Explicit formulas for Laplace transforms of stochastic integrals

T. R. Hurd* and A. Kuznetsov*

Dept. of Mathematics and Statistics

McMaster University

Hamilton ON L8S 4K1

Canada

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Abstract

In this article we provide a general identity useful for computing expectations of the form

$$E\left[e^{-\int_0^T \phi(X_s)ds} g(X_T)\right]$$

for diffusion processes X_t and certain functions ϕ. In the case of CIR and Jacobi diffusions, this identity leads to explicit formulas for the Laplace transform of a multidimensional family of random variables constructed from X_t and its integrals.

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1 Introduction

The modelling of financial time series such as stock prices, interest rates, and foreign exchange rates has been important in the development of the theory of stochastic processes. The solution to practical problems such as option pricing, model calibration, and portfolio selection relies to a great extent on the solvability properties of the underlying one-dimensional stochastic processes. Since financial mathematics is not a fundamental science like physics, the criteria for useful models are mostly pragmatic: models are of interest if they fit certain "stylized facts", and retain an adequate degree of computational tractability.

For example, the recent popularity of Lévy models for stock returns is due in large part to the possibility of using the fast Fourier transform to compute option prices (see [11]). Similarly, the class of affine processes (see [6]) derives its popularity for interest rate theory, stock modelling, and other financial applications in large part because explicit or close to explicit formulas are available for key functionals of the underlying processes. The so-called solvable Markov models, studied by [1], constitute a further distinct family of models that includes geometric Brownian motion, the Ornstein-Uhlenbeck processes, the Cox-Ingersoll-Ross (CIR) model, and the Jacobi process: these processes have found hundreds of applications in finance and other areas of stochastic phenomena.

It is this last family of solvable diffusion models that provide the natural setting for the present paper. For a one-dimensional stationary diffusion process X_t on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\leq T}, \mathrm{P})$, we present an argument involving a measure change from P to Q which yields an identity of the form

$$E^{\mathrm{P}}\left[e^{-\int_0^T \phi(X_s)ds} g(X_T)\right] = f(X_0)E^{\mathrm{Q}}\left[\frac{g(X_T)}{f(X_T)}\right] \tag{1}$$

for certain functions ϕ, f, g. We are then able to provide a number of interesting cases where the new expectation can be evaluated in terms of special functions.

The identity (1) is closely related to results contained in a recent article [2], where the authors present the complete classification scheme of diffusion processes (and also birth and death processes) X_t and functions $\phi(x)$, for which the expectation (1) can be computed in terms of hypergeometric functions. By using a PDE approach, non-singular transformations and spectral resolution, they were able not only to give a complete classification of these processes, but also to provide explicit series formulas in many important cases.

Our results, while of intrinsic interest in the theory of stochastic processes and partial differential equations, are motivated by financial mathematics, and they suggest a number of new applications in finance that we hope to explore in future papers. As just one example, we note here a connection between our new results and the theory of utility based pricing and hedging in incomplete markets. Utility theory, the proper economic foundation for financial decision making and contract valuation in general markets, has been under rapid development in the past decade. However, the list of explicitly solvable problems in utility theory is not long, despite the efforts of a large number of researchers, a fact which has hampered both the development of the theory, and the adoption of utility methods by finance practitioners. A recent paper [9] shows how the utility based prices of important volatility derivatives in certain stochastic volatility models including the classic Heston model boil down to natural Feynman-Kac type expectations. Serendipitously, these expectations turn out to be of the precise form we address here and thus can be expressed explicitly in terms of our new formulas.

Section 2 provides two alternative heuristic derivations of the formula for general diffusion processes, one purely probabilistic involving the Girsanov theorem, the other by partial differential equations. When the underlying process X_t is a Cox-Ingersoll-Ross (CIR) process (a positive mean reverting diffusion), and we choose $f(x) = e^{-v_1 x} x^{v_2}, g(x) = e^{-w_1 x} x^{w_2}$, Theorem 3.1 provides the precise conditions under which

$$E^{\mathrm{P}} \left[e^{-\int_0^T \left(d_1 X_s + \frac{d_2}{X_s} \right) ds} e^{-w_1 X_T} X_T^{w_2} \right]$$

can be computed in closed form in terms of the confluent hypergeometric function. To facilitate computations of this formula over a wide range of parameter values, in Appendix 6 we provide a simple asymptotic expansion which complements the standard power series expansion of the hypergeometric function.

