N asymptotics will continue to hold for a wide range of more complex network models as long as they have a generalized LTI property. Note also that this strategy clearly works to yield rigorous proofs for the Watts model of Section 4.5.2 and the basic G-K model of Section 5.1.

5.3 Measures of Cascade Impact

In cascade modelling, it is always important to have a number of economic measures of the impact of such hypothetical crises. In this section, we explore some simple statistical measures that can be computed analytically using the cascade mapping of Proposition 12. The first global measure of cascade impact is of course the unconditional probability of eventual default, computed by

$$p^* := \mathbb{P}[v \in \mathscr{D}_\infty] = \sum_{jk} P_{jk} p^{(\infty)}_{jk} \ . \tag{5.30}$$

Also of interest are two variants of default probability that condition on a bank being either a debtor or creditor of another bank. That is, if $w \in \mathscr{N}_v^-$,

$$p^{*-} = \mathbb{P}[w \in \mathscr{D}_\infty | w \in \mathscr{N}_v^-] = \sum_{jk} P_{j|k} \, Q^+_k \, p^{(\infty)}_{jk} \tag{5.31}$$

$$p^{*+} = \mathbb{P}[v \in \mathscr{D}_\infty | w \in \mathscr{N}_v^-] = \sum_{jk} P_{k|j} \, Q^-_j \, p^{(\infty)}_{jk} \ . \tag{5.32}$$

Recall from Section 2.1.3 that the zero-recovery assumption implies there is a large cost to the system at the time any bank defaults, and we are certainly interested in computing measures of this impact. The first measure was defined by (2.16):

$$\textbf{Default Cascade impact:} \ \mathrm{DCI} = N^{-1} \sum_v \bar{\mathrm{X}}_v \, \mathbf{1}(v \in \mathscr{D}_\infty) \ .$$

The expected default cascade impact per bank can easily be computed in terms of the fixed point π^* and the eventual default probabilities $p^* = F(\pi^*)$:

$$\mathbb{E}[\mathrm{DCI}] = \sum_{jk} P_{jk} \mathbb{E}\left[\sum_{\ell \in \mathscr{E}_v^+} \bar{\Omega}_\ell \, \mathbf{1}(v \in \mathscr{D}_\infty) \middle| v \in \mathscr{N}_{jk} \right] \ .$$

In an LTI model, the collection $\bar{\Omega}_\ell, \ell \in \mathscr{E}_v^+$ and $\mathbf{1}(v \in \mathscr{D}_\infty)$ is mutually independent conditioned on $v \in \mathscr{N}_{jk}$, and for large N we have the asymptotic formula

$$\mathbb{E}[\mathrm{DCI}] = \sum_{jk} kP_{jk}p_{jk}^{(\infty)} \sum_{j'} Q_{j'|k}\mathbb{E}\bar{\Omega}_{kj'} \tag{5.33}$$

where $\mathbb{E}\bar{\Omega}_{kj} := \mathbb{E}[\bar{\Omega}_\ell | \ell \in \mathscr{E}_{kj}] = \int_{\mathbb{R}^+} x\, dW_{kj}(x)$.

By comparing default cascade impact to the average buffer size $\mathbb{E}[\bar{\Delta}_v] = \sum_{jk} P_{jk} \int_{\mathbb{R}^+} x\, dD_{jk}(x)$ we also get a measure of the average eventual buffer size:

$$\mathbb{E}[\Delta_v^{(\infty)}] = N^{-1}\mathbb{E}\left[\sum_v \left(\bar{\Delta}_v - \sum_{w\in\mathcal{N}_v^-} \bar{\Omega}_{wv}\mathbf{1}(w\in\mathscr{D}_\infty)\right)\right] = E[\bar{\Delta}_v - \mathrm{DCI}]\ . \tag{5.34}$$

A slightly different measure of the cascade impact is the *average shortfall* of banks' buffers compared to the default shocks impacting them:

$$\textbf{Average Shortfall:}\ \ \mathrm{AS} = N^{-1}\sum_v \left(\sum_{w\in\mathcal{N}_v^-} \bar{\Omega}_{wv}\mathbf{1}(w\in\mathscr{D}_\infty) - \bar{\Delta}_v\right)^+ .$$

The expected average shortfall also has a nice formula which we can compute using a formula related to (5.20): for two independent random variables X, Z,

$$\mathbb{E}[(Z-X)^+] = \int_{\mathbb{R}^2} (z-x)^+ f_X(x)\, f_Z(z)\, dxdz = \int_{\mathbb{R}} \tilde{F}_X(z)\, f_Z(z)\, dz = \langle \tilde{F}_X, f_Z\rangle \tag{5.35}$$

where $\tilde{F}_X(z) = \int_{-\infty}^z (z-x) f_X(x)dx$ is the integrated CDF of X. Thus we find the expected average shortfall is

$$\mathrm{EAS} = \sum_{jk} P_{jk}\ \langle \tilde{D}_{jk}, (\widetilde{w}_j^{(\infty)})^{\circledast j}\rangle \tag{5.36}$$

where $\tilde{D}'_{jk}(x) = D_{jk}(x)$ and $\widetilde{w}_j^{(\infty)}$ is the limit of (5.22) .

Default Correlation: We are also interested in *default correlation* and more refined measures of joint default such as *CoVaR* introduced by [4]. The most basic network measure of dependence is the joint probability of eventual default for counterparty pairs (w, v):

$$p_{joint}^{(\infty)} := \mathbb{P}[w\in\mathscr{D}_\infty, v\in\mathscr{D}_\infty | w\in\mathcal{N}_v^-]\ .$$

Equivalently, one can compute the default probabilities of a bank conditioned on default of one of its counterparties:

$$\mathbb{P}[v\in\mathscr{D}_\infty | w\in\mathscr{D}_\infty, w\in\mathcal{N}_v^-] = \frac{p_{joint}^{(\infty)}}{p^{*-}},\ \mathbb{P}[w\in\mathscr{D}_\infty | v\in\mathscr{D}_\infty, w\in\mathcal{N}_v^-] = \frac{p_{joint}^{(\infty)}}{p^{*+}} \tag{5.37}$$

where the denominators are the two natural conditional marginal default probabilities from (5.31),(5.32). The easiest way to compute such quantities is first to disaggregate the joint probability of *nondefault* over the types of w and v. We then have

$$\mathbb{P}[w \notin \mathscr{D}_\infty, v \notin \mathscr{D}_\infty | w \in \mathscr{N}_v^- \cap \mathscr{N}_{j'k'},\ v \in \mathscr{N}_{jk}]$$

$$= \mathbb{P}\left[\Delta_w \le \sum_{w' \in \mathscr{N}_w^-} \Omega_{w'w} D_{w'}^\infty, \Delta_v \le \sum_{w' \in \mathscr{N}_v^-} \Omega_{w'v} D_{w'}^\infty \mathbf{1}(w' \ne w) \middle| w \in \mathscr{N}_v^- \cap \mathscr{N}_{j'k'},\ v \in \mathscr{N}_{jk}\right]$$

$$= \left(1 - \langle D_{j'k'}, (\widetilde{w}_{j'}^{(\infty)})^{\circledast j'} \rangle\right)\left(1 - \langle D_{jk}, (\widetilde{w}_j^{(\infty)})^{\circledast j-1} \rangle\right) .$$

Note that the convolution power in the second factor is reduced by 1 since it is known that one counterparty of v has definitely not defaulted. Therefore, the unconditional joint probability of *non-default* is

$$\mathbb{P}[w \notin \mathscr{D}_\infty, v \notin \mathscr{D}_\infty | v \in \mathscr{N}_w^+]$$

$$= \sum_{j'k',jk} P_{j'|k'} Q_{k'j} P_{k|j} \left(1 - \langle D_{j'k'}, (\widetilde{w}_{j'}^{(\infty)})^{\circledast j'} \rangle\right)\left(1 - \langle D_{jk}, (\widetilde{w}_j^{(\infty)})^{j-1} \rangle\right)$$

$$= \sum_{jk} P_{k|j}\,(1 - \tilde{\pi}_j^*)\,\left(1 - \langle D_{jk}, (\widetilde{w}_j^{(\infty)})^{\circledast j-1} \rangle\right)$$

where $\tilde{\pi}_j^* = \sum_{k'} \pi_{k'}^* Q_{k'j}$. Since the marginal probabilities of default are p^{*-} and p^{*+} one can see that

$$p_{joint}^{(\infty)} = \mathbb{P}[w \notin \mathscr{D}_\infty, v \notin \mathscr{D}_\infty | v \in \mathscr{N}_w^+] + p^{*-} + p^{*+} - 1 . \tag{5.38}$$

5.4 The Cascade Condition

Just as in Section 4.5.3 we derived a cascade condition that characterizes the growth of small cascades in the Watts model, we now consider the possibilities for cascades on the Gai-Kapadia RFN that start with small initial default probabilities. That is, we let $D_{jk}(0) = p_{jk}^{(0)} := \varepsilon_{jk} \ge 0$ and suppose these are uniformly bounded by a small positive number $\bar{\varepsilon} > 0$:

$$|\varepsilon| := \max_{jk} |\varepsilon_{jk}| \le \bar{\varepsilon} .$$

Then the default buffer CDF (5.39) becomes

$$D_{jk}(x) = \varepsilon_{jk} + (1 - \varepsilon_{jk})\tilde{D}_{jk}(x),\ x \ge 0 \tag{5.39}$$

with $\tilde{D}$ interpreted as the CDF of $\bar{\Delta}_v$ conditioned on v not initially defaulted.

