Portfolio Choice in Markets with Contagion

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September 26, 2012

Abstract

We consider the problem of optimal investment and consumption in a class of multidimensional jump-diffusion models in which asset prices are subject to mutually exciting jump processes. This captures a type of contagion where each downward jump in an asset's price results in increased likelihood of further jumps, both in that asset and in the other assets. We solve in closed-form the dynamic consumption-investment problem of a log-utility investor in such a contagion model, prove a theorem verifying its optimality and discuss features of the solution, including flight-to-quality. The exponential and power utility investors are also considered: in these cases, the optimal strategy can be characterized as a distortion of the strategy of a corresponding non-contagion investor.

Keywords: Merton problem, jumps, Hawkes process, mutual excitation, contagion, flight-to-quality.
JEL Classification: G11, G01.

*Research supported by the NSF under grants DMS-0532370 and NES-0850533.
†Research supported by NSERC and MITACS Canada.

1 Introduction

The recent financial crisis has emphasized the relevance of jumps for understanding the various forms of risk inherent in asset returns and their implications for asset allocation and diversification. Most portfolios, from those of individual investors to those of more sophisticated institutional investors including University endowments, suffered badly during the latest crisis episode, with many commonly employed asset allocation strategies resulting in large losses in 2008-09. Reasonably diversified portfolios can survive a single isolated negative jump in asset returns. However, jumps that tend to affect most or all asset classes together are difficult to hedge by diversification alone. Moreover, additional jumps of this nature seemed to happen in close succession, as if the very occurrence of a jump substantially increased the likelihood of future jumps.

Motivated by these events, we consider in this paper the issue of optimal portfolio construction when assets are subject to jumps that share the qualitative features experienced most vividly during the recent financial crisis. These salient features include the fact that multiple jumps were observed, at a rate that was markedly higher than the long term unconditional arrival rate; these jumps affected multiple asset classes and markets; and they affected them not necessarily at the same time, but typically in close succession over days or weeks.

To capture these key elements, we consider a model for asset returns where a jump in one asset class or region raises the probability of future jumps in both the same asset class or region, and the other classes or regions. Jump processes of this type were first introduced by Hawkes (1971) with further developments due to Hawkes and Oakes (1974) and Oakes (1975). Models of this type have been used in epidemiology, neurophysiology and seismology (see, e.g., Brillinger (1988) and Ogata and Akaike (1982)), genome analysis (see Reynaud-Bouret and Schbath (2010)), credit derivatives (Errais et al. (2010)), to model transaction times and price changes at high frequency (Bowsher (2007)), trading activity at different maturities of the yield curve (Salmon and Tham (2007)) and propagation phenomena in social interactions (Crane and Sornette (2008).)

We extend the pure jump Hawkes model employed in the above applications, in order to better represent financial asset returns. We add a drift to capture the assets' expected returns and a standard, Brownian-driven, volatility component to capture their day-to-day normal variations. We call this model a "Hawkes jump-diffusion" by analogy with the Poisson jump-diffusion of Merton (1976). Unlike models typically employed in finance, the jump part of this model is no longer a Lévy process since excitation introduces a departure from independence of the increments of the jumps.

In the model, jump intensities are stochastic and react to recent jumps: a jump increases the rate of incidence (or intensity) of future jumps; running counter to this is mean reversion, which pulls the jump intensities back down in the absence of further excitation. In the univariate case, only "self excitation" can take place, whereas in the multivariate case "mutual excitation" consisting of both self- and cross-excitation (from one asset to another) can take place. We now illustrate the presence of the mutual excitation phenomenon by filtering jump intensities for jumps in the US financial sector stock index during the recent crisis. Figure 1 plots the estimated intensity of US jumps over time, filtered from the observed returns on indices of financial stocks in the US, UK, Eurozone and Asia. The excitation mechanism is apparent in the Fall of 2008, when jump intensities increase rapidly in response to each jump, most of them originating in the US, and to a lesser extent in the Winter of 2009, which looks more like a slow train wreck.

The bottom panel of the plot shows that the filtered intensities contain information that is different from other measures of market stress, VIX and the CDS rate on financial stocks. In particular, VIX is a measure of total quadratic variation and as a result captures the total risk of the assets instead of just their jump risk. So the same jumps which cause the jump intensity to increase in the middle panel also cause VIX to increase in the bottom panel, but VIX includes Brownian volatility, making it a much noisier measure of jump risk.

The purpose of the paper is to solve for the optimal portfolio of an investor who faces this type of risk in his/her investment opportunity set. By considering a more realistic model for jumps, incorporating mutual excitation, we are able to study the optimal portfolio of an investor in a realistic setting where a jump that occurs somewhere will increase the probability of further jumps in other asset classes or markets. The model generates jumps that will tend to be clustered (as a result of the time series self-excitation), systematic (as a result of the cross-sectional excitation), but neither exactly simultaneous nor certain, since the excitation

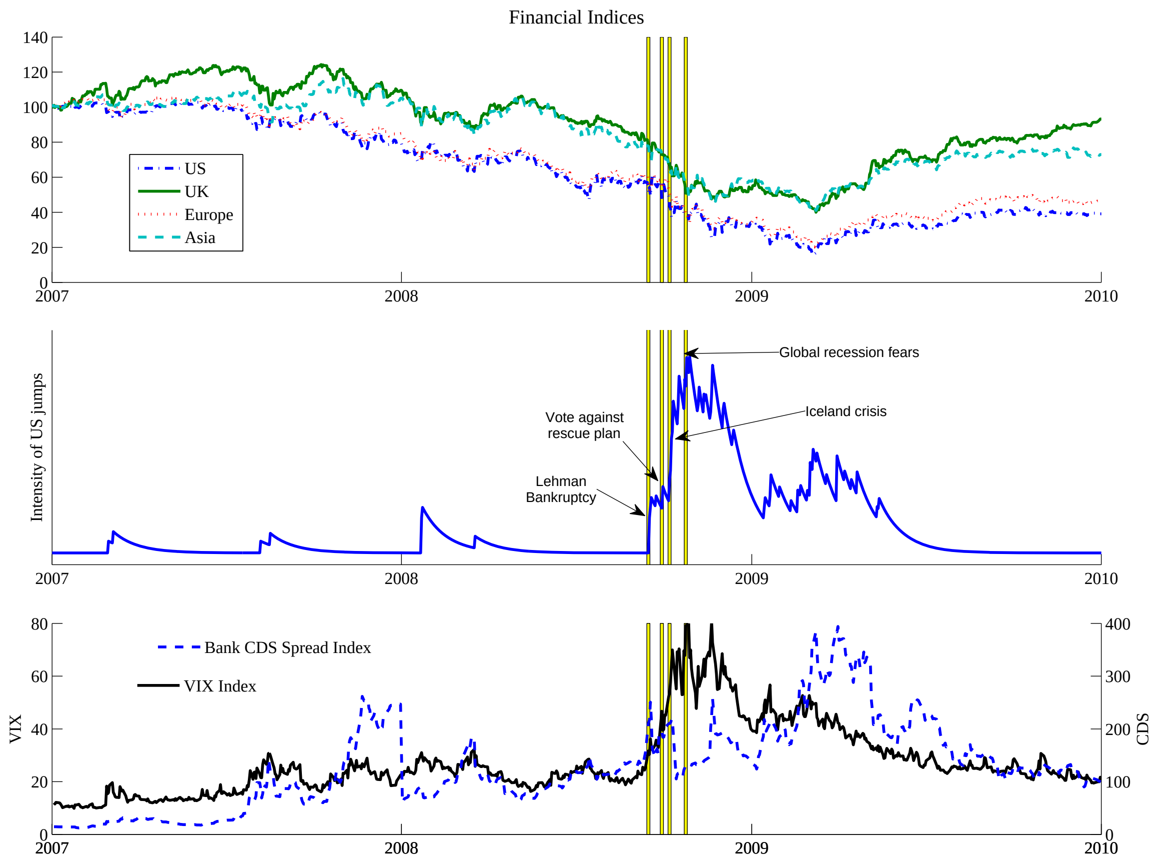

Figure 1: Time-varying Jump Intensities During the Financial Crisis. The top panel shows time series of stock indices for the financial sector in the four regions, 2007-2009. The middle panel shows the intensity of US jumps filtered from the model, based on each 3σ and above event identified as a jump. Each jump leads to an increase in the jump intensity, followed by mean reversion until the next jump. The bottom panel shows the time series of two alternative measures of financial distress, the VIX index and Markit's CDS index for the banking sector in the US.

phenomenon merely raises the probability of future jump occurrence. By analogy with epidemics, the probability of getting infected increases in a pandemy but does not typically reach one, and there is an incubation period which can range in the case of financial markets from hours to days, depending upon subsequent news arrival, and once established the pandemy does not go away immediately. Furthermore, the model is multivariate and the contagion can be asymmetric, with jumps occurring in one asset class or market having a greater excitation potential for the other sectors or regions than jumps that originate elsewhere: for instance, most financial crises that originate in or transit through the US tend to have greater ramifications in the rest of the world than crises that originate outside the US. Poisson jumps, whose intensities are constant, are not able to reproduce these empirical features, and this motivates our inclusion of the more general class of Hawkes jumps in the model.

This paper is part of a literature that has investigated the properties of optimal portfolios when asset returns can jump (see, e.g., Aase (1984), Jeanblanc-Picqué and Pontier (1990), Shirakawa (1990), Han and Rachev (2000), Ortobelli et al. (2003), Kallsen (2000), Carr et al. (2001), Liu et al. (2003), Das and Uppal (2004), Emmer and Klüppelberg (2004), Madan (2004), Cvitanić et al. (2008), Delong and Klüppelberg (2008) and Aït-Sahalia et al. (2009)). The novel aspect in the present paper is the inclusion of Hawkes jumps in asset returns: such jumps share the dual characteristics of being systematic, meaning that they affect multiple assets or asset classes at the same time, and mutually exciting, meaning that they affect the rate at which future jumps occur in each asset class. By contrast, in the earlier model of Aït-Sahalia et al. (2009), assets were subject to random jumps which could affect one or more asset or asset classes, but when they occurred, they were simultaneous and every asset in that sector or region would jump. Such jumps were also "Poissonian", in the sense that the arrival of jumps today did not influence the future arrival of jumps.