In Section 4, we investigate the lesser known Jacobi process (a mean reverting diffusion taking its values on $[0, 1]$, see [5]). When we take $f(x) = x^{v_1}(1-x)^{v_2}, g(x) = x^{w_1}(1-x)^{w_2}$, we are able to show conditions under which the method leads to a convergent and tractable representation of

$$E^{\mathrm{P}} \left[e^{-d_1 \int_0^T \left(\frac{1-X_s}{X_s} \right) ds - d_2 \int_0^T \left(\frac{X_s}{1-X_s} \right) ds} e^{-w_1 X_T} X_T^{w_2} \right]$$

in terms of hypergeometric functions. In this case, the formula is in terms of a rapidly convergent series of Jacobi polynomials (see [10]). Appendix 7 examines the asymptotic properties of this expansion as $t \to 0$.

2 The general method

In this section, we present two heuristic derivations of the basic identity, one probabilistic, the other via partial differential equations. At the end of the section, we discuss the general technical conditions under which the formula holds. Let $X_t \in D, t \in [0,T],\ T \geq 0$ be a one-dimensional stationary diffusion process under the measure P, defined by its initial condition $X_0 = x$, and its Markov generator

$$\mathcal{L}^{\mathrm{P}} = \mu(x)\partial_x + \frac{1}{2}\sigma^2(x)\partial_x^2 \ . \tag{2}$$

Here D is a (possibly infinite) interval on the real line.

From the generator we identify the stochastic differential equation followed by X_t:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t^{\mathrm{P}}, X_0 = x \ ,$$

where W^{P} is a P-Brownian motion. We also assume that $\sigma(x) > 0$ inside domain D.

Identity 1. *For any "good" functions $f(x) > 0$ and $g(x)$ we have*

$$E_x^{\mathrm{P}}\left[e^{-\int\limits_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)}ds} g(X_T)\right] = f(x)E_x^{\mathrm{Q}}\left[\frac{g(X_T)}{f(X_T)}\right], \tag{3}$$

where under the measure Q *the Markov generator of X_t has a modified drift:*

$$\mathcal{L}^{\mathrm{Q}} = \left(\mu(x) + \sigma^2(x)\frac{f'(x)}{f(x)}\right)\partial_x + \frac{1}{2}\sigma^2(x)\partial_x^2. \tag{4}$$

For a probabilistic justification of this identity, we start with the integrated Itô formula for the process $\log(f(X_t))$:

$$\log(f(X_T)) = \log(f(x)) + \int_0^T \frac{f'(X_s)}{f(X_s)} \left(\mu(X_s)ds + \sigma(X_s)dW_s^{\mathrm{P}}\right)$$
$$+\frac{1}{2}\int_0^T \left(\frac{f''(X_s)}{f(X_s)} - \left(\frac{f'(X_s)}{f(X_s)}\right)^2\right)\sigma^2(X_s)ds.$$

By exponentiation, multiplication by $g(X_T)$, and rearrangement of factors, one finds

$$e^{-\int_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)}ds} g(X_T) = \frac{f(X_0)}{f(X_T)} g(X_T) e^{Z_T - \frac{1}{2}\langle Z,Z\rangle_T}$$

where $Z_t = \int_0^t \frac{f'(X_s)}{f(X_s)}\sigma(X_s)dW_s^{\mathrm{P}}$ is a P local martingale. Application of the Girsanov theorem with Radon–Nikodym derivative

$$\frac{d\mathrm{Q}}{d\mathrm{P}} = e^{Z_T - \frac{1}{2}\langle Z,Z\rangle_T}, \tag{5}$$

leads finally to the desired expectation formula. The Girsanov theorem also implies that

$$dW_t^{\mathrm{Q}} = dW_t^{\mathrm{P}} - \sigma(X_t)\frac{f'(X_t)}{f(X_t)}dt,$$

is a Brownian motion under the measure Q, and one obtains that the dynamics of X_t under Q is given by (4).

We observe that the probabilistic argument can be made rigorous by imposing two-times differentiability of $f(x)$ (for the Itô formula), plus the condition $f(x) \neq 0$ for any x inside domain D, which we need for the process Z_t to be well defined. Using the same argument as in [3], [4] and [6], one can prove that a sufficient condition to justify the use of the Girsanov theorem is the following:

Assumption 1. *The boundaries of the interval D are unattainable both under measure* P *and under measure* Q.