As we have remarked already, a very low density of initially defaulted banks means they are likely to be far apart in the network and the only probable way for a large cascade to develop is that there should be a positive probability for any single bank to trigger an increasing sequence of defaults, without regard to other initially defaulted banks. In Section 4.5.3 we found for the Watts model that this statement can be related to the existence of a giant connected "vulnerable" cluster.

We again write $G(\pi) = G(\pi;\varepsilon)$ for the cascade mapping, to highlight the dependence on the parameters ε and suppress the dependence on $\tilde{D}$. Now the sequence $\pi^{(n)}$ starting from the initial values $\pi^{(0)} = G(0;\varepsilon)$ must converge to the least fixed point $\pi^*(\varepsilon)$. The question now is: is $\pi^*(\varepsilon)$ of order ε or of order 1 as $\varepsilon \to 0$? In other words, is there a *cascade condition* that determines if an infinitesimally small initial "seed" fraction will grow to a large-scale cascade? In view of the vector valued nature of the cascade mapping, it turns out that the answer depends on the spectral radius of the derivative matrix ∇G with $\nabla G_{k,k'} = \partial G_k / \partial \pi_{k'}$ evaluated at $\pi = 0; \varepsilon = 0$. Recall that the *spectral radius* of ∇G, defined by $||\nabla G|| := \max_{a:|a|=1} |\nabla G \cdot a|$, is the largest eigenvalue of ∇G in absolute value.

In our framework, the derivative ∇G is easy to calculate:

$$\nabla G_{k,k'} = \sum_j \; j \left(\langle \tilde{D}_{jk}, w_{k'j} \rangle - \tilde{D}_{jk}(0) \right) \; Q_{k'|j} \; P_{j|k} \; . \tag{5.40}$$

Note each component of ∇G is non-negative: To enable an elementary proof of the following result, we assume each component is strictly positive and the degrees are bounded.

Proposition 5 (Cascade Condition). *Suppose the sequence of G-K financial networks $(N, P, Q, \bar{\Delta}, \bar{\Omega})$ has P, Q supported on the finite set $\{0, 1, \ldots, K\}^2$, and G is the $N = \infty$ cascade mapping defined by Proposition 12. Then ∇G defined by* (5.40) *is a component-wise positive $(K+1)$-dimensional matrix such that:*

1. *If $||\nabla G|| > 1$, then there is $\bar{\varepsilon} > 0$ such that for all ε with $0 < |\varepsilon| < \bar{\varepsilon}$, $|\pi^*(\varepsilon)| > \bar{\varepsilon}$. That is, in the $N = \infty$ network, any uniform seed with a positive fraction will trigger a cascade with default fraction bigger than $\bar{\varepsilon}$ almost surely.*
2. *If $||\nabla G|| < 1$, then there is $\bar{\varepsilon} > 0$ and C such that for all ε with $0 < |\varepsilon| < \bar{\varepsilon}$, $|\pi^*(\varepsilon)| \le C\varepsilon$. That is, the $N = \infty$ network will almost surely not exhibit large scale cascades for any infinitesimal seed.*

Proof: Part 1: We write $\nabla G = M_0$ where $M_\varepsilon = \partial G / \partial \pi|_{\pi = \mathbf{0}, \varepsilon}$. By continuous dependence in ε, there are values $\bar{\varepsilon}_1 > 0$ and $\lambda > 1$ such that the matrix M_ε is positive and has spectral radius $||M_\varepsilon|| \ge \lambda$ for all ε with $0 \le \varepsilon_{jk} < \bar{\varepsilon}_1$. Let us fix any such ε.

By the Perron-Frobenius Theorem for positive matrices, there is a unique normalized eigenvector v such that $M_\varepsilon \cdot v = |M_\varepsilon| v$: it has all positive entries and normalization $|v| = 1$. Taylor's Theorem with a second order remainder implies that for $\bar{\varepsilon}_1$ small enough there is $C' > 0$ such that

$$G(a;\varepsilon) = G(0;\varepsilon) + M_\varepsilon \cdot a + R(a), \quad |R(a)| \le C' |a|^2$$

for all $a \in [0,1]^{\mathbb{Z}_+}$ with $|a| \le \bar{\varepsilon}_1$ (note we drop the $\cdot$ notation in the following).

Now we show that the sequence $a^{(1)} = G(0;\varepsilon), a^{(n+1)} = G(a^{(n)};\varepsilon)$ leaves the set $|a| \le \bar{\varepsilon}$ provided $\bar{\varepsilon}$ is chosen small enough (independently of ε). For this, since $\bar{\varepsilon}_1 > 0$ there is $\beta_1 > 0$ and a non-negative vector y_1 such that $a^{(1)} = \beta_1 v + y_1$. Assuming inductively that $a^{(n)} = \beta_n v + y_n$ for some $\beta_n > 0$ and a non-negative vector y_n and that $|a^{(n)}| \le \bar\varepsilon_1$, the monotonic property of G combined with Taylor's Theorem implies

$$\begin{aligned} a^{(n+1)} &= G(a^{(n)};\varepsilon) \ge G(\beta_n v;\varepsilon) \\ &= G(0;\varepsilon) + \beta_n M_\varepsilon \cdot v + R(\beta_n v) \\ &\ge \beta_1 v + y_1 + \frac{1}{2}(1+\lambda)\beta_n v + \left(\frac{1}{2}(\lambda-1)\beta_n v + R(\beta_n v)\right) \end{aligned}$$

Let $\beta_{n+1} = \beta_1 + \frac{1}{2}(1+\lambda)\beta_n$ and note that $y_{n+1} = a^{(n+1)} - \beta_{n+1} v \ge 0$ provided $\bar\varepsilon \le \min(\bar\varepsilon_1, \frac{1}{2C'}(\lambda-1)\min_j v_j)$. Since the sequence β_n increases without bound, we can iterate the inductive argument only a finite number of steps before $|a^{(n+1)}| > \bar\varepsilon$.

Part 2: By continuous dependence in both a and ε, there are now values $\bar\varepsilon > 0$ and $\lambda = \frac{1}{2}(1+||\nabla G||) < 1$ such that the matrix $M_{a;\varepsilon} = \partial G/\partial a|_{\mathbf{a};\varepsilon}$ has spectral radius $||M_{a;\varepsilon}|| \le \lambda$ for all $0 \le \varepsilon < \bar\varepsilon$ and $|a| \le \bar\varepsilon$. Fix any such ε. Now we note that for vectors a,b with $|a|,|b| \le \bar\varepsilon$ we can use Taylor's Theorem again to write

$$G(a;\varepsilon) - G(b;\varepsilon) = M_\varepsilon \cdot (a-b) + R(a,b)$$

where the remainder has bound $C''|a-b|^2$ for some $C'' > 0$. Then provided $|a^{(n+1)}|, |a^{(n)}| \le \bar\varepsilon$ and $\bar\varepsilon \le \frac{1-||\nabla G||}{4C''}$

$$\begin{aligned} |a^{(n+1)} - a^{(n)}| &= |G(a^{(n)};\varepsilon) - G(a^{(n-1)};\varepsilon)| \\ &\le \frac{1}{2}(\lambda+1)|a^{(n)} - a^{(n-1)}| + \left(\frac{1}{2}(\lambda-1)|a^{(n)} - a^{(n-1)}| + |R(a^{(n)}, a^{(n-1)})|\right) \\ &\le \frac{1}{2}(\lambda+1)|a^{(n)} - a^{(n-1)}| \end{aligned}$$

for all $n \ge 1$. Since $|G(0;\varepsilon)| \le C'\varepsilon$ for some $C' > 0$, we can iterate this inequality to show $|a^{(\infty)}| \le C\varepsilon$ with $C = \frac{4C'}{1-||\nabla G||}$. □

We can understand the cascade condition more clearly by introducing the notion of *vulnerable edge* which means a directed edge $\ell = (wv)$ whose weight $\bar\Omega_\ell$ exceeds the default buffer of its downstream node $v = \mathscr{N}_\ell^+$. For the following argument, we suppose the network has only solvent banks, i.e. $D_{jk}(0) = 0$ for all j,k. An edge $\ell = (wv)$ is thus vulnerable if and only if $\bar\Delta_v \le \bar\Omega_\ell$. The matrix element $\nabla G_{kk'}$ has a simple explanation that gives more intuition about the nature of the cascade condition: it is the expected number of vulnerable edges ℓ with $k_\ell = k'$ that enter a node v with $k_v = k$. Then for small values of π, one has a linear approximation for the change in π in a single cascade step:

$$\pi_k^{(m+1)} - \pi_k^{(m)} = \sum_{k'} G_{k,k'} \ (\pi^{(m)} m_{k'} - \pi_{k'}^{(m-1)}) + O(|\pi|^2)\ . \qquad (5.41)$$

The condition for a global cascade starting from an infinitesimal seed is that the matrix ∇G must have an expanding direction, i.e. an eigenvalue with magnitude bigger than 1.