We show that the optimal portfolio solution in the model can be obtained in full closed-form in the log utility case, and in quasi-closed-form in other cases. Importantly, we show that the optimal solution becomes time-varying, with the investor reacting to changes in the intensity of the jumps. For a log-utility investor, the solution remains myopic, as in the classical Merton (1971) problem with log-utility, in the sense that the investor does not need to take into account the full dynamics of the state variables. The log investor in our model holds at each point in time the same portfolio as a log investor who believes that jump intensities are constant, but his/her optimal portfolio weight is now constantly changing to reflect the time-variation in jump intensities. One consequence of this result is that each time a market shock occurs, the investor perceives an increase in jump intensities, and sells some amount of *each* risky asset, and invests the proceeds in the riskless asset, a behavior we interpret as a "flight to quality."

Formal computations also work for both power and exponential utility investors, although we have not proved the appropriate verification theorem for these utilities. Nevertheless, the resultant investment strategy under contagion can be interpreted in terms of the equivalent strategy under the "non-contagion" assumption that jump intensities are constant. We find that under the contagion conditions of the model, the investor will choose a portfolio that is optimal for a non-contagion investor who has a specific distorted value of the intensities, which we characterize. This distortion of intensities has the effect of magnifying the investment in the risky assets: the contagion investor will go "longer" when the non-contagion investor is long in the risky asset, and will go "shorter" when the non-contagion investor is short in the risky asset.

The paper is organized as follows. Section 2 presents the model for asset returns. Section 3 introduces the optimal portfolio problem when jumps are mutually exciting in the general case. Section 4 specializes the solution to the case of an investor with log-utility and derives the optimal portfolio and consumption policy in closed-form, including a complete verification theorem that supports this policy. Section 5 develops some interesting market specifications which exhibit an explicit closed-form log-optimal policy. Section 6 explores some of the properties of such explicit asset allocation policies. The exponential and power utility investment problems are outlined in Section 7. In these problems, the solutions can be characterized as distorted versions of the non-contagion solutions. Section 8 concludes.

2 Mutually Exciting Jumps

In this paper, jumps in one asset class not only increase the probability of future jumps in that asset class (self excitation) but also in other asset classes (cross excitation). In a mutually exciting model, the intensity of a jump counting process N ramps up in response to past jumps.

A mutually exciting process is a special case of path-dependent point process, whose intensity depends on the path of the underlying process. Mutually exciting counting processes, $N_{l,t}$ $(l = 1, ..., m)$, form an m-vector $\boldsymbol{N}_t = [N_{1,t}, ..., N_{m,t}]'$ such that[1]

$$\begin{aligned}\mathbb{P}\left[N_{l,t+\Delta t} - N_{l,t} = 1 | \mathcal{F}_t\right] &= \lambda_{l,t}\Delta t + o\left(\Delta t\right), \\ \mathbb{P}\left[N_{l,t+\Delta t} - N_{l,t} > 0 | \mathcal{F}_t\right] &= o\left(\Delta t\right)\end{aligned} \tag{2.1}$$

independently for each l. In the standard model specification we adopt in this paper, the Hawkes intensity processes $\boldsymbol{\lambda}_t = [\lambda_{1,t}, ..., \lambda_{m,t}]'$ have the integrated form

$$\lambda_{l,t} = e^{-\alpha_l t}\lambda_{l,0} + (1 - e^{-\alpha_l t})\lambda_{l,\infty} + \sum_{j=1}^{m} \int_0^t d_{lj} e^{-\alpha_l (t-s)} dN_{j,s}, \;\; l = 1, ..., m \tag{2.2}$$

For all $l, j = 1, \ldots, m$ the parameters $\lambda_{l,\infty}, d_{lj} \geq 0$ and $\lambda_{l,0}, \alpha_l > 0$ are constants. These parameter restrictions ensure the positivity of the intensity processes with probability one.

Differentiation of equation (2.2) shows that the intensity in asset class l has dynamics given by

$$d\lambda_{l,t} = \alpha_l \left(\lambda_{l,\infty} - \lambda_{l,t}\right) dt + \sum_{j=1}^{m} d_{lj} dN_{j,t}. \tag{2.3}$$

In other words, a jump $dN_{j,s}$, occurring at time $s \in [0, t)$ in asset class $j = 1, \ldots, m$, raises each of the jump intensities $\lambda_{l,t}$, $l = 1, \ldots, m$, by a constant amount d_{lj}. The lth jump intensity then mean-reverts to level $\lambda_{l,\infty}$ at speed α_l until the next jump occurs. Equation 2.3 also reveals that $(\boldsymbol{N}, \boldsymbol{\lambda})$ is a $2m$-dimensional Markov process, while $(\boldsymbol{\lambda})$ alone is an m-dimensional Markov process.

As a result, our model generates clusters of jumps over time, and jumps can propagate at different speed and with different intensities in the different asset classes depending on where they originate and which path they take to reach a given asset class or market. Free parameters in the model control the extent to which the two forms of excitation take place, the relative strength of the contagion phenomenon in different directions and the speed with which the excitation takes place and then relaxes.

The model produces both cross-asset class and time series excitation. In the univariate self-exciting case, a typical sample path of one component of $\boldsymbol{\lambda}$ is illustrated in Figure 2. Each jump increases the jump intensity, followed by mean-reversion.

[1] In this paper, we work in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ that satisfies "the usual conditions."

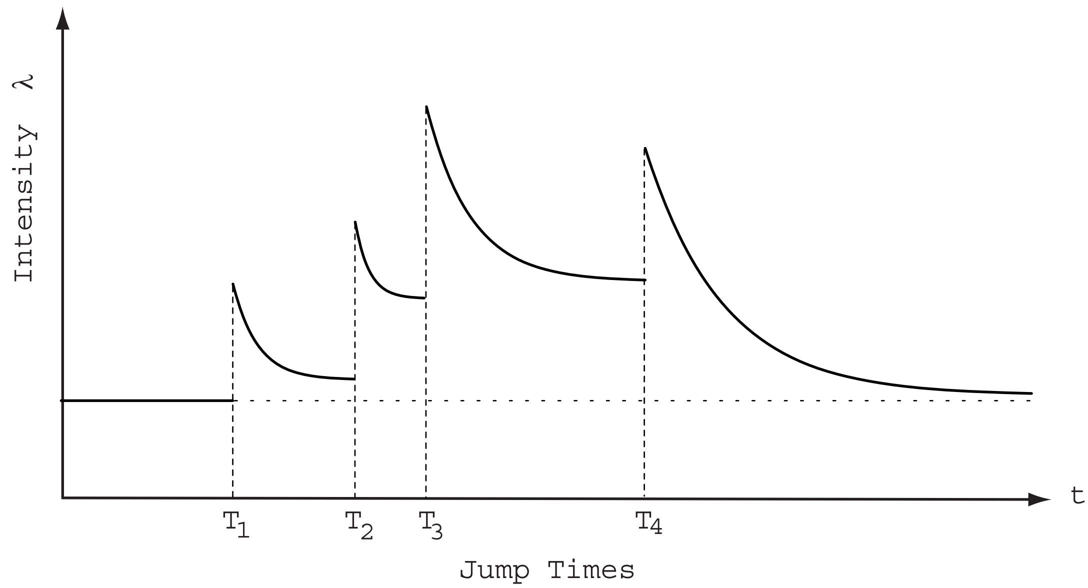

Figure 2: Sample path of a Hawkes intensity, $\lambda_{l,t}$.

This model also produces cross-asset, or mutual, excitation. Jumps in asset class l that occurred u units of time into the past raise the intensity of jumps in asset class j by $d_{jl}e^{-\alpha_j u}$, while conversely jumps in asset class j raise the intensity of jumps in asset class l by $d_{lj}e^{-\alpha_l u}$. Jumps in a given asset class i also raise the intensity of future jumps in the same asset class. Figure 3 illustrates this with two assets. At time T_1 there is a jump in the first asset class value, S_1. This jump self-excites the jump intensity λ_1. This increase in λ_1 raises the probability of observing another jump in S_1 at the future time T_2. These jumps have a contagious effect on S_2 since a jump in S_1 cross-excites the jump intensity of S_2. This, in turn, raises the probability of seeing a jump in S_2 at time T_3. Latter on, at time T_4, the jump in S_2 raises the probability of seeing a jump in S_1 at some future time T_5. The degree to which self- and cross-excitation matter in the model, and their relative strengths, is controlled by the parameters in d and $\boldsymbol{\alpha}$.

We note that each compensated process $N_{l,t} - \int_0^t \lambda_{l,s} ds$ is a local martingale. A number of important additional properties hold for this model:

1. **Markov Generator:** The Markov generator of this process acting on differentiable

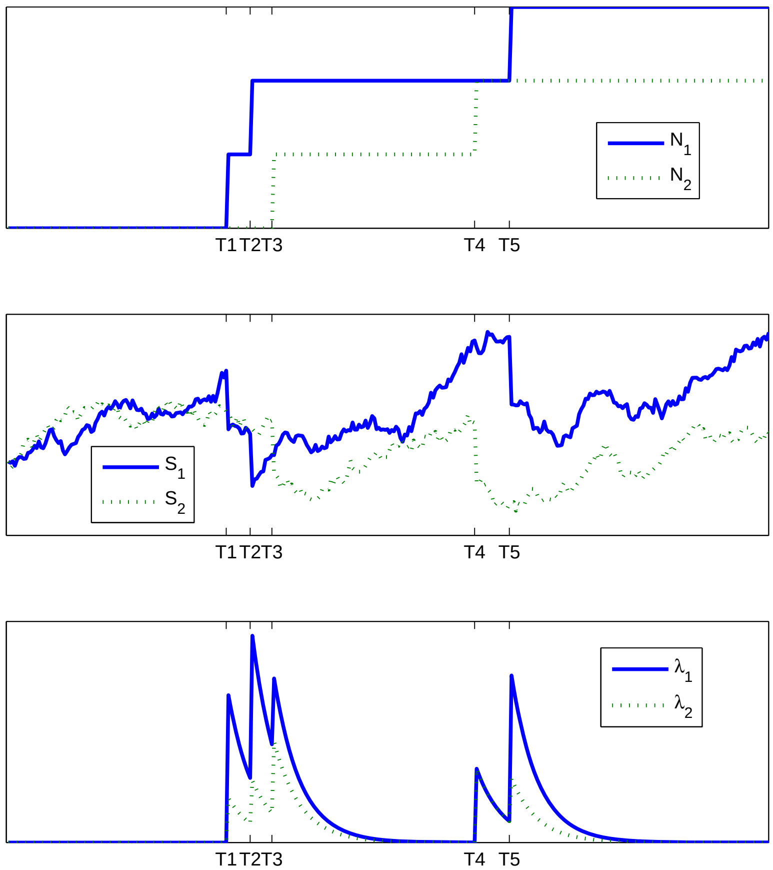

Figure 3: Cross- and self-excitation in a two asset-class world: Sample paths of the jumps, asset prices and jump intensities.