It is also of interest to give the heuristic derivation of Identity 1 by partial differential equations. The Feynman-Kac formula implies that under suitable conditions the left side of (3), which we denote by $\phi(T,x)$, can be obtained as the $t = T$ value of the solution $\phi(t,x)$ of partial differential equation:

$$\partial_t \phi - \mathcal{L}^{\mathrm{P}} \phi + \left(\frac{\mathcal{L}^{\mathrm{P}} f}{f} \right) \phi = 0, \qquad t \in (0,T] \tag{6}$$

with initial condition $\phi(0,x) = g(x)$. One can now easily check that if $\phi(t,x) = f(x)\psi(t,x)$, then

$$\mathcal{L}^{\mathrm{P}} \phi = f(\mathcal{L}^{\mathrm{Q}} \psi) + (\mathcal{L}^{\mathrm{P}} f)\psi \ .$$

Therefore $\psi(T,x)$ is the $t = T$ value of the solution of

$$\partial_t \psi - \mathcal{L}^{\mathrm{Q}} \psi = 0, \qquad t \in (0,T] \tag{7}$$

with initial condition $\psi(0,x) = g(x)/f(x)$, which again by the Feynman-Kac formula equals the right side of (3) divided by $f(x)$, thus justifying the identity.

For the PDE argument to be made rigorous, the double use of the Feynman-Kac formula requires careful attention to the boundary conditions of the partial differential equation.

In the next two sections, we investigate the CIR diffusion and the Jacobi diffusion, for which the probabilistic argument can be made rigorous, and moreover the identity leads to new and useful formulas. The paper [2] gives a general classification of processes for which similar results may be expected to hold. Examples discussed there include geometric Brownian motion, Ornstein-Uhlenbeck process, CIR process, Jacobi process, and a number of discrete Markov processes, for which they prove a number of results related to our formulas.

3 The CIR process

Let $X_t \in [0,\infty)$ be the CIR process with generator

$$\mathcal{L}^{\mathrm{P}} = (a-bx)\partial_x + \frac{1}{2}c^2 x \partial_x^2. \tag{8}$$

Assume $\alpha = \frac{2a}{c^2} - 1 \geq 0$, $\beta = \frac{2b}{c^2} \geq 0$: it is known that the process X_t has an unattainable boundary at 0 if and only if $\alpha \geq 0$.

In this section we will give a closed form expression for the function

$$G^{\mathrm{CIR}}(T,x;d_1,d_2,w_1,w_2) = E^{\mathrm{P}}_{0,x}\left[e^{-\int_0^T \left(d_1 X_s + \frac{d_2}{X_s}\right)ds} e^{-w_1 X_T} X_T^{w_2} \right]. \tag{9}$$

Theorem 3.1. *Suppose $d_1 \geq -\frac{b^2}{2c^2}$ and $d_2 \geq -\frac{\alpha^2 c^2}{8}$. Then the measure* Q *obtained from* (5) *by choosing $f(x) = e^{-v_1 x} x^{v_2}$ with $v_1 = \frac{1}{2}\left(-\beta + \sqrt{\beta^2 + \frac{8d_1}{c^2}}\right)$, $v_2 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + \frac{8d_2}{c^2}}\right)$ is well defined and equivalent to* P*. Define $\gamma_T = (\beta + 2v_1)\left(1 - e^{-(\beta/2+v_1)c^2 T}\right)^{-1}$.*

1. *If $\alpha + v_2 + w_2 + 1 \leq 0$ or $\gamma_T + w_1 - v_1 \leq 0$ then $G^{\mathrm{CIR}}(T,x;d_1,d_2,w_1,w_2) = \infty$.*

2. *If $\alpha + w_2 + v_2 + 1 > 0$ and $\gamma_T + w_1 - v_1 > 0$ then*

$$\begin{aligned} G^{\mathrm{CIR}}(T,x;d_1,d_2,w_1,w_2) &= e^{-\left(av_1 + bv_2 + c^2 v_1 v_2\right)T} x^{v_2} \left(\gamma_T + w_1 - v_1\right)^{-\alpha - v_2 - w_2 - 1} \gamma_T^{\alpha + 2v_2 + 1} \\ &\times \exp\left(-x\left(v_1 + \frac{\gamma_T(w_1 - v_1)}{\gamma_T + w_1 - v_1} e^{-(\beta/2+v_1)c^2 T}\right)\right) \\ &\times \frac{\Gamma(\alpha + v_2 + w_2 + 1)}{\Gamma(\alpha + 2v_2 + 1)} {}_1F_1\left(v_2 - w_2, \alpha + 2v_2 + 1; -\frac{\gamma_T^2 x e^{-(\beta/2+v_1)c^2 T}}{\gamma_T + w_1 - v_1}\right). \end{aligned} \tag{10}$$

Remarks:

1. Formula 10 is convenient when the argument of the confluent hypergeometric function ${}_1F_1$ is small, that is when either x is small or T is large. One then uses the Taylor series for ${}_1F_1$

$$ {}_1F_1(a,b;z) = 1 + \frac{a}{b}z + \frac{1}{2!}\frac{a(a+1)}{b(b+1)}z^2 + \dots $$

When the argument is large (i.e. when x is large or T is small or both), one should instead use an asymptotic formula such as the one given in Appendix 6.