It turns out that the cascade condition is indeed a strong measure of systemic risk in simulated networks. One can check that in the setting of independent edge probabilities $Q_{kj} = Q^+_k Q^-_j$ and deterministic edge weights $\bar{\Omega}_{wv} = \bar{\Omega}_{jk}$ when $v \in \mathscr{N}_{jk}$, the spectral radius becomes

$$||\nabla G|| = \sum_{jk} \frac{jk}{z} P_{jk} \mathbb{P}[\bar{\Delta}_{j'k} \le \bar{\Omega}_{j'k}] \ ,$$

a result that has been derived in a rather different fashion by Gai and Kapadia [42] and Amini, Cont and Minca [7]. [42] also extends Watt's percolation theory approach [82] from undirected networks to directed nonassortative networks. We will see in the next section that the percolation approach to the cascade condition further extends to the general setting of directed assortative networks with random edge weights.

5.4.1 Frequency and Size of Global Cascades

We learned in Section 4.5.3 that the possibility of a large scale cascade in the Watts model depends on the connectivity of the directed subnetwork of vulnerable edges and nodes, a problem related to *site percolation*. The previous section addressed the potential for a small seed to grow into a global G-K cascade, and now we wish to understand how the frequency of global cascades in large random networks is related to the so-called extended in-component associated to the giant vulnerable cluster. In the present context, a vulnerable cluster has the meaning of a connected subgraph of the network consisting of *vulnerable* directed edges, where a vulnerable directed edge is one whose weight is sufficient to exceed the default buffer of its downstream node. We define:

- $\mathscr{E}_V \subset \mathscr{E}$, the set of vulnerable directed edges;
- $\mathscr{E}_s$, the largest strongly connected set of vulnerable edges (the *giant vulnerable cluster* of $\mathscr{E}_V$);
- $\mathscr{E}_i$ and $\mathscr{E}_o$, the *in-component* and *out-component* of the giant vulnerable cluster, i.e. the set of vulnerable edges that are connected to or from $\mathscr{E}_s$ by a directed path of vulnerable edges;
- $1 - b_k := \mathbb{P}[\ell \in \mathscr{E}_i | k_\ell = k]$, a conditional probability of an edge being in $\mathscr{E}_i$;
- $a_{k,jk'} = \mathbb{P}[\bar{\Delta}_v \le \bar{\Omega}_{wv} | \ell \in \mathscr{E}^-_v, k_\ell = k, v \in \mathscr{N}_{jk'}]$, the conditional probability of an edge being vulnerable.

Now note that $\ell = (w,v) \in \mathscr{E}^c_i$ (i.e. the complement of $\mathscr{E}_i$) means either $\bar{\Delta}_v > \bar{\Omega}_{wv}$ or $\bar{\Delta}_v \le \bar{\Omega}_{wv}$ and all the k_v "downstream" directed edges $\ell' \in \mathscr{E}^+_v$ are in the set $\mathscr{E}^c_i$. Thus, invoking the LTI property of the model, one determines that for all $k \in \mathbb{Z}_+$

$$b_k = \sum_j Q_{j|k} \sum_{k'} P_{k'|j} \left(1 - a_{k,jk'} + a_{k,jk'} (b_{k'})^{k'}\right) := H_k(\mathbf{b}) \tag{5.42}$$

where $a_{k,jk'} = \langle \tilde{D}_{jk'}, w_{kj} \rangle$. In other words, the vector $\mathbf{b} = (b_k)$ satisfies the fixed point equation $\mathbf{b} = H(\mathbf{b})$ where

$$H_k(b) = \sum_{jk'} Q_{j|k}\, P_{k'|j} \left(1 - \langle \tilde{D}_{jk'}, w_{kj} \rangle + \langle \tilde{D}_{jk'}, w_{kj} \rangle (b_{k'})^{k'}\right), \quad k \in \mathbb{Z}_+ \,. \tag{5.43}$$

The equation $\mathbf{b} = H(\mathbf{b})$ has a trivial fixed point $\mathbf{e} = (1, 1, \dots)$. In case $\mathbf{e}$ is a stable fixed point, we expect that the set $\mathscr{E}_i$ will have probability zero. We now verify that the cascade condition $||\nabla G|| > 1$ of Proposition 5 is equivalent to the condition that $\mathbf{e}$ is an unstable fixed point, in which case there will be a nontrivial fixed point $0 \le \mathbf{b}_\infty < \mathbf{e}$ that corresponds to the set $\mathscr{E}_i$ having positive probability. A sufficient condition for $\mathbf{e}$ to be an unstable fixed point is that $||\nabla H|| > 1$ where the derivative $\nabla H_{kk'} = (\partial H_k / \partial b_{k'})|_{\mathbf{b}=\mathbf{e}}$ is given by

$$\nabla H_{kk'} = \sum_j k' Q_{j|k} P_{k'|j} \langle \tilde{D}_{jk'}, w_{kj} \rangle \,. \tag{5.44}$$

One can verify directly that

$$\nabla H = \Lambda^{-1} \cdot (\nabla G)' \cdot \Lambda$$

for the diagonal matrix $\Lambda_{kk'} = \delta_{kk'} k Q^+_k$ and from this it follows that the spectrum, and hence the spectral radii and spectral norms, of ∇H and ∇G are equal. Hence $||\nabla H|| > 1$ if and only if $||\nabla G|| > 1$.

As long as the cascade condition $||\nabla H|| > 1$ is satisfied, a global cascade will arise from a random single seed v if it triggers at least one edge $(v, v') \in \mathscr{E}_i$. The *cascade frequency* f is at least as large as the probability that this occurs, and is therefore bounded from below:

$$f \ge \sum_k (1 - (b_k)^k) P^+_k \,. \tag{5.45}$$

Given that the single seed triggers an edge in the giant in-cluster $\mathscr{E}_i$, how large will the resultant global cascade be? Well, certainly, the cascade will grow to the strongly-connected giant cluster $\mathscr{E}_s$, and continue to include all of the extended out-component $\mathscr{E}_o$ of the giant cluster. From this point, higher order defaults become likely, so the cascade may grow much further. But, without restriction, we can say that when the cascade condition holds, whenever the giant vulnerable cluster is triggered, the resultant cascade will include all of $\mathscr{E}_o$. To compute the fractional size of this set, it is convenient to introduce the conditional probability

$$c_k = \mathbb{P}[v \notin \mathscr{E}_o | k_v = k] \tag{5.46}$$

where the event $v \notin \mathscr{E}_o$ is defined to mean that it has no in-edges ℓ that are in $\mathscr{E}_o$. For this calculation, we recognize that the event $\ell = (w,v) \notin \mathscr{E}_o$ means either $w \notin \mathscr{E}_0$ or $w \in \mathscr{E}_0$ and $\bar{\Omega}_{wv} < \bar{\Delta}_v$. The events $\ell = (w,v) \notin \mathscr{E}_o$ for all $w \in \mathscr{N}_v^-$ are mutually independent only when conditioned on the state of v and $\bar{\Delta}_v$, and conditioning carefully leads to the fixed point equation for the vector of probabilities $\mathbf{c} = (c_k)$:

$$c_k = \sum_j P_{j|k} \int d\tilde{D}_{jk}(x) \Big[\sum_{k'} Q_{k'|j} \left(W_{k'j}(x) + (1 - W_{k'j}(x)) c_{k'} \right) \Big]^j . \tag{5.47}$$

Again, the cascade condition $||\nabla G|| > 1$ is sufficient for the trivial fixed point $\mathbf{c} = \mathbf{e}$ to be unstable, meaning the size of the vulnerable out-component $\mathscr{E}_o$ is a positive fraction of the network, computable by the formula

$$\mathbb{P}[v \in \mathscr{E}_o] = 1 - \sum_{jk} P_{jk} \int d\tilde{D}_{jk}(x) \Big[\sum_{k'} Q_{k'|j} \left(W_{k'j}(x) + (1 - W_{k'j}(x)) c_{k'} \right) \Big]^j \tag{5.48}$$

This is a lower bound on the size of the default cascade that results when the initial seed triggers a global cascade. It is interesting that the form of the expectation in (5.47) involves the point-wise power of the W distribution rather the convolution power that appears in (5.23).

5.5 Testing and Using the Cascade Mapping

The cascade mapping framework, both mathematical and conceptual, was developed for a larger purpose, namely to help understand the nature of real cascades in real world networks. So, how well does it work after all this effort?

While we have considered in this chapter only the simplest cascade mechanism, namely the zero-recovery default cascade of Gai-Kapadia, we have placed this cascade mechanism on RFNs that have a rich complexity. Determining what our cascade mapping has to say about actual cascades is still a question of experimental computation. Since we have made a number of uncontrolled approximations, the analytical method should be validated by comparing with the results of Monte Carlo simulation experiments under a broad range of model specifications. Before continuing, it is helpful to consider the types of questions we wish to address.

Robustness of the formalism: The LTI formalism on RFNs, and the resultant cascade mapping results, rest on bold assumptions that need to be checked. First, we can gain from the experience of others who have studied the Locally Treelike ansatz for a great number of models on random networks. As a general rule, validated for example by [63], when benchmarked against Monte Carlo simulation experiments, the LT analytical formulas have been found to work effectively under a broad range of conditions. This is hard, slow work that must continue to push back the modelling frontiers as new models are introduced.