functions $g : (\mathbb{Z}_+)^m \times (\mathbb{R}_+)^m \to \mathbb{R}$ is given by

$$[\mathcal{A}g](\boldsymbol{n}, \boldsymbol{\lambda}) = \sum_{l=1}^{m} \left[\alpha_l \left(\lambda_{l,\infty} - \lambda_l\right) \frac{\partial g}{\partial \lambda_l} + \lambda_l \left(g(\boldsymbol{n} + \boldsymbol{e}_l, \boldsymbol{\lambda} + \boldsymbol{d_l}) - g(\boldsymbol{n}, \boldsymbol{\lambda})\right) \right]$$

where $\boldsymbol{e}_l = [\delta_{1l}, \ldots, \delta_{ml}]'$ and $\boldsymbol{d}_l = [d_{1l}, \ldots, d_{ml}]'$. If

$$\mathbb{E}\left[\int_0^t \Big| \, [\mathcal{A}g](\boldsymbol{N}_s, \boldsymbol{\lambda}_s) - \sum_{\ell=1}^{m} \big[\alpha_\ell \left(\lambda_{\ell,\infty} - \lambda_{\ell,s}\right) \frac{\partial g(\boldsymbol{N}_s, \boldsymbol{\lambda}_s)}{\partial \lambda_l} \big] \, \Big| ds \right] < \infty$$

for all t and g is differentiable in λ, the Dynkin formula says that for each $t \le T$:

$$\mathbb{E}[g(\boldsymbol{N}_T, \boldsymbol{\lambda}_T)|\mathcal{F}_t] = g(\boldsymbol{N}_t, \boldsymbol{\lambda}_t) + \mathbb{E}\left[\int_t^T [\mathcal{A}g](\boldsymbol{N}_s, \boldsymbol{\lambda}_s) \; ds | \mathcal{F}_t \right] . \tag{2.4}$$

2. **Stationarity Assumption:** Let the process $\boldsymbol{\lambda}_t$ satisfy (2.3) with $\boldsymbol{\lambda}_0 \in \mathbb{R}_+^m$; $\alpha_i > 0$; $d_{lj} \ge 0$ and with $\boldsymbol{\Gamma} = (\Gamma_{lj})_{l,j=1,\ldots,m}$, where $\Gamma_{lj} = \alpha_j \delta_{lj} - d_{lj}$, a positive (hence invertible) matrix where δ_{lj} is the Kronecker symbol. Then the intensities $\boldsymbol{\lambda}$ are stationary processes with bounded moments. Using the Dynkin formula, one can show that under this assumption the first moments $f_l(t) = \mathbb{E}\left[\lambda_l(t)\right]$ converge as $t \to \infty$ to the non-negative values $f_l(\infty) = \sum_{j=1}^{m} (\delta_{lj} - \alpha_l^{-1} d_{lj}) \lambda_{j,\infty}$. Similar formulas can be derived for the large time limits of the higher moment functions. Using these bounds, the ergodic theorem for semimartingales (see e.g. Khasminskii (1960)) then implies that for any measurable function $K(\lambda)$ that satisfies a bound $|K(\lambda)| \le M(1 + \|\lambda\|^2)$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T K(\lambda_t) dt = \lim_{t\to\infty} \mathbb{E}[K(\lambda_t)] = \int_{\mathbb{R}_+^n} K(\lambda) \mu(d\lambda) \tag{2.5}$$

a.s. where $\mu(d\lambda)$ is the invariant (infinite time) measure of λ_t.

3. **Affine Structure:** The joint characteristic function has the affine form

$$\mathbb{E}_{0,\lambda_0}[e^{i\boldsymbol{u}\boldsymbol{N_T} + i\boldsymbol{v}\boldsymbol{\lambda_T}}] = \exp\left[iA(T; \boldsymbol{u}, \boldsymbol{v}) + i \sum_{l=1}^{m} B_l(T; \boldsymbol{u}, \boldsymbol{v}) \lambda_{l,0} \right]$$

where the deterministic functions A, B_l satisfy the Riccati equations

$$\frac{\partial A}{\partial T} = \sum_{l=1}^{m} \alpha_l \lambda_{l,\infty} B_\ell; \quad A(0) = 0; \tag{2.6}$$

$$\frac{\partial B_l}{\partial T} = -\alpha_l B_l - i \left(e^{iu_l + i \sum_{j=1}^{m} d_{jl} B_j} - 1 \right); \quad B_l(0) = v_l; \quad l = 1, \ldots, m \; . \tag{2.7}$$

3 Optimal Portfolio Selection When Jumps Are Mutually Exciting

We now solve Merton's problem: the investor maximizes the expected utility of consumption by investing in a set of n risky assets and a riskless asset over the infinite time horizon $t \in [0, \infty)$. The innovation is that the risky assets are now subject to shocks generated by an m-dimensional Hawkes jump-diffusion process.

3.1 Asset Return Dynamics

The riskless asset with price $S_{0,t}$ is assumed to earn a constant rate of interest $r \geq 0$. The n risky assets with prices $\boldsymbol{S}_t = [S_{1,t}, \ldots, S_{n,t}]'$ follow a semimartingale dynamics with asset shocks generated by an m-dimensional Hawkes process. Specifically, we assume

$$\frac{dS_{0,t}}{S_{0,t}} = r dt, \tag{3.1}$$

$$\frac{dS_{i,t}}{S_{i,t-}} = (r + R_i)\, dt + \sum_{j=1}^{n} \sigma_{i,j} dW_{j,t} + \sum_{l=1}^{m} J_{i,l} Z_{l,t} dN_{l,t}, \;\; i = 1, ..., n \tag{3.2}$$

Here $\boldsymbol{N}_t = [N_{1,t}, \ldots, N_{m,t}]'$ is an $m-$dimensional, $m \leq n$, vector of mutually exciting Hawkes processes with intensities $\boldsymbol{\lambda}_t = [\lambda_{1,t}, \ldots, \lambda_{m,t}]'$ that follow the Markovian dynamics

$$d\lambda_{l,t} = \alpha_l \left(\lambda_{l,\infty} - \lambda_{l,t}\right) dt + \sum\nolimits_{j=1}^{m} d_{lj} dN_{j,t}, \quad l = 1, \ldots, m, \tag{3.3}$$

with constant parameters $\alpha_l > 0$, $\lambda_{l,\infty} \geq 0$, and $d_{lj} \geq 0$. Under the condition that $\boldsymbol{\Gamma}$ is a positive matrix, the $\boldsymbol{\lambda}$ process is stationary.

The vector $\boldsymbol{W}_t = [W_{1,t}, \ldots, W_{n,t}]'$ is an $n-$dimensional standard Brownian motion. $J_{i,l} Z_{l,t}$ is the response of asset i to the lth shock where $Z_{l,t}$, a scalar random variable with probability measure $\nu_l(dz)$ on $[0, 1]$, is scaled on an asset-by-asset basis by the deterministic scaling factor $J_{i,l} \in [-1, 0]$. For clarity, we chose to include only negative asset jumps in the asset price dynamics, since those are the more relevant ones from both a portfolio risk management perspective and their contribution to mutual excitation.

We assume that the individual Brownian motions, the Hawkes process and the random variables Z_l are mutually independent. The quantities R_i, σ_{ij} and jump scaling factors $J_{i,l}$ are constant parameters. We write $\boldsymbol{R} = [R_1, \ldots, R_n]'$, $\boldsymbol{J} = (J_{i,l})_{i=1,\ldots,n;l=1,\ldots,m}$, and $\boldsymbol{\sigma} = (\sigma_{i,j})_{i,j=1,\ldots,n}$ and we assume that the matrix $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}'$ is nonsingular.

In Section 5, we will make further assumptions on the structure of the matrix $\boldsymbol{\Sigma}$ to facilitate the derivation of an explicit solution, assuming in particular that it possesses a factor structure. But the existence and structure of the optimal portfolio solution can be determined without further specialization, and we now turn to this problem.

3.2 Wealth Dynamics and Expected Utility

Let $\omega_{0,t}$ denote the percentage of wealth (or portfolio weight) invested at time t in the riskless asset and $\boldsymbol{\omega}_t = [\omega_{1,t}, \ldots, \omega_{n,t}]'$ denote the vector of portfolio weights in each of the n risky assets, assumed to be adapted cáglád processes since the portfolio weights cannot anticipate the jumps. The portfolio weights satisfy

$$\omega_{0,t} + \sum\nolimits_{i=1}^{n} \omega_{i,t} = 1. \tag{3.4}$$

The investor consumes C_t at time t. In the absence of any income derived outside his investments in these assets, the investor's wealth, starting with the initial endowment X_0, follows the dynamics

$$\begin{aligned} dX_t &= -C_t dt + \omega_{0,t} X_t \frac{dS_{0,t}}{S_{0,t-}} + \sum\nolimits_{i=1}^{n} \omega_{i,t} X_t \frac{dS_{i,t}}{S_{i,t-}} \\ &= \left(rX_t + \boldsymbol{\omega}_t' \boldsymbol{R} X_t - C_t\right) dt \;+\; X_t \boldsymbol{\omega}_t' \boldsymbol{\sigma} d\boldsymbol{W}_t + X_t \sum\nolimits_{l=1}^{m} \left(\boldsymbol{\omega}_t' \boldsymbol{J}\right)_l Z_{l,t} dN_{l,t}. \end{aligned} \tag{3.5}$$

We consider an investor with time-separable utility of consumption $U(\cdot)$ and subjective discount rate or "impatience" parameter $\beta > 0$. The investor's problem at any time $t \geq 0$ is then to pick the consumption and portfolio weight processes $\{C_s, \boldsymbol{\omega}_s\}_{t \leq s \leq \infty}$ which maximize the infinite–horizon discounted expected utility of consumption. The optimal policies $\{C_s, \boldsymbol{\omega}_s\}_{t \leq s \leq \infty}$ are subject to the *admissibility condition* that the discounted wealth process remains positive almost surely.