2. If $w_2 - v_2 = n$, where n is a nonnegative integer, then formula 10 can be simplified, since

$$ \frac{\Gamma(\alpha + v_2 + w_2 + 1)}{\Gamma(\alpha + 2v_2 + 1)} {}_1F_1\left(v_2 - w_2, \alpha + 2v_2 + 1; -z\right) = n! L_n^{(\alpha+2v_2)}\left(-z\right), $$

where $L_n^{(a)}(z)$ is a *Laguerre polynomial* (see [10]), defined as

$$ L_n^{(a)}(z) = \sum_{k=0}^{n} (-1)^k \binom{n+a}{n-k} \frac{z^k}{k!} = \frac{(a+1)_n}{n!} {}_1F_1\left(-n, a+1; z\right). $$

In particular, when $v_2 = w_2 = 0$, we obtain the well known exponential affine formula.

Proof: Let $f(x) = e^{-v_1 x} x^{v_2}$ and $g = e^{-w_1 x} x^{w_2}$. Since $d_1 = bv_1 + \frac{1}{2}c^2 v_1^2$ and $d_2 = av_2 + \frac{1}{2}c^2 v_2(v_2 - 1)$, we can compute

$$ \begin{aligned} \frac{\mathcal{L}^{\mathrm{P}} f}{f} &= -(av_1 + bv_2 + c^2 v_1 v_2) + x\left(bv_1 + \frac{1}{2}c^2 v_1^2\right) + \frac{1}{x}\left(av_2 + \frac{1}{2}c^2 v_2(v_2 - 1)\right) \\ &= -(av_1 + bv_2 + c^2 v_1 v_2) + d_1 x + \frac{d_2}{x}. \end{aligned} $$

Thus $G^{\mathrm{CIR}}(T,x;d_1,d_2,w_1,w_2)$ equals

$$e^{-(av_1+bv_2+c^2v_1v_2)T}E^{\mathrm{P}}_{0,x}\left[e^{-\int\limits_0^T \frac{\mathcal{L}^{\mathrm{P}}f(X_s)}{f(X_s)}ds}e^{-w_1X_T}X_T^{w_2}\right] .$$

By following the argument of Section 2, one computes from (4) that X_t under the measure Q has the generator

$$\mathcal{L}^{\mathrm{Q}} = (\tilde{a}-\tilde{b}x)\partial_x + \frac{1}{2}c^2x\partial_x^2 ,$$

where $\tilde{a} = a + c^2v_2$ and $\tilde{b} = b + c^2v_1$. By the assumptions, $\tilde{\alpha} = \frac{2\tilde{a}}{c^2} - 1 = \alpha + 2v_2 \geq 0$. Therefore Assumption 1 holds, the measure change is justified, and we have the following identity:

$$\begin{aligned} E^{\mathrm{P}}_{0,x}\left[e^{-\int\limits_0^T \frac{\mathcal{L}^{\mathrm{P}}f(X_s)}{f(X_s)}ds}e^{-w_1X_T}X_T^{w_2}\right] &= x^{v_2}e^{-v_1x}E^{\mathrm{Q}}_{0,x}\left[X_T^{w_2-v_2}e^{(v_1-w_1)X_T}\right] \\ &= x^{v_2}e^{-v_1x}\int\limits_0^\infty y^{w_2-v_2}e^{(v_1-w_1)y}p^{\mathrm{Q}}_T(x,y)dy . \qquad (11) \end{aligned}$$

The transition probability density function $p^{\mathrm{Q}}_t(x,y)$ of the CIR process X_t is given by

$$p^{\mathrm{Q}}_t(x,y) = \gamma_t\left(\frac{ye^{\tilde{b}t}}{x}\right)^{\frac{\tilde{\alpha}}{2}}\exp\left[-\gamma_t(xe^{-\tilde{b}t}+y)\right]I_{\tilde{\alpha}}\left(2\gamma_t\sqrt{xye^{-\tilde{b}t}}\right),$$