The framework developed in this chapter extends the original Gai-Kapadia paper in many respects. We have extended the skeleton construction to allow configuration graphs with general node and edge degree distributions. We have allowed buffers and exposures to be random with arbitrary marginal distributions. We have new formulas for the cascade condition and the economic impact of the cascade. The corresponding Monte Carlo validation experiments have not yet been completed, and will take considerably more time and effort.

Usefulness of the formalism: While a systematic Monte Carlo survey to validate the LTI analytic framework is pending, which will build confidence in the reliability of the framework, it is worthwhile to press forward using the formal analytical framework as a freestanding tool. Without the need for Monte Carlo, there are many purely analytical experiments on simple networks that are quick and easy to implement on a computer. The scope of our default cascade formalism now includes flexibility in new dimensions never before explored. The skeletons are now assortative. Our balance sheets and exposures are now stochastic, and their variances are key parameters that represent our uncertainty in the network. These new features have unknown implications on the cascades that can result in simple models.

Learning about real cascades: As network data for financial systems grows and becomes available, network models will grow in complexity to incorporate new, more realistic features drawn from the data. Moreover, the problem of calibrating network models to data will become increasingly challenging. Our framework has been designed to scale up in complexity to adapt to such needs.

Economic and financial implications: Analytical models can provide insight into purely economic problems. One important example is capital adequacy: Regulators want to know how a fixed amount of regulatory capital can be spread optimally across different banks to reduce systemic risk. The answer to such a question can be used to decide how much excess capital should be held by systemically important financial institutions (SIFIs). The behaviour of bankers is complex: they use evolving strategies and game theory continuously in time to navigate their firm through tortuous waters. Analytical models make it possible for policy makers, regulators and market participants to test such strategies under stressful scenarios.

5.6 A New Model for Default Cascades with Asymmetric Shocks

It turns out that the Watts model of Chapter 4 can be unified with the Gai-Kapadia zero recovery mechanism of the present chapter to give an economically natural default cascade model on a undirected network with bivariate link exposures that represent the unnetted positive exposures between counterparties.

The links in the Gai-Kapadia model are directed, and represent an idealized debtor-creditor relationship that is usually described in terms of unsecured overnight lending. The reality of banking is that counterparty relations are arbitrarily complex, and certainly cannot be boiled down to a single value at a moment in time. Counterparty banks will likely have overnight lending relations, will likely share a portfolio of interest rate and FX swap exposures, will owe each other certificates of deposit and the like, will trade repos, and so on. Determining the value of such exposures at any moment is exceedingly complicated. Banks exert major efforts to compute their *potential future exposures* to other banks at all times, following regulatory guidelines on counterparty credit risk. If at some moment one bank were to default, these exposures will often have opposite signs, and it is the positive part of the unnetted exposure that will impact the nondefaulted counterparty. To reduce counterparty risk, banks enter into *master netting agreements* (MNAs) with all of their important counterparties. This standardized agreement aims to specify exactly how the positive and negative exposures in the portfolio of different contracts between them can be offset in the event one bank defaults. An ideal fully netted position would leave only a single exposure in one direction at any time. It follows that the pervasiveness of MNAs in the interbank network provides a partial justification for taking edges to point in only one direction and for neglecting reflexive edges (those with links of both directions), as we have done in the Gai-Kapadia model. However, despite the existence of MNAs, allowing counterparty exposures to be unnetted or partially netted, and therefore bi-directional, is clearly a very natural modeling generalization.

In this section, we introduce a default cascade model that combines features of the Watts and Gai-Kapadia models while allowing edges to take on a more nuanced meaning. It views the financial system as a network of banks connected by *undirected edges*, with edges placed where there are deemed to be strong counterparty relations. For example, these edges should certainly include all pairs of banks that share a master netting agreement. It is known that building and maintaining counterparty relations is expensive for banks, particularly when the relationship is governed by the MNA. Thus it is reasonable to expect the network to be sparse, and that the existence of edges may be slowly varying while the exposures they carry might change quickly.

Given an edge $\ell = (w, v)$ between two banks v and w, the exposure $\bar{\Omega}_\ell$ carried by the edge will now be assumed to be multi-dimensional to represent the aggregated exposures across different types of securities. In the simplest variant we now consider, the multidimensional vector $\bar{\Omega}_\ell$ can be reduced to a pair of non-negative exposures $(\bar{\Omega}_{w,v}, \bar{\Omega}_{v,w})$ where $\bar{\Omega}_{w,v}$ is the loss to v given the default of w. The model is then akin to the Watts model, but with asymmetric shocks that may be transmitted in either direction across edges. If $\min(\bar{\Omega}_{w,v}, \bar{\Omega}_{v,w}) = 0$ (i.e. only one direction is ever non-zero), this new setting reduces to a slightly non-standard specification of the Gai-Kapadia model.

With an aim to develop large N asymptotics, we now provide an LTI-compatible RFN specification for this model:

Assumptions 7 (Watts-Gai-Kapadia Default Cascade Model).

1. Banks have limited liability and receive zero recovery of interbank liabilities from any defaulted bank.

2. The skeleton consists of a random graph $(\mathcal{N},\mathcal{E})$ on N banks which is an undirected assortative configuration model with node and edge type distributions $(P_k, Q_{kk'})$ with mean degree $z=\sum_k\ k\ P_k$, bounded degrees $k\le K$ and satisfying the consistency conditions:

$$
\begin{aligned}
Q_{kk'} \ &= \ Q_{k'k}, \quad \sum_{k\le k'} Q_{kk'} = 1 \\
Q_k \ &:= kP_k/z = \frac{1}{2}\sum_{k'\neq k} Q_{k'k} + Q_{kk} \\
Q_{k'|k} \ &:= (1+\delta_{kk'})\frac{Q_{kk'}}{2Q_k}\ .
\end{aligned}
\tag{5.49}
$$

3. Conditionally on the skeleton, the default buffers $\bar{\Delta}_v$ are a collection of independent non-negative random variables whose distributions depend only on the degree k_v:
$$\mathbb{P}[\bar{\Delta}_v \le x | v\in\mathcal{N}_k] := D_k(x), \quad k,x\ge 0$$
for cumulative probability distributions $D_k(\cdot)$ parametrized by k.
4. For each undirected link $\ell=(w,v)\in\mathcal{E}$, the exposure is a bivariate random variable $(\bar{\Omega}_{w,v},\bar{\Omega}_{v,w})$ on $\mathbb{R}^2_+$. Conditioned on the skeleton, the collection of edge exposures is independent with a bivariate distribution function that depends only on the bi-degree (k_w,k_v)
$$\mathbb{P}[\bar{\Omega}_{w,v}\le x, \bar{\Omega}_{v,w}\le y | w\in\mathcal{N}_k\cap\mathcal{N}_v, v\in\mathcal{N}_{k'}] := W_{kk'}(x,y) = W_{k'k}(y,x), \quad x,y\ge 0\ .$$
The conditional marginals are
$$W_{kk'}(x) := W_{kk'}(x,\infty) = W_{k'k}(\infty,x)\ .$$
5. The remaining balance sheet quantities are freely specified.

Since $\bar{\Omega}_{v,w}$ represents the shock that will be transmitted from v to w at the moment v defaults, and $\bar{\Delta}_w\ge 0$ represents the threshold for the default of w, the set of defaulted banks$\mathcal{D}_n$ and its indicator $D^n_v=\mathbf{1}(v\in\mathcal{D}_n)$ after n steps of the cascade again follows a recursion for all $n\ge 0$ and $v\in\mathcal{N}$:

$$D^n_v = \mathbf{1}\left(\bar{\Delta}_v \le \sum_{w\in\mathcal{N}_v}\bar{\Omega}_{w,v}D^{n-1}_w\right) \tag{5.50}$$

starting with $D^{-1}_v=0$. Just as Proposition 4 shows for the Watts model, we can show that this model has the WOR threshold property. First we define WOR default events for directed edges (v,w) by the recursion

$$D^n_{v,w} := \mathbf{1}(v\in\mathcal{D}_n \text{ WOR } w) = \mathbf{1}\left(\bar{\Delta}_v \le \sum_{w'\in\mathcal{N}_v}\bar{\Omega}_{w',v}D^{n-1}_{w',v}\mathbf{1}(w'\neq w)\right) \tag{5.51}$$

starting with $D^{-1}_{v,w} = 0$ for all directed edges. Then, one finds that the default cascade has the WOR form because for all $n \geq 0$

$$D^n_v = \tilde{D}^n_v \tag{5.52}$$

where

$$\tilde{D}^{(n)}_v := \mathbf{1}\left(\bar{\Delta}_v \leq \sum_{w\in\mathcal{N}_v} \bar{\Omega}_{w,v} D^{n-1}_{w,v}\right) . \tag{5.53}$$

The following proposition is analogous to Proposition 12 and can be justified by similar formal arguments:

Formal Proposition 14 (Watts-Gai-Kapadia Cascade)**.** *Consider the LTI sequence of Watts-Gai-Kapadia financial networks* $(N,P,Q,\bar{\Delta},\bar{\Omega})$ *satisfying Assumptions 7. Let* $\hat{p}^{(0)}_k := \mathbb{P}[w \in \mathscr{D}_0 \text{ WOR } v | w \in \mathcal{N}_v \cap \mathcal{N}_k] = D_k(0)$ *denote initial WOR default probabilities. Then the following formulas hold in the limit as* $N \to \infty$*:*