Stochastic dynamic programming (see, e.g., Fleming and Soner (2006)) leads at time t to the discounted expected utility of consumption in the form $V(X_t, \boldsymbol{\lambda}_t, t)$ where the value function is defined by

$$V(x, \boldsymbol{\lambda}, t) = \max_{\{C_s, \boldsymbol{\omega}_s;\, t \leq s \leq \infty\}} \mathbb{E}_{x, \boldsymbol{\lambda}, t} \left[\int_t^{\infty} e^{-\beta s} U(C_s) ds \right] \tag{3.6}$$

Here, the discounted wealth and intensities satisfy (3.5) and (3.3) over $[t, \infty)$ with initial conditions $X_t = x$, $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}$. Under the assumption that V is sufficiently differentiable, the appropriate form of Itô's lemma (see, e.g., Protter (2004)) for semi-martingale processes leads to the Hamilton-Jacobi-Bellman equation that characterizes the optimal solution to the investor's problem:

$$0 = \max_{\{C,\boldsymbol{\omega}\}} \left\{ \frac{\partial V(x,\boldsymbol{\lambda},t)}{\partial t} + \sum_{l=1}^{m} \alpha_l (\lambda_{l,\infty} - \lambda_l) \frac{\partial V(x,\boldsymbol{\lambda},t)}{\partial \lambda_i} + e^{-\beta t} U(C) \right. \\ + \frac{\partial V(x,\boldsymbol{\lambda},t)}{\partial x} (rx + \boldsymbol{\omega}' \boldsymbol{R} x - C) + \frac{1}{2} \frac{\partial^2 V(x,\boldsymbol{\lambda},t)}{\partial x^2} \boldsymbol{\omega}' \boldsymbol{\Sigma} \boldsymbol{\omega} x^2 + \\ \left. \sum_{l=1}^{m} \lambda_l \int \left[V\left(x + (\boldsymbol{\omega}' \boldsymbol{J})_l zx, \boldsymbol{\lambda} + \boldsymbol{d}_l, t\right) - V(x,\boldsymbol{\lambda},t) \right] \nu_l(dz) \right\} \tag{3.7}$$

with the transversality condition $\lim_{t\to\infty} \mathbb{E}\left[V(X_t, \boldsymbol{\lambda}_t, t)\right] = 0$.

Using the standard time-homogeneity argument for infinite horizon problems, we have that

$$\begin{aligned} e^{\beta t} V(x,\boldsymbol{\lambda},t) &= \max_{\{C_s,\boldsymbol{\omega}_s;\ t\le s\le\infty\}} \mathbb{E}_{x,\boldsymbol{\lambda},t} \left[\int_t^\infty e^{-\beta(s-t)} U(C_s) ds \right] \\ &= \max_{\{C_s,\boldsymbol{\omega}_s;\ t\le s\le\infty\}} \mathbb{E}_{x,\boldsymbol{\lambda},t} \left[\int_0^\infty e^{-\beta u} U(C_{t+u}) du \right] \\ &= \max_{\{C_s,\boldsymbol{\omega}_s;\ 0\le s\le\infty\}} \mathbb{E}_{x,\boldsymbol{\lambda},0} \left[\int_0^\infty e^{-\beta u} U(C_u) du \right] \\ &= V(x,\boldsymbol{\lambda},0) \equiv L(x,\boldsymbol{\lambda}) \end{aligned}$$

is independent of time. Thus $V(x,\boldsymbol{\lambda},t) = e^{-\beta t} L(x,\boldsymbol{\lambda})$ and (3.7) reduces to the following time-independent equation for the value function L :

$$0 = \max_{\{C,\boldsymbol{\omega}\}} \left\{ U(C) - \beta L(x,\boldsymbol{\lambda}) + \sum_{l=1}^{m} \alpha_l (\lambda_{l,\infty} - \lambda_l) \frac{\partial L(x,\boldsymbol{\lambda})}{\partial \lambda_l} \right. \\ + \frac{\partial L(x,\boldsymbol{\lambda})}{\partial x} (rx + \boldsymbol{\omega}' \boldsymbol{R} x - C) + \frac{1}{2} \frac{\partial^2 L(x,\boldsymbol{\lambda})}{\partial x^2} \boldsymbol{\omega}' \boldsymbol{\Sigma} \boldsymbol{\omega} x^2 \\ \left. + \sum_{l=1}^{m} \lambda_l \int \left[L\left(x + (\boldsymbol{\omega}' \boldsymbol{J})_l zx, \boldsymbol{\lambda} + \boldsymbol{d}_l\right) - L(x,\boldsymbol{\lambda}) \right] \nu_l(dz) \right\} \tag{3.8}$$

with the transversality condition

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\beta t} L(X_t, \boldsymbol{\lambda}_t) \right] = 0. \tag{3.9}$$

The maximization problem in (3.8) separates into one for C, with first order condition

$$U'(C) = \frac{\partial L(x, \boldsymbol{\lambda})}{\partial x}$$

and one for $\boldsymbol{\omega}$:

$$\boldsymbol{\omega}^* = \boldsymbol{\omega}^*(x, \boldsymbol{\lambda}) := \text{argmax}_{\{\boldsymbol{\omega}\}} \left\{ \frac{\partial L(x, \boldsymbol{\lambda})}{\partial x} \boldsymbol{\omega}' \boldsymbol{R} x + \frac{1}{2} \frac{\partial^2 L(x, \boldsymbol{\lambda})}{\partial x^2} \boldsymbol{\omega}' \boldsymbol{\Sigma} \boldsymbol{\omega} x^2 \right.$$
$$\left. + \sum_{l=1}^{m} \lambda_l \int \left[L\left(x + (\boldsymbol{\omega}' \boldsymbol{J})_l zx, \boldsymbol{\lambda} + \boldsymbol{d}_l\right) - L(x, \boldsymbol{\lambda}) \right] \nu_l(dz) \right\} \quad (3.10)$$

At time $t \geq 0$, given wealth X_t and intensity vector $\boldsymbol{\lambda}_t$, the optimal consumption choice is therefore $C_t^* = C^*(X_t, \boldsymbol{\lambda}_t)$ where

$$C^*(x, \boldsymbol{\lambda}) \equiv [U']^{-1} (\partial L(x, \boldsymbol{\lambda}) / \partial x). \quad (3.11)$$

In order to determine the optimal portfolio weights, wealth and value function, we need to be more specific about the utility function U.

4 Log-Utility Investors

There are three classic utility functions for which one may hope to make further analytical progress, namely the log investor whose utility of consumption is the logarithm function, the power investor and the exponential investor. Collectively, these examples are known as HARA utilities. The optimal investment problem for various simpler types of market dynamics with these utilities lead to separable forms for the value functions. As we shall now show in this and Section 7, this separation property extends to the Hawkes-diffusion model, albeit with some extra twists. In this section, we concentrate on the log-investor, for whom we are able to prove strong results on the existence and uniqueness of the optimal strategy.

4.1 Optimal Investment with Log-Utility

We now specialize the problem to that faced by an investor with logarithmic utility, $U(x) = \log(x)$. To start, we look for a candidate solution to (3.8) in the form

$$L(x, \boldsymbol{\lambda}) = f(\boldsymbol{\lambda}) + M^{-1} \log(x) \quad (4.1)$$

for some positive function f and constant M. Then

$$\frac{\partial L(x,\boldsymbol{\lambda})}{\partial \lambda_l} = f_{\lambda_l}(\boldsymbol{\lambda}), \quad \frac{\partial L(x,\boldsymbol{\lambda})}{\partial x} = M^{-1}x^{-1},$$
$$\frac{\partial^2 L(x,\boldsymbol{\lambda})}{\partial x^2} = -M^{-1}x^{-2}. \tag{4.2}$$

and the optimal policy for the portfolio weights at time $t \geq 0$ is $\boldsymbol{\omega}_t^* = \boldsymbol{\omega}^*(\boldsymbol{\lambda}_t)$ where

$$\boldsymbol{\omega}^*(\boldsymbol{\lambda}) = \operatorname{argmin}_{\boldsymbol{\omega}} K_l(\boldsymbol{\omega},\boldsymbol{\lambda})$$
$$K_l(\boldsymbol{\omega},\boldsymbol{\lambda}) \equiv \left\{ -\boldsymbol{\omega}'\boldsymbol{R} + \frac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} - \sum_{l=1}^{m} \lambda_l \int \log\left(1 + (\boldsymbol{\omega}'\boldsymbol{J})_l z\right) \nu_l(dz) \right\}. \tag{4.3}$$

We note that the convexity of K_l implies this minimization has a unique solution $\boldsymbol{\omega}^*(\boldsymbol{\lambda}_t)$ for any $\boldsymbol{\lambda} \in \mathbb{R}_+^n$. As for the optimal consumption policy, from equation (3.11), and the facts that $[U']^{-1}(y) = y^{-1}$ and $\partial L(x,\boldsymbol{\lambda})/\partial x = M^{-1}x^{-1}$, we obtain $M = \beta$ and

$$C_t^* = \beta X_t \tag{4.4}$$

Next, we substitute the optimal $(C^*, \boldsymbol{\omega}^*)$ into (3.8) and determine that the function f must solve

$$[\mathcal{A}f](\boldsymbol{\lambda}) - \beta f(\boldsymbol{\lambda}) = F(\boldsymbol{\lambda}), \quad \boldsymbol{\lambda} \in \mathbb{R}_+^n \tag{4.5}$$

where the Markov generator $\mathcal{A}$ for the process $\boldsymbol{\lambda}_t$ is given by

$$[\mathcal{A}f](\boldsymbol{\lambda}) = \sum_{l=1}^{m} \left(\alpha_l (\lambda_{l,\infty} - \lambda_l) f_{\lambda_l}(\boldsymbol{\lambda}) + \lambda_l \left[f(\boldsymbol{\lambda} + \boldsymbol{d}_l) - f(\boldsymbol{\lambda}) \right] \right) \tag{4.6}$$

and the nonhomogeneous term is

$$F(\boldsymbol{\lambda}) = 1 - \frac{r}{\beta} - \log\beta + \beta^{-1} K_l(\boldsymbol{\omega}^*(\boldsymbol{\lambda}), \boldsymbol{\lambda}).$$

The following lemma gives a computable formula for the smooth solution of (4.5).