where $\gamma_t \equiv -2\tilde{b}/(c^2(e^{-\tilde{b}t}-1))$ and I_ν is the modified Bessel function of the first kind. Moreover, the integral on the right side of (11) is finite if $\alpha + w_2 + v_2 + 1 > 0$ and $\gamma_T + w_1 - v_1 > 0$, and given in closed form by using the formula (see [8]):

$$\int\limits_0^\infty y^{\mu-\frac{1}{2}}e^{-\alpha y}I_{2\nu}\left(2\beta\sqrt{y}\right)dy = e^{\frac{\beta^2}{\alpha}}\frac{\beta^{2\nu}}{\alpha^{\mu+\nu+\frac{1}{2}}}\frac{\Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(2\nu+1)}{}_1F_1\left(\nu-\mu+\frac{1}{2},2\nu+1;-\frac{\beta^2}{\alpha}\right). \qquad (12)$$

This proves (10). On the other hand, if either $\alpha + w_2 + v_2 + 1 > 0$ or $\gamma_T + w_1 - v_1 > 0$, then the right side of (11) is infinite.

$\square$

4 The Jacobi process

Let $X_t \in [0, 1]$ be the Jacobi diffusion process with generator

$$\mathcal{L}^{\mathrm{P}} = (a - bx)\partial_x + \frac{1}{2}c^2 x(1-x)\partial_x^2 \,. \tag{13}$$

Define $\alpha = \frac{2a}{c^2} - 1$ and $\beta = \frac{2(b-a)}{c^2} - 1$. It is known that the process X_t has unattainable boundaries if and only if $\alpha \geq 0,\ \beta \geq 0$.

In this section we provide methods to compute the function

$$G^{\mathrm{Jacobi}}(T, x; d_1, d_2, w_1, w_2) = E^{\mathrm{P}}_{0,x}\left[e^{-\int\limits_0^T d_1\left(\frac{1-X_s}{X_s}\right) + d_2\left(\frac{X_s}{1-X_s}\right) ds} X_T^{w_1}(1 - X_T)^{w_2} \right]. \tag{14}$$

Theorem 4.1. *Let the parameters of the Jacobi process satisfy* $\alpha \geq 0, \beta \geq 0$, *and assume* $d_1 \geq -\alpha^2 c^2/8, d_2 \geq -\beta^2 c^2/8$. *Then the measure* Q *obtained from* (5) *by choosing* $f(x) = x^{v_1}(1-x)^{v_2}$ *where* $v_1 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + \frac{8d_1}{c^2}}\right)$ *and* $v_2 = \frac{1}{2}\left(-\beta + \sqrt{\beta^2 + \frac{8d_2}{c^2}}\right)$ *is well defined, and equivalent to* P.

1. *If* $\alpha + w_1 + v_1 \leq -1$ *or* $\beta + w_2 + v_2 \leq -1$, *then* $G^{\mathrm{Jacobi}}(T, x; d_1, d_2, w_1, w_2) = \infty$.

2. If $\alpha + w_1 + v_1 > -1$ and $\beta + w_2 + v_2 > -1$, then

$$
\begin{aligned}
G^{\text{Jacobi}}(T, x; d_1, d_2, w_1, w_2) &= e^{-\left((b-a)v_1 + av_2 + c^2 v_1 v_2\right)T} x^{v_1}(1-x)^{v_2} \\
&\times \mathrm{B}(\alpha + w_1 + v_1 + 1, \beta + w_2 + v_2 + 1)\frac{\Gamma(\alpha+\beta+2v_1+2v_2+1)}{\Gamma(\alpha+2v_1+1)\Gamma(\beta+2v_2+1)} \\
&\times \sum_{n=0}^{\infty} e^{-n(n+\alpha+\beta+2v_1+2v_2+1)\frac{c^2 T}{2}} \ \frac{(\alpha+\beta+2v_1+2v_2+1)_n}{(\alpha+2v_1+1)_n} \\
&\times (2n+\alpha+\beta+2v_1+2v_2+1) q_n P_n^{(\alpha+2v_1, \beta+2v_2)}(2x-1) \ . \qquad (15)
\end{aligned}
$$