1. *For any* $n = 1,2,\ldots$*, the quantities* $\hat{p}^{(n)}_k = \mathbb{P}[w \in \mathscr{D}_n \text{ WOR } v | w \in \mathcal{N}_v \cap \mathcal{N}_k]$ *satisfy the recursive formulas*

$$\hat{p}^{(n)}_k = \langle D_k, (\widetilde{w}^{(n-1)}_k)^{\circledast k-1}\rangle , \tag{5.54}$$

and the full default probabilities $p^{(n)}_k = \mathbb{P}[w \in \mathscr{D}_n | w \in \mathcal{N}_k]$ *are given by*

$$p^{(n)}_k = \langle D_k, (\widetilde{w}^{(n-1)}_k)^{\circledast k}\rangle . \tag{5.55}$$

Here the marginal exposure PDFs $\widetilde{w}^{(n-1)}_k(x)$ *are given by*

$$\widetilde{w}^{(n-1)}_k(x) = \sum_{k'} Q_{k'|k}\left((1-\hat{p}^{(n-1)}_{k'})\delta_0(x) + \hat{p}^{(n-1)}_k W'^{+}_{k'k}(x)\right) \tag{5.56}$$

with $Q_{k'|k}$ *defined by* (5.49)*.*

2. *The new probabilities* $\hat{p}^{(n)} = (\hat{p}^{(n)}_k)$ *are given recursively by* $\hat{p}^{(n)} = G(\hat{p}^{(n-1)})$ *for a vector valued function which is explicit in terms of the specification* $(N,P,Q,\bar{\Delta},\bar{\Omega})$*. The cascade mapping* G *maps* $[0,1]^{\mathbb{Z}_+}$ *onto itself, and is monotonic. Since* $\hat{p}^{(0)} = G(0)$*, the sequence* $\hat{p}^{(n)}$ *converges to the least fixed point* $\hat{p}^* \in [0,1]^{\mathbb{Z}_+}$*, that is*

$$\hat{p}^* = G(\hat{p}^*) . \tag{5.57}$$

Note that a consequence of the WOR property is that the asymptotic cascade mapping depends only on the collection of marginal distributions $W^+_{kk'}(x)$, and not the full bivariate distribution for $\bar{\Omega}_{wv}, \bar{\Omega}_{vw}$. In other words, without affecting the cascade probabilities, $\bar{\Omega}_{wv}, \bar{\Omega}_{vw}$ can be taken to be conditionally independent, so that

$$W_{kk'}(x,y) = W^+_{kk'}(x) W^+_{k'k}(y)$$

for an arbitrary collection $W^+_{kk'}(x)$ of univariate CDFs. Alternatively, we may assume $\bar{\Omega}_{wv} = \bar{\Omega}_{vw}$ for all $(w,v) \in \mathscr{E}$. Then the theorem reduces to the cascade mapping theorem proved in [52] for the Watts model on an assortative skeleton with random edge weight CDFs

$$\mathbb{P}[\bar{\Omega}_{w,v} \le x | w \in \mathscr{N}_k, v \in \mathscr{N}_{k'}] = W_{kk'}(x), \quad x \ge 0 \ .$$

It is interesting that the more realistic meaning attached to exposures is consistent with a skeleton that is undirected instead of directed. One nice modeling feature is therefore that the node-degree k_v can be unambiguously correlated with the size of v's balance sheet. In contrast, by focussing on the directionality of edges, the Gai-Kapadia model on RFNs is forced to live on a directed skeleton, whose bi-degrees (j_v, k_v) need to be specified, creating an additional complexity that now seems of less importance.

We have seen already in the Watts and Gai-Kapadia models that the cascade mapping function G determines essential features beyond the eventual default probabilities $\hat{p}^*$. We will not be surprised that it is the spectral norm of $\nabla G_{kk'} = \partial_{p_{k'}} G_k|_{p=0;\varepsilon=0}$ that determines whether the model admits global cascades or not. Or that estimates of the frequency and size of global cascades can be computed using percolation ideas. We leave such exercises to the interested reader.

5.7 Cascade Computations

While the simplified version of the Gai-Kapadia cascade mapping given in Proposition 11 is straightforward to implement on a computer, the structure of the convolution power in (5.23) and similar equations at the heart of the generalized cascade mapping of Proposition 12 is more difficult from the point of view of numerical approximations. Numerical evaluation of the implied integrals leads to truncation errors and discretization errors, both of which will be awkward to handle in our setting. In this section, we follow the method developed in [52] for the case where the random variables $\{\bar{\Delta}_v, \bar{\Omega}_\ell\}$ all take values in the finite discrete set

$$\mathscr{M} = \{0, \delta x, \ldots, (M-1)\delta x\} \tag{5.58}$$

with a large value M and a common grid spacing δx. We can think of this as specifying both the truncation and discretization of non-negative continuous random variables. In such a situation, the convolutions in (5.23) can be performed exactly and efficiently by use of the discrete Fast Fourier Transform (FFT), whose detailed properties are summarized in Appendix A.2. For the moment, let us take $\delta x = 1$ for simplicity.

Let X, Y be two independent random variables with probability mass functions (PMF) p_X, p_Y taking values on the non-negative integers $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. Then the random variable $X + Y$ also takes values on this set and has the probability mass function (PMF) $p_{X+Y} = p_X \circledast p_Y$ where the convolution of two functions f, g is defined to be

$$(f \circledast g)(n) = \sum_{m=0}^{n} f(m)g(n-m), \quad n \in \mathbb{Z}_+ \ . \tag{5.59}$$

Note that p_{X+Y} will not necessarily have support on the finite set $\mathscr{M}$ if p_X, p_Y have support on $\mathscr{M}$, a fact that can lead to so-called *aliasing problems*.

We now consider a probability given as in (5.23)

$$P = \mathbb{P}[X \le \sum_{i=1}^{j} Y_i]$$

for independent random variables $X, Y_1, Y_2, \ldots, Y_j$ where each Y_i is distributed with PMF $g_i(m)$ and X has PMF $f(m)$. We suppose that f has support on $\mathscr{M}$ while each g_i has support on $\{0, 1, \ldots, \lfloor M-1/j \rfloor\}$. We also define the CDF $F(n) = \sum_{m=0}^{n} f(m) = \mathbb{P}(X \le n)$ for $n \in \mathscr{M}$. Then these M-vectors have M-dimensional FFTs $\hat{f} = \mathscr{F}(f)$, $\hat{g}_i = \mathscr{F}(g_i)$, $\hat{F} = \mathscr{F}(F)$. Using the FFT identities (A.1) to (A.4), we are led to a means to compute P for any j involving matrix operations and the FFT:

$$\mathbb{P}[X \le \sum_{i=1}^{j} Y_i] = \sum_{m \in \mathscr{M}} (g_1 \circledast \cdots \circledast g_j)(m) \sum_{n=0}^{m} f(n) = \sum_{m \in \mathscr{M}} (g_1 \circledast \cdots \circledast g_j)(m) F(m)$$
$$= \langle F, g_1 \circledast \cdots \circledast g_j \rangle = \frac{1}{M} \langle \hat{F}, (g_1 \widehat{\circledast \cdots \circledast} g_j) \rangle = \frac{1}{M} (\hat{F})' \cdot (\hat{g}_1 \cdot * \ldots * \hat{g}_j) \ . \tag{5.60}$$

Here in the final expression, A' denotes the conjugate transpose of a matrix A, $\cdot$ denotes matrix multiplication between a size $[1, M]$ matrix and a size $[M, 1]$ matrix, and $\cdot *$ denotes component-wise ("Hadamard") multiplication of vectors and matrices.

Remark 5. There is no aliasing problem and the identity (5.60) is true if $g_1 \circledast \cdots \circledast g_j$ has support on $\mathscr{M}$. We formalise this requirement as the Aliasing Assumption.

To summarize, as long as no aliasing errors arise, we can compute equation (5.23) efficiently using the FFT identity (5.60). Having realized this fact, it becomes sufficient to store as initial data only the Fourier transformed probability data for the random variables $\bar{\Delta}_v, \bar{\Omega}_\ell$, that is

$$\hat{D}_{jk} := \mathscr{F}(D_{jk}) \in \mathbb{C}^M, \quad \hat{w}_{kj} := \mathscr{F}(w_{kj}) \in \mathbb{C}^M, \quad j, k \in \mathbb{Z}_+ \ . \tag{5.61}$$

Then the node update step becomes

$$p_{jk} = \frac{1}{M} \left\langle \hat{D}_{jk}, \widehat{(\widetilde{w}_j)}^{\circledast j} \right\rangle \tag{5.62}$$

where

$$\widehat{(\widetilde{w}_j)} = \sum_{k'} [1 - \pi_{k'} + \pi_{k'} \hat{w}_{k'j}] \ Q_{k'|j} \ .$$

Remark 6. The overall computation time for any numerical implementation of the cascade formulas in Proposition 12 will be dominated by computing convolutions such as those in (5.23). In particular, one can check that negligible total time will be taken in precomputing FFTs (each of which take of the order of $M\log_2 M$ additions and multiplications). One can compare the efficiency of our recommended FFT approach to the direct approach by considering a single evaluation of the convolution: to compute $f \circledast g$ using (5.59) for M-vectors f,g requires M^2 additions and multiplications, whereas to compute $\hat{f} \cdot * \hat{g}$ requires only M multiplications. All else being equal, we can expect a speedup by a factor of $O(M)$ when the FFT method is used instead of the direct method. Since in our implementations we often have $M \gtrsim 2^{10}$, this is a huge improvement. This speedup factor is too optimistic in practice when one takes care of the aliasing problem by taking a large, conservative value for M.