Lemma 1. *The function $f : \mathbb{R}_+^m \to \mathbb{R}_+$ defined by (4.5) is differentiable and given by the absolutely convergent integral*

$$f(\boldsymbol{\lambda}) = \int_0^\infty e^{-\beta s} \mathbb{E}_{0,\boldsymbol{\lambda}} \left[F(\boldsymbol{\lambda}_s) \right] ds \ . \tag{4.7}$$

Proof. By the ergodic property, we know both that $\mathbb{E}_{0,\boldsymbol{\lambda}}\left[F(\boldsymbol{\lambda}_s)\right] \to \mathbb{E}_{0,\boldsymbol{\lambda}}\left[F(\boldsymbol{\lambda}_\infty)\right]$ and $\frac{\partial}{\partial \lambda_\ell}\mathbb{E}_{0,\boldsymbol{\lambda}}\left[F(\boldsymbol{\lambda}_s)\right] \to 0$ as $s \to \infty$. One can also verify directly that there is a constant $\tilde{M} > 0$ such that

$$|F(\lambda)| \leq \tilde{M}(1 + \|\lambda\|^2) \ . \tag{4.8}$$

From these facts follows the absolute convergence both of the integral in (4.7) and the integral

$$\frac{\partial f}{\partial \lambda_\ell} = \int_0^\infty e^{-\beta s} \frac{\partial}{\partial \lambda_\ell} \mathbb{E}_{0,\boldsymbol{\lambda}}\left[F(\boldsymbol{\lambda}_s)\right] ds \ .$$

Since the right hand side of (4.7) is differentiable, the Feynman-Kac formula implies it satisfies (4.5). □

4.2 A Verification Result for the Log Investor

The following verification theorem follows the logic outlined in Section III.9 of Fleming and Soner (2006) and ensures that the above argument correctly characterizes both the optimal strategy and the associated value function. A more general verification result for log investors can be found in Goll and Kallsen (2000).

Theorem 1. *Consider the optimal problem (3.6) for the log investor with impatience parameter $\beta > 0$, investing in the asset price model defined by (3.1), (3.2), (3.3) and satisfying the Stationarity Assumption that the matrix $\Gamma = (\alpha_j \delta_{ij} - d_{ij})$ is positive.*

1. *The candidate solution $\tilde{V}(x, \boldsymbol{\lambda}, t) = e^{-\beta t}\left[f(\boldsymbol{\lambda}) + \beta^{-1}\log(x)\right]$ is a classical (i.e. differentiable) solution of the HJB equation (3.7).*

2. *For all initial conditions $x > 0$, $\boldsymbol{\lambda} \in \mathbb{R}^n_+$, the pair of processes $(\boldsymbol{\omega}^*_t, C^*_t), t \geq 0$ defined by (4.3) and (4.4) is an admissible policy, in the sense that they are progressively measurable and the process X^*_t remains finite and positive (t, ω) almost surely and solves the appropriate SDE.*

3. *Let $\mathcal{C}$ denote the class of admissible policies $(\boldsymbol{\omega}_t, C_t), t \geq 0$ that satisfy*

$$\lim_{t\to\infty} \mathbb{E}_{x,\boldsymbol{\lambda}}\left[\tilde{V}\left(X_t, \boldsymbol{\lambda}_t, t\right)\right] \geq 0 \tag{4.9}$$

For any $(\boldsymbol{\omega}, C) \in \mathcal{C}$,

$$\mathbb{E}_{x,\boldsymbol{\lambda}}\left[\int_0^\infty e^{-\beta s}U(C_s)ds\right] \leq \tilde{V}(x, \boldsymbol{\lambda}, 0). \tag{4.10}$$

4. *Let* V_{AS} *denote the value function*

$$V_{AS}\left(x, \boldsymbol{\lambda}\right) = \max_{(C,\boldsymbol{\omega})\in\mathcal{C}} \mathbb{E}_{x,\boldsymbol{\lambda}}\left[\int_0^\infty e^{-\beta s}U(C_s)ds\right] . \tag{4.11}$$

The optimal policy $(\boldsymbol{\omega}_t^*, C_t^*), t \geq 0$ *satisfies*

$$\lim_{t\to\infty} \mathbb{E}_{x,\boldsymbol{\lambda}}\left[\tilde{V}\left(X_t^*, \boldsymbol{\lambda}_t, t\right)\right] = 0 \tag{4.12}$$

and the equality

$$\mathbb{E}_{x,\boldsymbol{\lambda}}\left[\int_0^\infty e^{-\beta s}U(C_s^*)ds\right] = \tilde{V}(x, \boldsymbol{\lambda}, 0). \tag{4.13}$$

Hence $\tilde{V}(x, \boldsymbol{\lambda}, 0) = V_{AS}(x, \boldsymbol{\lambda})$ *and* $(\boldsymbol{\omega}^*, C^*)$ *is the optimal portfolio policy in the class* $\mathcal{C}$.

Proof. That $\tilde{V}$ is a classical solution of (3.7) follows from Lemma 1 which shows that $f(\lambda)$ is a differentiable solution of (4.5). That $(\boldsymbol{\omega}_t^*, C_t^*), t \geq 0$ is admissible follows by general considerations.

Suppose $(\boldsymbol{\omega}, C) \in \mathcal{C}$ and consider the process $\xi_s = \tilde{V}(X_s, \boldsymbol{\lambda}_s, s)$. For any $t > 0$, the Dynkin formula implies that

$$\begin{aligned}
\mathbb{E}_{x,\boldsymbol{\lambda}}\left[\xi_t\right] = \xi_0 + \int_0^t \mathbb{E}_{x,\boldsymbol{\lambda}}\Bigg[&-\beta\tilde{V}(X_s, \boldsymbol{\lambda}_s, s) + \sum_{l=1}^m \alpha_l\left(\lambda_{l,\infty} - \lambda_{l,s}\right)\frac{\partial\tilde{V}\left(X_s, \boldsymbol{\lambda}_s, s\right)}{\partial\lambda_l} \\
&+ \frac{\partial\tilde{V}\left(X_s, \boldsymbol{\lambda}_s, s\right)}{\partial x}\left(rX_s + \boldsymbol{\omega}_s'\boldsymbol{R}X_s - C_s\right) + \frac{1}{2}\frac{\partial^2\tilde{V}\left(X_s, \boldsymbol{\lambda}_s, s\right)}{\partial x^2}\boldsymbol{\omega}_s'\boldsymbol{\Sigma}\boldsymbol{\omega}_s X_s^2 \\
&+ \sum_{l=1}^m \lambda_{l,s-}\int\left[\tilde{V}\left(X_s + \left(\boldsymbol{\omega}_s'\boldsymbol{J}\right)_l zX_s, \boldsymbol{\lambda}_{s-} + \boldsymbol{d}_l, s\right) - \tilde{V}\left(X_s, \boldsymbol{\lambda}_{s-}, s\right)\right]\nu_l\left(dz\right)\Bigg]ds
\end{aligned} \tag{4.14}$$

Using the fact that $\tilde{V}$ solves the HJB equation (3.7) leads in the usual way to the inequality

$$\mathbb{E}_{x,\boldsymbol{\lambda}}\left[\xi_t\right] \leq \xi_0 - \mathbb{E}_{x,\boldsymbol{\lambda}}\left[\int_0^t e^{-\beta s}U(C_s)ds\right] \tag{4.15}$$

Finally, one can use the transversality condition (4.9) to take the limit $t \to \infty$ and obtain the desired result

$$\mathbb{E}_{x,\boldsymbol{\lambda}}\left[\int_0^\infty e^{-\beta s}U(C_s)ds\right] \leq \xi_0 = \tilde{V}(x, \lambda, 0).$$

The strategy $(\boldsymbol{\omega}_t^*, C_t^*), t \geq 0$ maximizes (3.7) for almost every (t, ω), which means (4.15) holds as an equality for every $t > 0$. Finally, one needs to verify (4.12) in order to conclude that $\xi_0 = \tilde{V}_0 = V_{AS}$. First, by the ergodic property (2.5) one has

$$\lim_{t\to\infty} \mathbb{E}_{x,\boldsymbol{\lambda}}\left[e^{-\beta t}|f(\boldsymbol{\lambda}_t)|\right] = 0$$

Next, by plugging in the optimal consumption $C^* = \beta X^*$ one then finds one can solve the problem of optimizing the expected utility of terminal wealth over the interval $[0, t]$ with the adjusted interest rate $\hat{r} = r - \beta$ to show that

$$\mathbb{E}_{x,\boldsymbol{\lambda}}\left[e^{-\beta t}\log(X_t^*)\right] = [te^{-\beta t}] \cdot \frac{1}{t}\int_0^t \mathbb{E}_{x,\boldsymbol{\lambda}}[\hat{r} + K_l(\boldsymbol{\omega}^*(\boldsymbol{\lambda}_{\boldsymbol{s}}), \boldsymbol{\lambda}_{\boldsymbol{s}})]ds$$

From the bounds (4.8) on the functions F, K this can be seen to converge to 0 as $t \to \infty$, again by the ergodic property (2.5). □

4.3 Properties of the Solution

The log investor is often described as "myopic" because she acts at each moment in time as if the dynamical variables are in fact static parameters. Thus we should not be surprised that the optimal consumption and portfolio weights at any time given by (4.4) and (4.3) are independent of the coefficients of the SDEs driving the asset returns. In particular, at any time t, the policy $(\boldsymbol{\omega}_t^*, C_t^*, \boldsymbol{\lambda}_t)$ is precisely the same as the policy when the jump frequency is treated as a constant $\lambda = \lambda_t$, although that "constant" is changed at each instant.

A second observation is that the policy $(\boldsymbol{\omega}_t^*, C_t^*, \boldsymbol{\lambda}_t), t \geq 0$ does not require knowledge of the function $f(\lambda)$. However, determining the function f requires solving the non-homogeneous equation (4.5), or equivalently evaluating the integral in (4.7). For this, we need some further structure on the problem, which we now add in order to derive the complete form of the solution.

5 Additional Structure on the Diffusive and Jump Risks

To find interesting examples of closed form optimal portfolio solutions, it is convenient to model the variance-covariance matrix Σ of the diffusive part of asset returns in such a way that its inverse is explicit. For this purpose, we adopt a modelling approach that is common in asset pricing, namely to assume a factor structure. That is, we specify a block-structure for Σ consisting of k blocks of dimension m, with $n = mk$:

$$\underset{n\times n}{\boldsymbol{\Sigma}} = \boldsymbol{\sigma}\boldsymbol{\sigma}' = \begin{pmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \cdots \\ \Sigma_{2,1} & \ddots & \Sigma_{2,m} \\ \cdots & \Sigma_{m,m-1} & \Sigma_{m,m} \end{pmatrix} \tag{5.1}$$

with diagonal (or within-asset class) blocks

$$\underset{k\times k}{\boldsymbol{\Sigma}_{l,l}} = \upsilon_l^2 \begin{pmatrix} 1 & \rho_{l,l} & \cdots \\ \rho_{l,l} & \ddots & \rho_{l,l} \\ \cdots & \rho_{l,l} & 1 \end{pmatrix} \tag{5.2}$$

and off-diagonal (or across-asset class) blocks

$$\underset{k\times k}{\boldsymbol{\Sigma}_{l,s}} = 0 \tag{5.3}$$

where $1 > \rho_{l,l} > -1/(k-1)$.