The coefficients q_n are defined as

$$
q_n \;=\; {}_3F_2\left(\begin{matrix} -n, & \alpha+\beta+2v_1+2v_2+n+1, & \beta+v_2+w_2+1 \\ & \alpha+\beta+w_1+v_1+w_2+v_2+2, & \beta+2v_2+1 \end{matrix}\ ;1\right) \qquad (16)
$$

and can be computed via a three term recurrence relation:

$$
A_n q_{n+1} = (\beta + v_2 + 1 + A_n + C_n) q_n - C_n q_{n-1} \ , \quad q_{-1} = 0, \quad q_0 = 1 \ , \qquad (17)
$$

where A_n and C_n are given by the following formulas

$$
\begin{aligned}
A_n &= -\frac{(n+\alpha+\beta+2v_1+2v_2+1)(n+\beta+2v_2+1)(n+\alpha+\beta+w_1+v_1+w_2+v_2+2)}{(2n+\alpha+\beta+2v_1+2v_2+1)(2n+\alpha+\beta+2v_1+2v_2+2)} \\
C_n &= \frac{n(n+v_1+v_2-w_1-w_2-1)(n+\alpha+2v_1)}{(2n+\alpha+\beta+2v_1+2v_2+1)(2n+\alpha+\beta+2v_1+2v_2)} \ .
\end{aligned}
$$

Remark:

If $d_2 = w_2 = 0$, then also $v_2 = 0$, and (15) can be simplified, since the ${}_3F_2$ function collapses to ${}_2F_1$:

$$
q_n = {}_2F_1(-n, \alpha+\beta+2v_1+n+1; \alpha+\beta+w_1+v_1+2; 1) \ ,
$$

which can be computed explicitly (see [8])

$$q_n = (-1)^n \frac{(v_1 - w_1)_n(\alpha + \beta + 2v_1 + 1)_n}{(\alpha + \beta + w_1 + v_1 + 2)_n(\alpha + 2v_1 + 1)_n} \ .$$

This leads to

$$\begin{aligned} G^{\mathrm{Jacobi}}(T, x; d_1, 0, w_1, 0) &= x^{v_1} e^{((a-b)v_1 T)} \frac{\Gamma(\alpha + w_1 + v_1 + 1)\Gamma(\alpha + \beta + 2v_1 + 1)}{\Gamma(\alpha + \beta + w_1 + v_1 + 2)\Gamma(\alpha + 2v_1 + 1)} \\ &\times \sum_{n=0}^{\infty} (-1)^n e^{-n(n+\alpha+\beta+2v_1+1)\frac{c^2 T}{2}} \frac{(v_1 - w_1)_n(\alpha + \beta + 2v_1 + 1)_n}{(\alpha + \beta + w_1 + v_1 + 2)_n(\alpha + 2v_1 + 1)_n} \\ &\times (2n + \alpha + \beta + 2v_1 + 1) P_n^{(\alpha+2v_1,\beta)}(2x - 1) \ . \end{aligned} \tag{18}$$

Proof: Let $f(x) = x^{v_1}(1-x)^{v_2}$, and $g(x) = x^{w_1}(1-x)^{w_2}$. Since $d_1 = av_1 + \frac{1}{2}c^2 v_1(v_1 - 1)$ and $d_2 = (b-a)v_2 + \frac{1}{2}c^2 v_2(v_2 - 1)$, we can compute

$$\begin{aligned} \frac{\mathcal{L}^{\mathrm{P}} f}{f} &= -\left(b(v_1 + v_2) + c^2 v_1 v_2 + \frac{1}{2}c^2(v_1(v_1 - 1) + v_2(v_2 - 1)) \right) + \\ &+ \frac{1}{x}\left(av_1 + \frac{1}{2}c^2 v_1(v_1 - 1) \right) + \frac{1}{1-x}\left((b-a)v_2 + \frac{1}{2}c^2 v_2(v_2 - 1) \right) \\ &= d_1 \left(\frac{1}{x} - 1 \right) + d_2 \left(\frac{1}{1-x} - 1 \right) - ((b-a)v_1 + av_2 + c^2 v_1 v_2) \ . \end{aligned}$$

Thus $G^{\mathrm{Jacobi}}(T, x; d_1, d_2, w_1, w_2)$ equals

$$e^{-((b-a)v_1 + av_2 + c^2 v_1 v_2)T} E^{\mathrm{P}}_{0,x} \left[e^{-\int_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)} ds} X_T^{w_1}(1 - X_T)^{w_2} \right] \ .$$