5.8 Numerical Experiments

The formalism developed in this chapter already raises a great variety of questions and unanswered issues concerning the G-K model, which is of course the simplest of all default cascade models. Most such questions can only be studied with computer experiments. In this section, we report briefly on some of the simplest experiments that test the usefulness and validity of the LTI analytics developed so far in this chapter.

5.8.1 Experiment 1: Benchmark Gai-Kapadia Model

We choose as our benchmark the model specification given in the Gai and Kapadia paper [42], and develop variations on the theme that explore certain interesting dimensions away from this benchmark. Here is their original model specification:

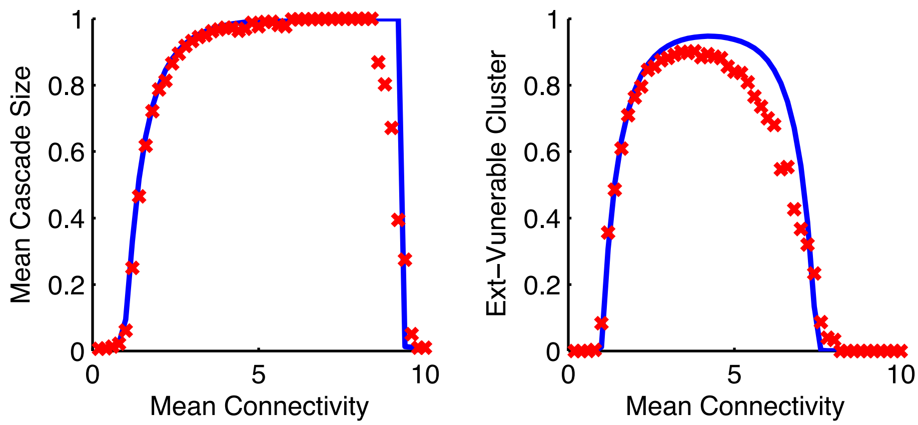
1. The skeleton graph comprises $N = 1000$ banks (which we change to $N = 10000$) taken from the Poisson random directed graph model with mean in and out degree z, and thus $P = \mathrm{Bin}(N, z/(N-1)) \times \mathrm{Bin}(N, z/(N-1))$ and $Q = Q^+Q^-$.
2. Capital buffers and assets are identical across banks, with $\bar{\Delta}_v = 4\%$ and $\bar{\mathrm{Z}}_v = 20\%$.
3. Exposures are equal across the debtors of each bank, and so $\bar{\Omega}_{wv} = \frac{20}{j_v}$.
4. The number of Monte Carlo simulations performed is Nsim $= 1000$.

It is also interesting to use the invariance property of the cascade mapping to rescale exposures $\bar{\Omega}_{wv}$ and buffers $\bar{\Delta}_v$ by $j_v/20$, leading to (i) $\bar{\Delta}_v = j_v/5$; (ii) $\bar{\mathrm{Z}}_v = j_v$; (iii) $\bar{\Omega}_{wv} = 1$. This rescaled specification is very similar to the benchmark parametrization used in the Watts model.

Figure 5.1(a) shows the dependence of the mean cascade size as a function of the mean degree z computed both analytically using Proposition 11 and by Monte Carlo simulation. The analytical results were obtained with seed probabilities $d_{k,0} = 10^{-3}$

for all k and for the Monte Carlo simulation we took the initial seed to be 1% of the network (100 banks).

Figure 5.1(b) shows how the analytical formula (5.45) for the frequency of global cascades compares to the frequency of global Monte Carlo cascades that started from a single random seed, where in the Monte Carlo simulations, a global cascade was deemed to be one that exceeds 50 banks (0.5% of the network).



(a) (b)

Fig. 5.1 These two graphs show (a) the mean fractional cascade size, and (b), the fractional extended vulnerable cluster size (analytic) and global cascade frequency (Monte Carlo), in the benchmark Gai-Kapadia model as a function of mean degree z, as computed using the large N analytics of Theorem 9 (blue curve) and by Monte Carlo simulation (red crosses). In Figure (a), the Monte Carlo computation involved Nsim $= 50$ realizations of the $N = 10000$ node graph, each with an initial seed generated by selecting nodes independently with probability 0.5%. In Figure (b), the simulation involved Nsim $= 2500$ realizations of the $N = 10000$ node graph and a single random initial seed node.

Comparison of these figures with the Watts model experiments in Chapter 4 exhibits striking similarities. Clearly this G-K model specification is very similar to the benchmark Watts 2002 model implemented in [82]. In fact, in both models, nodes with in-degree j are vulnerable if and only if $j \leq 5$. Apart from differences stemming from the directed nature of the G-K model, the cascade results shown in Figure 5.1 are almost identical to those found in Chapter 4. When z is smaller than 2 or 3, almost all nodes are vulnerable, and thus the contagion condition is essentially the condition for the giant cluster to exist, which suggests a phase transition at the value $z = 1$. On the other hand, when z is larger than about 7, most nodes are not vulnerable, and there is no giant vulnerable cluster. As z increases into this range it becomes harder for a single seed default to trigger a cascade. Occasionally, however a seed may have a large degree, opening the possibility for finite size effects causing higher order defaults that can lead to a large scale cascade. Although infrequent, when they occur such cascades usually trigger almost 100% of the network to default.

5.8.2 Experiment 2: Assortative Networks

For various reasons, configuration graphs with general edge assortativity parametrized by the Q matrix have been little studied in network science. On the other hand, numerous statistical studies of financial networks in the real world, notably [78], [11] and [27], have pointed out their strongly negative assortativity and speculated that this property is likely to strongly influence the strength of cascades that the network will exhibit. In this experiment, we explore the GK model close to the benchmark specification but with different parametrizations of the edge type matrix Q. Addressing this question provides us with a good starting point to test a number of points. How effective is the simulation algorithm for assortative configuration graphs given in Section 3.3.1? Does edge assortativity have a strong effect on the possibility of cascades?

We twist the benchmark GK model described above by replacing the independent edge probability matrix $Q_{kj} = Q^+_k Q^-_j$ by an interpolated matrix that exhibits positive or negative assortativity

$$Q(\alpha)_{kj} = \begin{cases} \alpha Q^+_{kj} + (1-\alpha) Q^+_k Q^-_j & \alpha \geq 0 \\ |\alpha| Q^-_{kj} + (1-|\alpha|) Q^+_k Q^-_j & \alpha < 0 \end{cases}$$

Here $Q^+_{kj} = \delta_{kj} Q^+_k$ has 100% correlation and is the maximally assortative matrix with the prescribed marginals. On the other hand, the maximally negative assortative matrix turns out to be given by

$$Q^-_{kj} = \mathbb{P}\Big[U \in (b_{k-1}, b_k], 1-U \in (b_{j-1}, b_j]\Big]$$

where U is uniform $[0,1]$ and $b_k = \sum_{k' \leq k} Q^+_{k'}$ for $k = 0, 1, \ldots, K_{max}$.

With some surprise we noticed that with P_{jk} given as in Experiment 1, the mean cascade size had no variation with α! After a moment's thought, however, we realize that because this model has an independent bivariate distribution of node degrees, any possible dependence on α is wiped out. We need some dependence between in and out node degrees to allow for the cascade to depend on assortativity. Figure 5.2 shows the dependence of the mean cascade size on the parameter α for four values of z, when we replace the independent matrix P by the fully correlated matrix

$$P_{jk} = P^+_k \delta_{jk} .$$

As we hoped, we see a strong nonmonotone dependence of cascade frequency on the assortativity parametrized by α, with a maximum effect for a positive assortativity value $\alpha \sim 0.4$.

Throughout this experiment, we need to pay attention to discrepancies between the Monte Carlo results and the analytics. While for the most part the match is quite good, surprisingly it breaks down completely for $z = 7.5$ and negative assortativity where the Monte Carlo results are particularly erratic. One important "rule of thumb" that can partly account for such anomalies is that they seem likely to occur in the neighbourhood of discontinuous phase transitions, in this case the one near $z \sim 7$. It seems likely we will see even larger discrepancies when the finite Monte Carlo samples are taken from a less heterogeneous model.

This experiment also shows that Monte Carlo simulation can become quite challenging as the random graph model becomes more complex, and provides motivation to explore widely using large N analytics which are relatively straightforward to implement.