The spectral decomposition of the Σ matrix is

$$\boldsymbol{\Sigma} \quad = \quad \underbrace{\sum_{l=1}^{m} \kappa_{1l} \frac{1}{k} \mathbf{1}_l \mathbf{1}_l'}_{=\ \bar{\Sigma}} \quad + \quad \underbrace{\sum_{l=1}^{m} \kappa_{2l} \left(\boldsymbol{M}_l - \frac{1}{k} \mathbf{1}_l \mathbf{1}_l' \right)}_{=\ \Sigma^{\perp}} \tag{5.4}$$

where

$$\kappa_{1l} = \upsilon_l^2 \left(1 + (k-1)\, \rho_{l,l}\right) \tag{5.5}$$

$$\kappa_{2l} = \upsilon_l^2 \left(1 - \rho_{l,l}\right) \tag{5.6}$$

are the $2m$ distinct eigenvalues of Σ. The multiplicity of each κ_{1l} is 1, and the multiplicity of each κ_{2l} is $k-1$. The eigenvector for κ_{1l} is $\mathbf{1}_l$, the $n-$vector with ones placed in the k rows corresponding to the $l-$block and zeros everywhere else, that is

$$\mathbf{1}_l = [0, \ldots, 0, \underbrace{1, \ldots, 1}_{\text{asset class } l}, 0, \ldots, 0]', \tag{5.7}$$

where the first 1 is located in the $k\,(l-1)+1$ coordinate. $\boldsymbol{M}_l$ is an $n \times n$ block diagonal matrix with a $k \times k$ identity matrix $\boldsymbol{I}_k$ placed in the $l-$block and zeros everywhere else:

$$\underset{n\times n}{\boldsymbol{M}_l} \quad = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \boldsymbol{I}_k & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \tag{5.8}$$

Corresponding to the above spectral structure, we have the orthogonal decomposition $\mathbb{R}^n = \bar{V} \oplus V^\perp$ where $\bar{V}$ is the span of $\{\mathbf{1}_l\}_{l=1,..,m}$ and $V^\perp$ is the orthogonal space.

As for the vector $\boldsymbol{J}$ of jump amplification coefficients, we assume that

$$\underset{n\times m}{\boldsymbol{J}} = [\boldsymbol{J}_1, ..., \boldsymbol{J}_m] = \begin{pmatrix} J_{1,1} & \cdots & J_{1,m} \\ \vdots & \ddots & \vdots \\ J_{n,1} & \cdots & J_{n,m} \end{pmatrix} \tag{5.9}$$

where

$$\boldsymbol{J}_l = j_l \mathbf{1}_l = [\underbrace{0, \ldots, 0}_{\text{asset class 1}}, \ldots, \underbrace{j_l, \ldots, j_l}_{\text{asset class } l}, \cdots \underbrace{, 0, \ldots, 0}_{\text{asset class } m}]' \tag{5.10}$$

for $l = 1, ..., m$. This structure means that assets within a given class l have the same response to the arrival of a jump, i.e., to a change in $\boldsymbol{N}_t$. But the proportional response j_l of assets of different classes to the arrival of a jump can be different ($j_l \neq j_h$ for $l \neq h$).

Finally, we assume that the vector of expected excess returns has the form

$$\boldsymbol{R} = \sum_{l=1}^{m} \bar{R}_l \mathbf{1}_l + \boldsymbol{R}^\perp = \bar{\boldsymbol{R}} + \boldsymbol{R}^\perp. \tag{5.11}$$

Here, we allow the expected excess returns to differ both within and across asset classes, by allowing $\boldsymbol{R}^\perp \neq \mathbf{0}$. The components of $\boldsymbol{R}$ play the role of the assets' alphas. The general $\boldsymbol{R}^\perp$ is orthogonal to each $\mathbf{1}_l$ and has the form

$$\boldsymbol{R}^\perp = [\boldsymbol{R}_1^{\perp\prime}, ..., \boldsymbol{R}_m^{\perp\prime}]'$$

where each of the k-vectors $\boldsymbol{R}_l^\perp$ is orthogonal to the k-vector $\mathbf{1}$.

5.1 Closed-Form Optimal Portfolio Solution

We now look for a vector of optimal portfolio weights $\boldsymbol{\omega}$**,** and it is convenient to look for it in the form of its decomposition using the same basis as above,

$$\boldsymbol{\omega} = \sum_{l=1}^{m} \bar{\omega}_l \mathbf{1}_l + \boldsymbol{\omega}^\perp = \bar{\boldsymbol{\omega}} + \boldsymbol{\omega}^\perp \tag{5.12}$$

where $\boldsymbol{\omega}^{\perp} = [\boldsymbol{\omega}_1^{\perp\prime}, ..., \boldsymbol{\omega}_l^{\perp\prime}]'$. The objective function $K_2(\boldsymbol{\omega})$ in (4.3) to be minimized reduces to

$$
\begin{aligned}
K_2(\boldsymbol{\omega}) &= \left\{ -\boldsymbol{\omega}'\boldsymbol{R} + \frac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} - \sum_{l=1}^{n} \lambda_l \int \log\left(1 + (\boldsymbol{\omega}'\boldsymbol{J})_l\, z\right) \nu_l\,(dz) \right\} \\
&= -\boldsymbol{\omega}^{\perp\prime}\boldsymbol{R}^{\perp} - \sum\nolimits_{l=1}^{m} \bar{\omega}_l \bar{R}_l \mathbf{1}_l'\mathbf{1}_l \\
&+ \frac{1}{2}\boldsymbol{\omega}^{\perp\prime}\Sigma^{\perp}\boldsymbol{\omega}^{\perp} + \frac{1}{2}\sum_{l=1}^{m} \bar{\omega}_l^2 \kappa_{1l} \frac{1}{k} \mathbf{1}_l'\mathbf{1}_l\mathbf{1}_l'\mathbf{1}_l \\
&- \sum_{l=1}^{n} \lambda_l \int \log\left(1 + k\bar{\omega}_l j_l z\right) \nu_l\,(dz)\,.
\end{aligned}
$$

The minimization problem then separates as

$$
\left(\boldsymbol{\omega}^{\perp *}, \bar{\boldsymbol{\omega}}^*\right) = \operatorname{argmin}_{\left\{\boldsymbol{\omega}^{\perp}, \bar{\boldsymbol{\omega}}\right\}} \left\{ g^{\perp}(\boldsymbol{\omega}^{\perp}) + \bar{g}(\bar{\omega}) \right\} \tag{5.13}
$$

where

$$
g^{\perp}(\boldsymbol{\omega}^{\perp}) = -\boldsymbol{\omega}^{\perp\prime}\boldsymbol{R}^{\perp} + \frac{1}{2}\boldsymbol{\omega}^{\perp\prime}\Sigma^{\perp}\boldsymbol{\omega}^{\perp} \tag{5.14}
$$

$$
\bar{g}(\bar{\boldsymbol{\omega}}) = -k\sum_{l=1}^{m} \bar{\omega}_l \bar{R}_l + \frac{1}{2}k\sum_{l=1}^{m} \bar{\omega}_l^2 \kappa_{1l} - \sum_{l=1}^{m} \lambda_{l,t} \int \log\left(1 + k\bar{\omega}_l j_l z\right) \nu_l\,(dz)\,. \tag{5.15}
$$

For the first part, minimizing $g^{\perp}(\boldsymbol{\omega}^{\perp})$, the structure of $\Sigma^{\perp}$ implies that

$$
\begin{aligned}
g^{\perp}(\boldsymbol{\omega}^{\perp}) &= -\boldsymbol{\omega}^{\perp\prime}\boldsymbol{R}^{\perp} + \frac{1}{2}\sum\nolimits_{l=1}^{m} \kappa_{2l}\boldsymbol{\omega}^{\perp\prime}\left(\boldsymbol{M}_l - \frac{1}{k}\mathbf{1}_l\mathbf{1}_l'\right)\boldsymbol{\omega}^{\perp} \\
&= -\boldsymbol{\omega}^{\perp\prime}\boldsymbol{R}^{\perp} + \frac{1}{2}\sum\nolimits_{l=1}^{m} \kappa_{2l}\boldsymbol{\omega}^{\perp\prime}\boldsymbol{M}_l\boldsymbol{\omega}^{\perp} \\
&= -\sum\nolimits_{l=1}^{m} \boldsymbol{\omega}_l^{\perp\prime}\boldsymbol{R}_l^{\perp} + \frac{1}{2}\sum\nolimits_{l=1}^{m} \kappa_{2l}\boldsymbol{\omega}_l^{\perp\prime}\boldsymbol{\omega}_l^{\perp}
\end{aligned}
$$

and therefore the optimal solution $\boldsymbol{\omega}^{\perp *}$ has blocks

$$
\boldsymbol{\omega}_l^{\perp *} = \frac{1}{\kappa_{2l}}\boldsymbol{R}_l^{\perp} \tag{5.16}
$$

for $l = 1, \ldots, m$. This part of the solution depends only on the diffusive characteristics (expected returns and variance-covariance) of the asset returns.

The problem of minimizing $\bar{g}(\bar{\boldsymbol{\omega}})$ separates itself into m separate minimization problems. With the change of variable

$$
\varpi_{ln} = k\bar{\omega}_l \tag{5.17}
$$

we see that

$$\varpi_{ln}^{*}=\mathrm{argmin}_{\{\varpi_{ln}\}}\left\{-\varpi_{ln}R_{l}+\frac{1}{2}\varpi_{ln}^{2}\kappa_{1l}/k-\lambda_{l,t}\int\log\left(1+\varpi_{ln}j_{l}z\right)\nu_{l}\left(dz\right)\right\} \tag{5.18}$$

The convexity of the objective function implies the existence of the minimizer. We can then determine ϖ_{ln}^{*} in closed form.