By following the argument of Section 2, one computes from (4) that X_t under the measure Q has the generator

$$\mathcal{L}^{\mathrm{Q}} = (\tilde{a} - \tilde{b}x)\partial_x + \frac{1}{2}c^2 x(1-x)\partial_x^2 \ , \tag{19}$$

where $\tilde{a} = a + c^2 v_1$ and $\tilde{b} = b + c^2(v_1 + v_2)$. By the assumptions, $\tilde{\alpha} = \frac{2\tilde{a}}{c^2} - 1 = \alpha + 2v_1 \geq 0$

and $\tilde{\beta} = \frac{2(\tilde{b}-\tilde{a})}{c^2} - 1 = \beta + 2v_2 \geq 0$. Therefore Assumption 1 holds, the measure change is justified, and we have the following identity:

$$E^{\mathrm{P}}_{0,x}\left[e^{-\int\limits_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)} ds} X_T^{w_1}(1-X_T)^{w_2}\right] = x^{v_1}(1-x)^{v_2} E^{\mathrm{Q}}_{0,x}\left[X_T^{w_1-v_1}(1-X_T)^{w_2-v_2}\right]$$
$$= x^{v_1}(1-x)^{v_2}\int\limits_0^1 y^{w_1-v_1}(1-y)^{w_2-v_2} p^{\mathrm{Q}}_T(x,y)dy. \quad (20)$$

The transition probability density function $p^{\mathrm{Q}}_t(x,y)$ of the Jacobi process is given by

$$p^{\mathrm{Q}}_t(x,y) = \frac{y^{\tilde{\alpha}}(1-y)^{\tilde{\beta}}}{B(\tilde{\alpha}+1,\tilde{\beta}+1)}\sum_{n=0}^{\infty}\frac{e^{-n(n+\tilde{\alpha}+\tilde{\beta}+1)\frac{c^2 t}{2}}}{p_n^2} P_n^{(\tilde{\alpha},\tilde{\beta})}(2x-1)P_n^{(\tilde{\alpha},\tilde{\beta})}(2y-1),$$

where

$$p_n^2 = \frac{(\tilde{\alpha}+1)_n(\tilde{\beta}+1)_n}{(\tilde{\alpha}+\tilde{\beta}+2)_{n-1}(2n+\tilde{\alpha}+\tilde{\beta}+1)n!}.$$

The integral on the right side of (20) is finite if and only if $\alpha+w_1+v_1 > -1, \beta+w_2+v_2 > -1$, and can be computed explicitly using the following formula (see [7] and [8]):

$$\int\limits_0^1 y^{\alpha+w_1+v_1}(1-y)^{\beta+w_2+v_2} P_n^{(\alpha+2v_1,\beta+2v_2)}(2y-1)dy$$
$$= \mathrm{B}(\alpha+w_1+v_1+1,\beta+w_2+v_2+1)\frac{(\beta+2v_2+1)_n}{n!}\; q_n\ .$$

The three term recurrence relation (17) for the coefficients q_n follows from the fact that q_n are related to the continuous Hahn polynomials

$$p_n(0;\beta+v_2+w_2+1, v_1-w_1, \alpha+v_1+w_1+1, v_2-w_2)$$

(see [10], page 31).

□

5 Conclusion

We have presented two families of diffusion processes for which explicit formulas for important expectations are proved. These results are consequences of a simple expectation identity derived for general diffusions. In the case where X_t is a CIR process, we have derived an explicit formula for the Laplace transform of the four dimensional random variable $(X_T, \log X_T, \int_0^T X_s ds, \int_0^T X_s^{-1} ds)$. The closest result we know is a well known closed formula for the Laplace transform of $(X_T, \int_0^T X_s ds)$. The more general formula (10) has potential uses in interest rate theory and credit risk.

For the less known Jacobi process, we have derived an explicit formula for the Laplace transform of the four dimensional random variable $(\log X_T, \log(1-X_T), \int_0^T \frac{1-X_s}{X_s} ds, \int_0^T \frac{X_s}{1-X_s} ds)$. This process has properties which may also prove useful in finance. For example, the related process $Y_t = \frac{1-X_s}{X_s}$ has the character of the spot interest rate or a default hazard rate. X_t itself could be taken as a stochastic recovery rate in credit risk modelling. In such approaches, (15) will no doubt prove to be very useful.

Our results are explicit realizations of integral formulas given in [2]. Their classification suggests that in the diffusion case, geometric Brownian motion and the Ornstein-Uhlenbeck process are the only further processes which admit identities of this type.