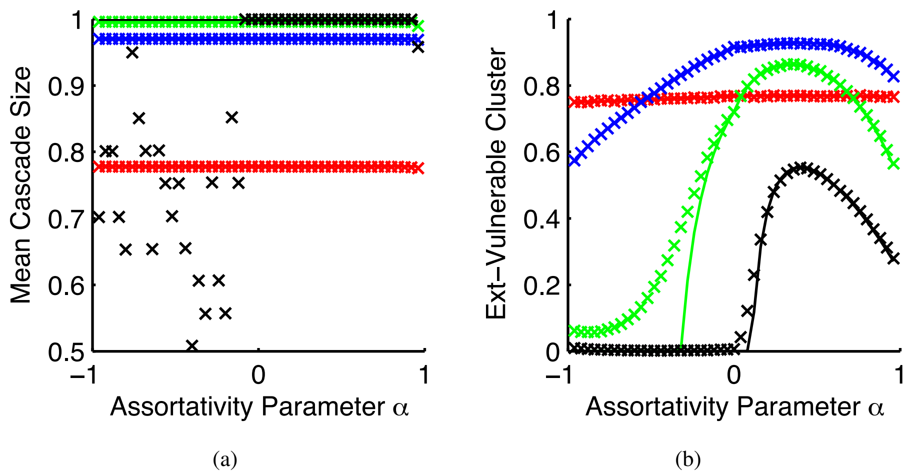


Fig. 5.2 These two graphs show (a) the mean cascade size, and (b) the fractional extended vulnerable cluster size (analytic) and global cascade frequency (Monte Carlo), in the GK model, as a function of positive and negative assortativity when P is maximally correlated. The analytical results are shown by the solid curves, with four values of z: 1.5 (red), 3.5 (blue), 5.5 (green), 7.5 (black). In Figure (a), the Monte Carlo computation involved Nsim $= 50$ realizations of the $N = 10000$ node graph, each with an initial seed generated by selecting nodes independently with probability 0.5%. In Figure (b), the analytical formula for the extended giant vulnerable cluster is compared to the result of an Nsim $\times$ Nfreq Monte Carlo simulation of global cascades consisting of Nsim $= 25$ realizations of the $N = 10000$ node graph, and for each graph realization Nfreq $= 2000$ random initial seed nodes.

5.8.3 Experiment 3: Random Buffers and Link Weights

What is the effect of uncertainty in the network? In reality, even with excellent databases we might never expect to know actual exposures and buffers with any precision. We can model this uncertainty by taking these to be random variables and testing how the variance parameters affect the resultant cascade sizes and frequencies. Furthermore, we can check whether the analytic approximations get much worse or not.

To test the impact of this idea, we can introduce collections of log normally distributed buffer and exposure random variables with four additional parameters $\sigma_\Delta, \sigma_\Omega \geq 0$ and $\rho_\Delta, \rho_\Omega \in (-1,1)$. That is,

$$\Delta_v = \frac{j_v}{5}\exp[\sigma_\Delta(\rho_\Delta Z_\Delta + \sqrt{1-\rho_\Delta^2}X_v) + \sigma_\Delta^2/2] \tag{5.63}$$

$$\Omega_\ell = \exp[\sigma_\Omega(\rho_\Omega Z_\Omega + \sqrt{1-\rho_\Omega^2}Y_{vw}) + \sigma_\Omega^2/2] \tag{5.64}$$

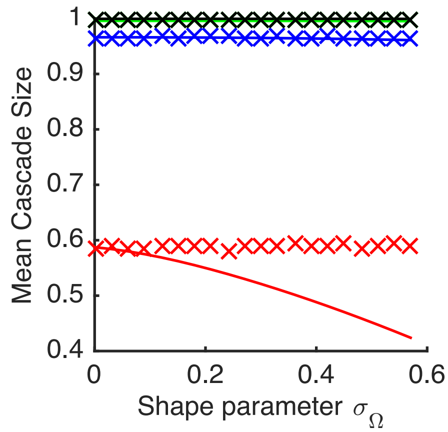
with a collection $\{Z_\Delta, X_v, Z_\Omega, Y_\ell\}$, $v \in \mathcal{N}$, $\ell \in \mathcal{E}$ of independent standard normals. Note that the benchmark Gai-Kapadia model of Experiment 1 arises by taking $\sigma_\Delta = \sigma_\Omega = 0$, and for all parameters, the new specification agrees with the benchmark in expectation. The case of $\sigma_\Omega = 0$ and varying $\sigma_\Delta, \rho_\Delta$ has been studied to some extent in [51] and so is of lesser interest. Also, by the rescaling property, the cascade mapping when $\rho_\Omega \neq 0$ can be transformed into the case $\rho_\Omega = 0$. However, the case of random exposures has not been studied previously, and therefore we take $\sigma_\Delta = 0, \rho_\Omega = 0$ and focus on the dependence of the default cascade on the shape parameter σ_Ω.

In our implementation of equations (5.23) and (5.24) of Proposition 12 we truncated and discretized the log normal exposure random variables by placing their values on a grid of $M = 2^{12}$ points with spacing $\delta x = 0.1$. Then we applied the FFT algorithm for computing the cascade equilibrium of Proposition 12. Prior to plotting our results we confirmed that they were insensitive to this approximation by comparing to the results when the grid was coarsened to $M = 2^{10}$ and $\delta x = 0.2$ (results not shown).

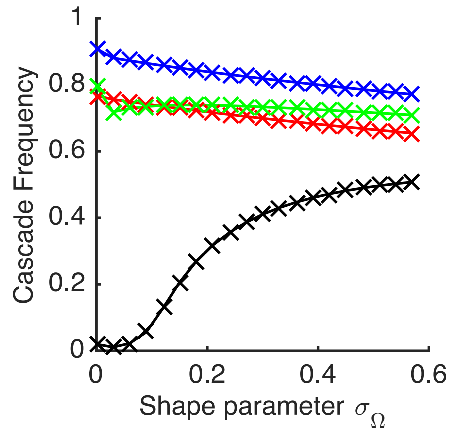
Figure 5.3 plots the dependence of the mean cascade size and cascade frequency as a function of the shape parameter of the exposure distribution, σ_Ω, for four values of the Poisson mean degree z: 1.5 (red), 3.5 (blue), 5.5 (green), 7.5 (black).

Again, we see a general agreement between the Monte Carlo and analytical results, but with some significant departures when $z = 1.5$. Further exploration shows that the lower phase transition at $z = 1$ is pushed to higher values of z as σ_Ω grows: it appears that the anomalous behaviour we observed when $z = 1.5$ is connected with this fact.

These three experiments illustrate only a very limited set of possible analytical experiments on the simplest of all default cascade models. Already we observe that while the analytical formulas agree with the Monte Carlo simulation results for the most part, there are always circumstances that undermine the accuracy of the



(a)



(b)

Fig. 5.3 These two graphs show (a) the mean cascade size, and (b) the fractional extended vulnerable cluster size (analytic) and global cascade frequency (Monte Carlo), in the G-K model, as a function of σ_Ω for $\sigma_\Delta = 0$ The analytical results are shown by the solid curves, with four values of z: 1.5 (red), 3.5 (blue), 5.5 (green), 7.5 (black). In Figure (a), the Monte Carlo computation involved Nsim $= 10$ realizations of the $N = 10000$ node graph, each with an initial seed generated by selecting nodes independently with probability 0.5%. In Figure (b), the analytical formula for the extended giant vulnerable cluster is compared to the result of Monte Carlo simulation of global cascades consisting of Nsim $= 10$ realizations of the $N = 10000$ node graph, and for each graph realization averaging ove Nfreq $= 1000$ random initial seed nodes.

agreement. It is important to try to develop “rules of thumb” that supply the intuition of the circumstances when the agreement is not acceptable.

Chapter 6
Future Directions for Cascade Models

Abstract The prospects are considered for extending the mathematical framework of cascade mechanisms on locally tree-like random financial networks to address problems of real financial importance.

Keywords: Random financial networks, local tree-like independence, cascade mechanism, balance sheet models, bank behaviour.

This book has set out to show that systemic risk and contagion in financial networks have a rich underlying mathematical structure: contagion shocks spread out in a way that can be understood in terms of percolation theory. Along the way, we have explored a variety of combinations of the three acronyms, a RFN (random financial network) with the LTI property (locally tree-like independence) on which we place a CM (cascade mechanism), to create contagion models of insolvency, illiquidity and asset fire sales. Towards the end of the book, we developed analytical methods in the context of default contagion, but which extend to other types of contagion as well. These models exhibit key stylized features of market contagion that economists have identified in financial crises from the past. In some circumstances, these ingredients fit together in a natural way that retains mathematical properties of percolation analysis as it has been developed in probability theory, leading to explicit cascade mappings and a host of associated concepts.

The concept of RFN, or random financial network, was introduced as a general stochastic setting appropriate either for hypothetical banking systems, or for simplifying the description of real-life banking systems that are sufficiently large and homogeneous. It was taken to comprise a skeleton of nodes representing banks connected by edges representing counterparty links, together with a stochastic specification of bank balance sheets and interbank exposures. Typical cascade models on RFNs can be computed using Monte Carlo simulation. What we have shown is that these models sometimes also admit an explicit analytical formulation from which we can understand much more. Ultimately, sych RFNs can be created with ever more complex combinations of random skeletons, balance sheets and cascade mechanisms.

LTI combines an assumption about the stochastic properties of the underlying skeleton related to sparsity of the edges together with an assumption about the multivariate dependence of balance sheet random variables. Roughly, it means that the dependence structure of the entire network is coded into the random skeleton: the multivariate distributions of remaining balance random variables are built with independence conditionally on the skeleton. Since there are a variety of approaches to modelling skeletons available in the random graph literature, such a framework provides a useful compromise between tractability, parsimony and flexibility.