Two cases are explicitly solvable. We first consider the case where the jump size is deterministic. In this situation, $\nu_l(dz)=\delta\left(z=\bar{z}_l\right)$ and the objective functions become:

$$\begin{aligned} f_{n}\left(\varpi\right)=&-\sum\nolimits_{l=1}^{m}\varpi_{ln}\bar{R}_{l}+\frac{1}{2}\sum\nolimits_{l=1}^{m}\varpi_{ln}^{2}\kappa_{1l}/k\\ &-\sum_{l=1}^{m}\lambda_{l,t}\log\left(1+\varpi_{ln}j_{l}\bar{z}_{l}\right). \end{aligned} \tag{5.19}$$

The first order conditions for the asset allocation parameters ϖ_{ln} are given by

$$-\bar{R}_{l}+\varpi_{ln}\kappa_{1l}/k-\lambda_{l,t}j_{l}\bar{z}_{l}\left(1+\varpi_{ln}j_{l}\bar{z}_{l}\right)^{-1}=0\text{ for }l=1,\ldots,m. \tag{5.20}$$

These first order conditions form a system of m independent quadratic equations. Each separate equation (5.20) admit a unique solution ϖ_{ln} satisfying the solvency constraint $\varpi_{ln}j_l\bar{z}_l>-1$. These are solvable in closed form:

$$\bar{\omega}_{l}^{*}=\frac{\varpi_{ln}^{*}}{k}=\frac{-\kappa_{1l}/k+j_{l}\bar{z}_{l}\bar{R}_{l}+\sqrt{\left(j_{l}\bar{z}_{l}\bar{R}_{l}+\kappa_{1l}/k\right)^{2}+4\lambda_{l,t}j_{l}^{2}\bar{z}_{l}^{2}\kappa_{1l}/k}}{2j_{l}\bar{z}_{l}\kappa_{1l}/k^{2}}. \tag{5.21}$$

A second case that is solvable in closed form is one where each jump term $Z_{l,t}$ has a binomial distribution (u_l with probability p_l or d_l with probability $1-p_l$), since the corresponding first-order conditions are cubic. The first order conditions are obtained by differentiation with respect to ϖ_{ln} of the objective function stated in (5.18), namely:

$$-\bar{R}_{l}+\varpi_{ln}\kappa_{1l}/k-\lambda_{l,t}j_{l}\int\left(1+\varpi_{ln}j_{l}z\right)^{-1}z\nu_{l}\left(dz\right)=0\text{ for }l=1,\ldots,m. \tag{5.22}$$

For binomially-distributed jumps, the conditions reduce to

$$-R_{l}+\varpi_{ln}\kappa_{1l}/k-\lambda_{l,t}j_{l}\left(p_{l}u_{l}\left(1+\varpi_{ln}j_{l}u_{l}\right)^{-1}+\left(1-p_{l}\right)d_{l}\left(1+\varpi_{ln}j_{l}d_{l}\right)^{-1}\right)=0 \tag{5.23}$$

which produce a cubic polynomial equation in ϖ_{ln}, again explicitly solvable, separately for each for $l=1,\ldots,m$.

6 Consequences for the Optimal Portfolio Allocation

We now investigate in more detail the consequences of the explicit portfolio weight formulae for an optimal asset allocation. The first element we note from (5.21) is the fact that the optimal portfolio is time-varying, since its composition changes with the jump intensities λ_{lt}.

In the case of purely diffusive risk, the optimal portfolio weights reduce to the classical Merton formula

$$\boldsymbol{\omega}^* = \boldsymbol{\Sigma}^{-1}\boldsymbol{R} \tag{6.1}$$

or, replacing $\boldsymbol{\Sigma}^{-1}$ by its explicit expression

$$\begin{cases} \boldsymbol{\omega}_l^{\perp *} = \frac{1}{\kappa_{2l}}\boldsymbol{R}_l^{\perp} \\ \bar{\omega}_l^* = \frac{1}{\kappa_{1l}}\bar{R}_l \end{cases} \tag{6.2}$$

for $l = 1, .., m$. As is well known, the solution in this case is constant.

In the case where Poissonian jumps are added to the model, the solution specializes to (5.21) but with $\lambda_{l,t}$ replaced by the constant Poissonian jump intensity. As in the purely-diffusive case, the solution becomes constant. A solution with similar qualitative features would be obtained in the mutually exciting case if one replaced each stochastic jump intensity by its unconditional expected value, although given that the portfolio weights are nonlinear functions of $\lambda_{l,t}$ these would not be the unconditional expected values of the portfolio weights.

Let us now return to the full solution in the mutually exciting case. A univariate model captures only part of the mutual excitation phenomenon: with a single asset, only time series self-excitation can take place. In order to investigate the full potential impact of mutual excitation on optimal portfolio holdings, we now specialize the results above to a two-asset model where both time series and cross-sectional excitation can arise. Assets 1 and 2 can excite each other, not necessarily in a symmetric fashion, depending on the 2×2 matrix of mutually exciting intensities with coefficients d_{ij}, $i, j = 1, 2$. The formulae above specialize with $n = 2$, $k = 1$ and $m = 2$, in which case $\kappa_{1l} = v_l^2$ for $l = 1, 2$ and hence:

$$\begin{cases} \boldsymbol{\omega}_l^{\perp *} = \frac{1}{\kappa_{2l}}\boldsymbol{R}_l^{\perp} \\ \bar{\omega}_l^* = \frac{-v_l^2 + j_l\bar{z}_l R_l + \sqrt{\left(j_l\bar{z}_l R_l + v_l^2\right)^2 + 4\lambda_{l,t} j_l^2 \bar{z}_l^2 v_l^2}}{2 j_l \bar{z}_l v_l^2} \end{cases} \tag{6.3}$$

Consider the change in the optimal portfolio allocation of an investor who observes a first shock, say to asset 1. For concreteness, let us return to the two-asset class scenario illustrated in Figure 3. In a Poissonian jump model, observing the first shock at time T_1 does not change the investor's optimal portfolio: since jumps' future arrivals are independent of past jumps, there is nothing to do going forward other than to absorb the losses from the first jump. In the mutually exciting model, however, the occurrence of the first jump at time T_1 self-excites the jump intensity $\lambda_{1,t}$ for $t > T_1$. This increase in $\lambda_{1,t}$ translates, if $\bar{z}_1 < 0$, into a reduced asset allocation to asset 1. Moreover, these jumps have a contagious effect on S_2 since a jump in asset 1 cross-excites the jump intensity $\lambda_{2,t}$ of asset 2. If $\bar{z}_2 < 0$, then the optimal policy is to reduce the asset allocation to asset 2. Note that the reduction to the position in both risky assets occurs immediately after T_1, without waiting for future jumps.

For the same sample paths as in Figure 3, Figure 4 shows the optimal portfolio weights. This is a flight to quality, in the sense that the occurrence of a single jump in asset 1 causes the investor to flee both risky assets (or all of them in the general n case) for the safety of the riskless asset. This phenomenon is well documented as an empirical reality in practical situations; we believe that this is the first portfolio choice model to actually capture it in a theoretical setting.

Increases in the jump intensities raise the probability of observing another jump in S_1 at the future time T_2. This, in turn, raises the probability of seeing a jump in S_2 at time T_3. Later on, at time T_4, the jump in S_2 raises the probability of seeing a jump in S_1 at some future time T_5, and so on. Mean reversion in the jump intensities at respective rates α_1 and α_2 counteracts these successive increases, keeping the intensities non-explosive (stationary, in fact). The optimal portfolio policy reacts to each change in jump intensity accordingly: increases lead to reduced asset allocation, decreases to increased asset allocation. Inevitably, this analysis is conducted in a partial equilibrium framework: it assumes among other things that expected returns do not change as jump intensities change, or at least not sufficiently to reverse the result.

Another interesting empirical phenomenon that can be revisited in light of these optimal portfolio policies is home bias. Home bias refers to the observed tendency of most investors' portfolios to be insufficiently diversified internationally. In the model, the benefits from diversi-

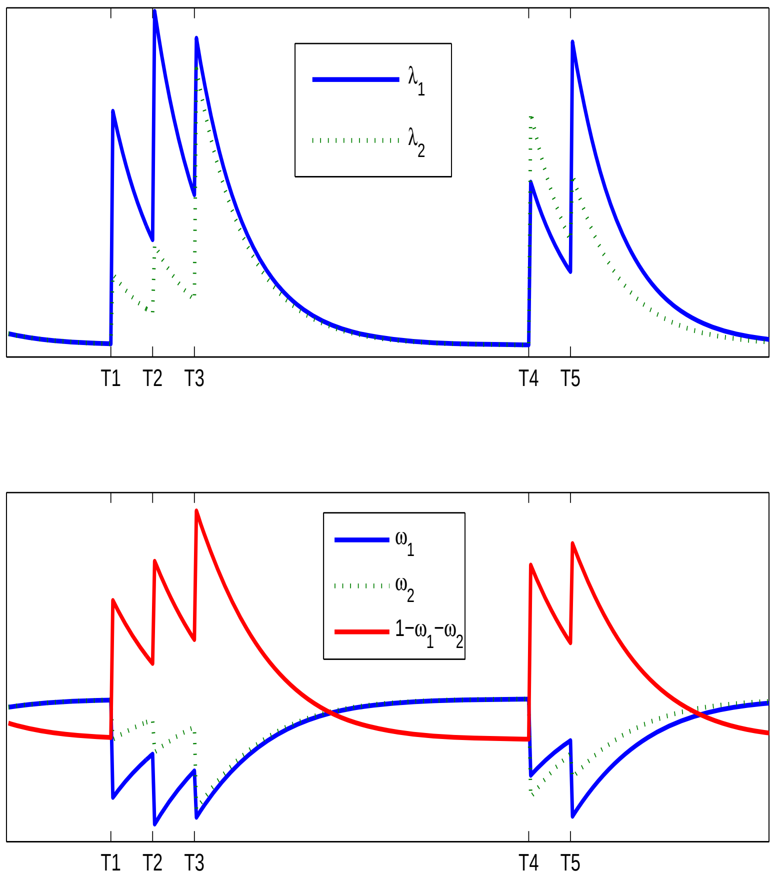

Figure 4: Mutual excitation in a two asset-class world: Jump intensities (top panel) and optimal portfolio weights (bottom panel).

fication are much less valuable than in the standard diffusive model since international assets do not protect as much against jumps in domestic assets in the presence of cross-sectional mutual excitation.

Finally, one phenomenon that is often documented in financial crises is the large increase in correlations between asset classes, with all of them increasing towards 1. In the context of the model, empirical correlations measured over a period where mutual excitation occurs will indeed be close to 1 as long as the jumps that result from mutual excitation are of the same sign (say both $\bar{z}_1 < 0$ and $\bar{z}_2 < 0$), at least on average. In such periods, the jumps' contribution to the observed correlation trumps the continuous contribution.