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6 CIR process: asymptotic expansion for small t

The following asymptotic formula [7] is useful when the argument of ${}_1F_1$ is large and negative (i.e. as $Re(z) \to -\infty$):

$$ {}_1F_1(a,b;z) = (-z)^{-a}\frac{\Gamma(b)}{\Gamma(b-a)}\sum_{k=0}^{N}\frac{(a)_n(a-b+1)_n}{n!}(-z)^{-n} + O\left(|z|^{-a-N-1}\right). $$

This leads to an approximation to (10):

$$ \begin{aligned} G^{\mathrm{CIR}}(t,x;d_1,d_2,w_1,w_2) &\approx x^{w_2}\exp\left(x\left(v_1-(w_1+v_1)\left(1+\frac{w_1+v_1}{\gamma_t}\right)^{-1}e^{-\left(\frac{\beta}{2}-v_1\right)c^2t}\right)\right) \\ &\times\exp\left(\frac{1}{2}\left((\alpha+1)v_1-w_2(\beta-2v_1)\right)c^2t\right)\left(1+\frac{w_1+v_1}{\gamma_t}\right)^{-(\alpha+2w_2+1)} \\ &\times\left[\sum_{n=0}^{N}\frac{(v_2-w_2)_n(-\alpha-v_2-w_2)_n}{n!}(-z)^{-n}+O\left(|z|^{w_2-v_2-N-1}\right)\right], \end{aligned} \tag{21} $$

where $z = -\gamma_t x e^{-\left(\frac{\beta}{2}-v_1\right)c^2t}\left(1+\frac{w_1+v_1}{\gamma_t}\right)^{-1}$ is the argument of ${}_1F_1$ function. We find that this formula is very convenient for computations when t is small and/or when x is large, all other parameters being bounded.

7 Jacobi process: asymptotic expansion for small t

In this section we will derive an asymptotic expansion for $G^{\text{Jacobi}}(t, x; d_1, d_2, w_1, w_2)$ for small t. For fixed $x \in (0, 1)$, we start with (20) and write

$$E_{0,x}^{\mathrm{Q}}\left[X_t^{w_1-v_1}(1-X_t)^{w_2-v_2}\right] = E_{0,x}^{\mathrm{Q}}\left[\sum_{n\geq 0} c_n(x)(X_t-x)^n\right] \approx \sum_{n\geq 0} c_n(x) M_n(t,x), \qquad (22)$$

where $c_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left[x^{w_1-v_1}(1-x)^{w_2-v_2}\right]$ and $M_n(t) := M_n(t,x) = E_{0,x}^{\mathrm{Q}}\left[(X_t-x)^n\right]$. Coefficients $c_n(x)$ can be computed explicitly as $c_n(x) = x^{w_1-v_1}(1-x)^{w_2-v_2}\hat{c}_n(x)$, where

$$\hat{c}_n(x) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(v_1-w_1)_k(v_2-w_2)_{n-k}}{x^k(1-x)^{n-k}}.$$

By applying Ito's lemma to the process $(X_t - x)^n$ we find that functions $M_n(t)$ can be found by recursively solving differential equations

$$\begin{aligned}\frac{d}{dt}M_n(t) = \left(\frac{c^2}{2}n(n-1) - n\tilde{b}\right) M_n(t) + \left(n(\tilde{a}-\tilde{b}x) + \frac{c^2}{2}n(n-1)(1-2x)\right) M_{n-1}(t) \\ +\frac{c^2}{2}n(n-1)x(1-x)M_{n-2}(t),\end{aligned} \qquad (23)$$

with initial conditions $M_n(0) = \delta_{n0}$. It follows at once from this equation that $M_n(t) = O(t^{[\frac{n+1}{2}]})$ as $t \to 0$.

Using this method one could obtain approximations of any order $O(t^N)$. Here, as an example, we provide the first four functions $M_n(t, x)$ which combined with (22) will give us a useful approximation to $G^{\text{Jacobi}}(t, x; d_1, d_2, w_1, w_2)$ of order $O(t^3)$.

$$\begin{cases} M_1(t) = (\tilde{a} - \tilde{b}x)\left(t - \tilde{b}\frac{t^2}{2}\right) + O(t^3) \\ M_2(t) = c^2 x(1-x)t + \left(\left(2(\tilde{a} - \tilde{b}x) + c^2(1-2x)\right)(\tilde{a} - \tilde{b}x) - c^2(2\tilde{b} + c^2)x(1-x)\right)\frac{t^2}{2} + O(t^3) \\ M_3(t) = 3\left(2(\tilde{a} - \tilde{b}x) + c^2(1-2x)\right)c^2 x(1-x)\frac{t^2}{2} + O(t^3) \\ M_4(t) = 6c^4 x^2(1-x)^2\frac{t^2}{2} + O(t^3) \end{cases}$$

Thus we have the following approximation

$$G^{\text{Jacobi}}(t, x; d_1, d_2, w_1, w_2) \approx x^{w_1}(1-x)^{w_2} e^{-((b-a)v_1 + av_2 + c^2 v_1 v_2)t}\left(1 + \sum_{n=1}^{4} \hat{c}_n(x) M_n(t)\right) + O(t^3).$$