A number of cascade mechanisms have been proposed in the literature, notably the model of Eisenberg and Noe [33]. Typically they amount to assumptions or behavioural rules that determine how banks adjust their balance sheets in response to the evolution of the financial crisis. We have reviewed in Chapter 2 different CMs describing a variety of effects such as liquidity shocks, bank defaults, and forced asset sales. The key observation was that these CMs tend to have a common threshold structure. Noticing this commonality is helpful in developing intuition about how different cascades will behave. Of course, such CMs provide only a caricature of the complex decision making of banks and must certainly develop in sophistication in future research, perhaps by following the theory of global games developed for example in [70].

What then are the advantages of having systemic risk models with such ingredients in which crises can be viewed as cascade mappings that lead to fixed points that describe new network equilibria? Typically, this structure leads to a number of advantageous features. Analytics in such models can often be easier to program and faster to compute than Monte Carlo simulations, facilitating exploration of sensitivities to key structural parameters. Sometimes simple specifications of such models can be directly understood by comparison with previously studied models. As well we have seen in Section 5.3 that certain systemic risk measures are computable directly from the fixed point.

Simple static cascade models are often criticized because they exclude important institutional elements, most notably *central clearing parties* (CCP), *central banks* and regulators. However, principles of model building in science suggest that it is important to understand the behaviour of uncontrolled isolated systems before trying to understand how to control them. Cascade mappings provide a level of understanding about the uncontrolled network that will guide the actions of regulators and governments. For a concrete example, it is now fully recognized in Basel III that systemically important financial institutions (SIFIs) must be subjected to regulatory capital surcharges. How such punitive measures are implemented should be defensible by economic and scientific principles. With a reliable cascade model, one can in principle provide such a rationale by finding the optimal distribution of regulatory capital amounts across the network, subject to a fixed aggregated budget.

The three elements, RFNs with an LTI specification upon which a CM is proposed, are intended to be a template for further explorations of systemic risk in ever more complex settings. So what are some of the most promising avenues to follow?

First, RFNs can be far from as simple as those investigated here. Nodes might take on richer characteristics: in our language, the *type* of the node will no longer mean the node degree, but may come to include information about the geographical location, the style of banking undertaken, the size of the institution and so on. For this, skeletons modelled by inhomogeneous random graphs generalizing the construction of Section 3.4 will be useful. Similarly, the network itself will expand to include insurance firms, hedge funds, mutual funds, pension funds, asset classes and even corporates. See [17] for an agent-based model of such an extended network. Describing *community structure*, also called *modularity*, is an active topic of network science that will directly impact systemic risk research. Similarly, the meaning of edges will become more nuanced, and depend in subtle ways on the types of the adjoined banks. Skeletons will evolve into hypergraphs to account for important classes of tri-party contracts such as credit default swaps, repos and asset return swaps.

The balance sheets of banks in this book all have a simple stylistic structure. Papers by [35] and [45] have shown how more complex debt seniority structures can be included in the cascade picture. Similar ideas applied to the asset side will allow rich structures that account for features of different asset classes such as credit rating, maturity and liquidity properties.

The structure of interbank contracts considered here is similarly stylistic: for the most part, these have been described as short term unsecured lending. In reality, such contracts are complex, and have structure and magnitude that are the result of bilateral strategic games played continuously between pairs of counterparty banks who seek to control risk and maximize returns.

This book has focussed on systemic risk, that is to say, the risk of large scale or even catastrophic disruptions of the financial markets. But good models of the financial system must account for the behaviour of markets in normal times as well. This is already recognized in macroprudential models used by central banks that link together modules describing parts of the economy, such as the *Systemic Risk Monitor* of the Austrian Central Bank, the RAMSI model used by the Bank of England and the MFRAF model used by the Bank of Canada. One important module in such economic models is always the financial network submodel that should behave realistically in normal times, and should generate endogenously an adequate spectrum of crisis scenarios.

Systemic risk modelling has advanced enormously on a range of fronts since the great financial crisis of 2007-08. It is my hope that this book will be viewed as a timely contribution to the field that provides researchers with two different things: first, a flexible set of mathematical techniques for analyzing cascades in general categories of networks, and second, a scientific modelling framework of the financial economy that can scale up to account for all the important dimensions of a type of risk that has the potential to critically impact the entire global economy.

Appendix A
Background Material

A.1 Special Notation

Matrix and Vector Notation: For vectors $x = [x_\nu]_{\nu=1,\dots,N}, y = [y_\nu]_{\nu=1,\dots,N} \in \mathbb{R}^N$ define relations

$$\begin{aligned} & x \le y \text{ means } \forall\ \nu,\ x_\nu \le y_\nu, \\ & x < y \text{ means } x \le y,\ \exists\ \nu : x_\nu < y_\nu, \\ \min(x,y) &= [\min(x_\nu, y_\nu)]_{\nu=1,\dots,N} \\ \max(x,y) &= [\max(x_\nu, y_\nu)]_{\nu=1,\dots,N} \\ (x)^+ &= \max(x,0), \\ (x)^- &= \max(-x,0)\ . \end{aligned}$$

Whenever $x \le y$ we can define the hyperinterval $[x,y] = \{z : x \le z \le y\}$. Column N-vectors are often treated as $[1,N]$ matrices, $X \cdot Y$ denotes the matrix product, and X' denotes the Hermitian (complex conjugate) transpose.

Function Notation: For functions f, g, $f \circ g$ denotes composition and $f \circledast g$ denotes convolution.

Graph Notation: For a complete description of the graphical notation used in this book, please see Section 3.1.

Set Notation and Indicators: For any set $A \subset \Omega$ in the sample space Ω, $A^c := \Omega \setminus A$ is its complement. For elements $x \in \Omega$ and Boolean propositions P:

$$\begin{aligned} \text{Indicator} \quad \mathbf{1}(P) \quad &= \quad \begin{cases} 1 \text{ if } P \text{ is true} \\ 0 \text{ if } P \text{ is false} \end{cases} \\ \text{Indicator function} \quad \mathbf{1}_A(x) \text{ means } \quad & \mathbf{1}(x \in A)\ . \end{aligned}$$

A.2 The Discrete Fourier Transform

We consider the space $\mathbb{C}^M$ of $\mathbb{C}$-valued functions on $\mathscr{M} = \{0,1,\ldots,M-1\}$. The discrete Fourier transform, or fast Fourier transform (FFT), is the linear mapping $\mathscr{F} : a = [a_0,\ldots,a_{M-1}] \in \mathbb{C}^M \to \hat{a} = \mathscr{F}(a) \in \mathbb{C}^M$ defined by

$$\hat{a}_k = \sum_{l\in\mathscr{M}} \zeta_{kl} a_l \ , k \in \mathscr{M} \ .$$

where the coefficient matrix $Z = (\zeta_{kl})$ has entries $\zeta_{kl} = e^{-2\pi ikl/M}$.

It is easy to prove that its inverse, the "inverse FFT" (IFFT), is given by the map $a \to \tilde{a} = \mathscr{G}(a)$ where

$$\tilde{a}_k = \frac{1}{M}\sum_{l\in\mathscr{M}} \bar{\zeta}_{kl} a_l \ , k \in \mathscr{M} \ .$$

If we let $\bar{a}$ denote the complex conjugate of a, we can define the Hermitian inner product between

$$\langle a,b\rangle := \sum_{m\in\mathscr{M}} \bar{a}_m b_m \ .$$

We also define the convolution product of two vectors:

$$(a \circledast b)(n) = \sum_{m\in\mathscr{M}} a(m)\, b(n-m \text{ modulo } M), \quad n \in \mathscr{M} \ .$$

Now we note the following easy-to-prove identities which hold for all $a,b \in \mathbb{C}^M$:

1. Inverse mappings:
$$a = \mathscr{G}(\mathscr{F}(a)) = \mathscr{F}(\mathscr{G}(a)) \ ; \tag{A.1}$$
2. Conjugation:
$$\overline{\mathscr{G}(a)} = \frac{1}{M}\mathscr{F}(\bar{a}) \ ; \tag{A.2}$$
3. Parseval Identity:
$$\langle a,b\rangle = M\langle \tilde{a},\tilde{b}\rangle = \frac{1}{M}\langle \hat{a},\hat{b}\rangle \ ; \tag{A.3}$$
4. Convolution Identities:
$$\tilde{a}\cdot*\tilde{b} = \widetilde{(a \circledast b)} \ ; \quad \hat{a}\cdot*\hat{b} = \widehat{(a \circledast b)} \tag{A.4}$$

   where $\cdot*$ denotes the component-wise product.

The primary application of the FFT in this book is its use in accelerating the computation of convolutions of probability mass functions supported on the set CM. If the support of the sum of two $\mathscr{M}$-valued random variables X,Y is itself in $\mathscr{M}$, that is, if $\text{supp}(X+Y) = \{n|\exists m \in \text{supp}(X) \text{s.t. } n-m \in \text{supp}(Y)\} \in \mathscr{M}$, then p_{X+Y}, the PMF of $X+Y$, is given by $p_X \circledast p_Y$. On the other hand, if there are $m \in \text{supp}(X), n \in \text{supp}(Y)$ such that $m+n \geq M$, then $p_{X+Y} - p_X \circledast p_Y$ is not zero. Such a difference is called an "aliasing error", and implies that $p_{X+Y} \neq \mathscr{G}(\mathscr{F}(p_X)\mathscr{F}(p_Y))$. In any application, we must take care to keep all such aliasing errors sufficiently small by choosing M sufficiently large but finite.

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