7 Other HARA Investors

The cases of power and exponential utility also lead to candidate value functions and optimal portfolios in separable form. Due to the complexity of the underlying HJB conditions, the relevant verification result requires a lengthy, and perhaps uninformative, analysis that we have not yet completed. Nonetheless, as we now show, the candidate solutions can be characterized in terms of a fixed point problem that in principle can be solved numerically.

7.1 Power Utility

In this section, we consider the power investor with $U(c) = c^\gamma/\gamma$, $\gamma \in (-\infty, 0) \cup (0, 1)$. The analysis now consists in verifying the consistency of the following form for the solution to (3.8) in the form

$$L(x, \boldsymbol{\lambda}) = x^\gamma g(\boldsymbol{\lambda})/\gamma$$

for some positive function g. Substitution into (3.8) leads to

$$0 = \max_{\{C,\boldsymbol{\omega}\}} \{U(C) - \beta L \tag{7.1}$$

$$+ \frac{L}{g} \sum_{l=1}^{m} \left[\alpha_l \left(\lambda_{l,\infty} - \lambda_l \right) \frac{\partial g\left(\boldsymbol{\lambda} \right)}{\partial \lambda_l} + \lambda_l \left(g(\boldsymbol{\lambda} + \boldsymbol{d}_l) - g(\boldsymbol{\lambda}) \right) \right]$$

$$+ \gamma L \left(r + \boldsymbol{\omega}' \boldsymbol{R} - C/x \right) + \frac{\gamma(\gamma - 1)L}{2} \boldsymbol{\omega}' \boldsymbol{\Sigma} \boldsymbol{\omega} \tag{7.2}$$

$$+ L \sum_{l=1}^{m} \lambda_l g(\boldsymbol{\lambda} + \boldsymbol{d}_l)/g(\boldsymbol{\lambda}) \int \left[(1 + \boldsymbol{\omega}' \boldsymbol{J} z)^{\gamma} - 1 \right] \nu_l \left(dz \right) \Big\}$$

The optimal policy for the portfolio weight at time $t \geq 0$ is $\boldsymbol{\omega}_t^* = \boldsymbol{\omega}^*(\boldsymbol{h}(\boldsymbol{\lambda}_t))$ where

$$\boldsymbol{\omega}^*(\boldsymbol{\lambda}) = \begin{cases} \arg\min_{\boldsymbol{\omega}} K^{\gamma}(\boldsymbol{\omega}, \boldsymbol{\lambda}) & \gamma > 0 \\ \arg\max_{\boldsymbol{\omega}} K^{\gamma}(\boldsymbol{\omega}, \boldsymbol{\lambda}) & \gamma < 0 \end{cases} \tag{7.3}$$

$$h_l(\boldsymbol{\lambda}) = \lambda_l g(\boldsymbol{\lambda} + \boldsymbol{d}_l)/g(\boldsymbol{\lambda}) \tag{7.4}$$

$$K^{\gamma}(\boldsymbol{\omega}, \boldsymbol{\lambda}) \equiv -\gamma \boldsymbol{\omega}' R - \frac{\gamma(\gamma - 1)}{2} \boldsymbol{\omega}' \Sigma \boldsymbol{\omega} - \sum_{l=1}^{m} \lambda_{l,t} \int_{(0,1]} \left[(1 + \boldsymbol{\omega}' \boldsymbol{J} z)^{\gamma} - 1 \right] \nu_l \left(dz \right). \tag{7.5}$$

Solving the first order condition for C leads to the optimal consumption $C^* = x \left(\gamma g(\boldsymbol{\lambda}) \right)^{1/(\gamma - 1)}$.

Finally, g is characterized by the implicit equation

$$[\mathcal{A}g](\boldsymbol{\lambda}) - (\beta - r\gamma) g\left(\boldsymbol{\lambda} \right) = G(\boldsymbol{\lambda}; g) \tag{7.6}$$

$$G(\boldsymbol{\lambda}; g) = g\left(\boldsymbol{\lambda} \right) K^{\gamma}\left(\boldsymbol{\omega}^*(\boldsymbol{h}(\boldsymbol{\lambda})), \boldsymbol{h}(\boldsymbol{\lambda}) \right) + (1 - \gamma)(\gamma g(\boldsymbol{\lambda}))^{\gamma/(\gamma - 1)}, \;\; \boldsymbol{\lambda} \in \mathbb{R}_+^n. \tag{7.7}$$

The Markov generator $\mathcal{A}$ is again given by (4.6). We use the Feynman-Kac formula to write the solution g as a fixed point of an infinite dimensional nonlinear mapping:

$$g = \mathcal{G}(g) \tag{7.8}$$

$$\left(\mathcal{G}(g) \right) \left(\boldsymbol{\lambda} \right) := \int_0^{\infty} e^{-(\beta - r\gamma)s} \mathbb{E}_{0,\boldsymbol{\lambda}} \left[G(\boldsymbol{\lambda}_s; g) \right] ds. \tag{7.9}$$

Remark 1. *Note that the power case differs from the log case in two distinct ways. The first term on the right side of (7.7) is a distorted version of the right side of (4.5), where the* λ *dependence in the function* K *is distorted in a* g *dependent fashion through the mapping* $\boldsymbol{h}$. *The second term does not arise in the log case, and introduces complications; a similar situation occurs and is dealt with in a different model, see Delong and Klüppelberg (2008).*

Following Delong and Klüppelberg (2008), one can attempt to verify that (7.8) is a contraction mapping, and that consequently the sequence of iterates $\{g^{(i)}, i = 0, 1, \dots\}$ with $g^{(0)} = 1$ and $g^{(i+1)} = \mathcal{G}(g^{(i)})$ converges to g. We do not attempt this here.

In examples for which the functions K and $\boldsymbol{\omega}^*_t(\boldsymbol{\lambda})$ are explicitly solvable, the iteration scheme can apparently be efficiently implemented numerically.

7.2 Exponential Utility

An investor with the exponential utility $U(x) = -e^{-\gamma x}/\gamma$ with risk aversion parameter $\gamma > 0$, unlike the log investor, can in principle consume at a negative rate, perhaps even reaching negative wealth, and thus in this setting the question of defining admissible strategies is more involved. We can however, search for candidate optimal strategies that solve the reduced HJB equation (3.8) in the form $L(x, \lambda) = -\exp[-\kappa x]g(\lambda)$ for some positive function g, and then attempt to interpret the result. The HJB equation turns into:

$$
\begin{aligned}
0 = \max_{\{C,\boldsymbol{\omega}\}} \Bigg\{ & U(C) - \beta L + \frac{L}{g(\boldsymbol{\lambda})}[\mathcal{A}g](\boldsymbol{\lambda}) \\
& - \kappa L\,(rx + \boldsymbol{\omega}'\boldsymbol{R}x - C) + \frac{\kappa^2 L}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}x^2 \\
& + L\sum_{l=1}^{m}\lambda_{l,t}\frac{g(\boldsymbol{\lambda}+\boldsymbol{d}_l)}{g(\boldsymbol{\lambda})}\int \left(\exp\left[-\kappa(\boldsymbol{\omega}'\boldsymbol{J}_l z x\right] - 1\right)\nu_l(dz)\Bigg\} ,
\end{aligned} \tag{7.10}
$$

where $\mathcal{A}g$ is given as before by (4.6). The first order conditions for C^* imply that $\gamma U(C^*) = \kappa L$ and hence

$$
C^* = \frac{1}{\gamma}\left(\kappa x - \log g - \log \kappa\right).
$$

The candidate optimal portfolio weights are best expressed in terms of dollar amounts $\pi_{i,t} = \omega_{i,t}X_t$ invested. We find $\boldsymbol{\pi}^*_t = \boldsymbol{\pi}^*(\boldsymbol{h}(\boldsymbol{\lambda})), \boldsymbol{h} = (h_1, \dots, h_m)$ where

$$
\boldsymbol{\pi}^*(\boldsymbol{\lambda}) = \operatorname{argmin}_{\boldsymbol{\pi}} K(\boldsymbol{\pi}, \boldsymbol{\lambda})) \tag{7.11}
$$

$$
h_l(\boldsymbol{\lambda}) = \lambda_l g(\boldsymbol{\lambda} + \boldsymbol{d}_l)/g(\boldsymbol{\lambda}) \tag{7.12}
$$

$$
K(\boldsymbol{\pi}, \boldsymbol{\lambda}) = \kappa\boldsymbol{\pi}'\boldsymbol{R} - \frac{\kappa^2}{2}\boldsymbol{\pi}'\boldsymbol{\Sigma}\boldsymbol{\pi} - \sum_{l=1}^{m}\lambda_l \int \left(\exp[-\kappa\left(\boldsymbol{\pi}'\boldsymbol{J}\right)_l z] - 1\right)\nu_l\,(dz)
$$

Substitution of $C^*, \boldsymbol{\pi}^*$ back into (7.10) leads to

$$
0 = (r - \beta)g + [\mathcal{A}g](\boldsymbol{\lambda}) - \kappa\,(rx - C^*) - g(\boldsymbol{\lambda})K(\boldsymbol{\pi}^*(\boldsymbol{\lambda}), \boldsymbol{h}(\boldsymbol{\lambda}))
$$

which in turn implies that $\kappa = r\gamma$. The condition on g is now implicit:

$$[\mathcal{A}g](\boldsymbol{\lambda}) - (\beta - r\gamma + r\log(r\gamma))g(\boldsymbol{\lambda}) = \tilde{G}(\boldsymbol{\lambda}; g) \tag{7.13}$$

$$\tilde{G}(\lambda; g) = g(\boldsymbol{\lambda})\left[r\log g + K(\boldsymbol{\pi}^*(\boldsymbol{\lambda}), \boldsymbol{h}(\boldsymbol{\lambda}))\right] \tag{7.14}$$

This is very similar to the characterization of g for the power utility case. In examples where K and $\boldsymbol{\pi}^*$ are explicitly known, one can attempt to solve numerically for g by iteration.

8 Conclusions

This paper extends the range of models for which solutions to the optimal dynamic portfolio-consumption problem are available, to one which includes mutually exciting jumps. We analyze features of the optimal solution and show that it differs from the usual case in important ways. In particular, it introduces an explicit time-variation in the optimal portfolio weights in response to changes in the jump intensity, providing a rare example of an explicit time-varying optimal portfolio solution. Moreover, power and exponential investors both adopt more aggressive strategies, characterized by a distortion function $\boldsymbol{h}$, than the corresponding investor who does not fully recognize the mutual excitation effect.

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