

# Pricing formulas, model error and hedging derivative portfolios

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## Abstract

We propose a method for extending a given asset pricing formula to account for two additional sources of risk: the risk associated with future changes in market-calibrated parameters and the remaining risk associated with idiosyncratic variations in the individual assets described by the formula. The paper makes simple and natural assumptions for how these risks behave. These extra risks should always be included when using the formula as a basis for portfolio management. We investigate an idealized typical portfolio problem, and argue that a rational and workable trading strategy can be based on minimizing the quadratic risk over the time intervals between trades. The example of the variance gamma pricing formula for equity derivatives is explored, and the method is seen to yield tractable decision strategies in this case.

## 1 Introduction

Any rational and practical approach to managing a financial portfolio naturally involves finding an optimal balance between a number of competing criteria. For example, a manager will wish to make good (but not excessive) use of available market data. They will also need to work with a pricing model in order to estimate risk and project to the future. However, the pricing model must balance complexity and accuracy against simplicity and implementability. They will want to trade but not too frequently, and before each trade will need to choose from many alternatives. They may need a set of investment decision strategies which are explicable to their boss, and will stand up to

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stringent scrutiny. They must be aware that strategies and parameters alike need to be updated continuously.

The paper focuses on an idealized market of derivatives on a single underlying asset  $S_t$  over a time period  $t \in [0, T]$ . We address the hypothetical situation of a manager who has written (sold) to a client a large and risky over-the-counter (nontradable) contingent claim  $F$  on  $S$ , with maturity date  $T$  (which we think of as quite large, say three years). Having accepted the large fee, and the large liability, the manager must now create a portfolio of investments in the exchange traded derivatives with the aim of adequately or even optimally hedging the risk. Thus the market consists of a number of different securities: a bank account, a non-dividend-paying stock, a family of derivatives on the stock and finally the over-the-counter claim. We assume

1. the bank account pays and charges a constant rate of interest  $r$ , which we take to be zero for simplicity. There is no limit on borrowing;
2. the stock price at time  $t$ , denoted  $S_t$ , is always positive;
3. the exchange traded derivatives are taken to be european puts and calls over a variety of strikes  $K^\alpha$  and maturities  $T^\alpha$  whose prices at time  $t$  are denoted  $D_t^\alpha$ ,  $\alpha = 2, \dots, M$ . By convention, we denote the stock itself by  $D^\alpha$  with  $\alpha = 1$ .
4. the over-the-counter claim price  $f_t$  is the value at time  $t$  of a claim given by a function  $F$  of the stock price history  $\{S_t\}_{t \leq T}$ . That is,  $F_T$  is an  $\mathcal{F}_T$  random variable. By convention, we denote the over-the-counter claim value by  $D^\alpha$  with  $\alpha = 0$ .
5. the market is frictionless: there is perfect liquidity, no transaction costs, unlimited shortselling, and the market is open at all times.

We assume the manager will set up an initial portfolio at  $t = 0$ , and then be making single trades at certain times  $t_1 < t_2 < \dots < t_k < \dots < T$ . In principle, these will be stopping times, but for this paper we take the deterministic values  $t_i = i\delta t$  for  $\delta t$  fixed. Our problem is to provide this manager with a consistent, workable strategy for trading, and to measure the performance of this strategy. The following two questions are addressed in this paper:

- Q1 What is the condition for optimal allocation at time  $t = 0$ ?
- Q2 What is the optimal single (or double or triple) trade which can be made at time  $t_i, i > 0$ ?

It is supposed that the rational manager will base their strategy on a derivative pricing model whose historical performance has been studied, benchmarked, and found to lead to acceptable modeling errors. This model is taken to be a finite parameter “risk neutral” pricing formula, whose parameters  $\theta^a$ ,  $a = 1, \dots, N$  are to be calibrated to the observed prices of the  $M - 1$  derivatives (where  $N \ll M$ ). Of course, in contradiction to the modeling assumptions, the calibrated values  $\hat{\theta}_t^a$  will depend on the

time  $t$ , reflecting changing market conditions. Furthermore, the calibrated model will generally fail to match all observed prices at a given time, the differences being thought of as “idiosyncratic errors”. The key point of the paper is to extend the original model by including minimal additional assumptions on changing parameters and idiosyncratic errors which are consistent with the observed performance of the model. These extra effects must be included in any rational portfolio strategy.

We argue that a rational strategy for the manager will be to minimize the total *risk-neutral* variance of the portfolio returns over each inter-trading time interval, without regard to the mean portfolio return. “Risk-neutral” refers to the measure which correctly prices the current market but not historical values. Variance optimization is equivalent to optimization with a quadratic utility function, which is an acceptable approximation to the general utility function over short time-intervals. Disregarding the mean return is consistent with a no-arbitrage condition true in an efficient market, but also justifiable pragmatically because the manager is profiting from the large fee collected for underwriting the claim  $F$ , and needs only consolidate that fee with minimal risk. Furthermore, estimating mean returns over short time intervals is a game for speculators and arbitrageurs, not hedgers.

Before each trade the manager will perform a number of steps:

1. place the payoff of newly expired options (if any) in the bank account;
2. observe the market prices;
3. recalibrate the risk-neutral parameters to the new data;
4. estimate the value of the liability  $F$  using the recalibrated pricing model;
5. find the optimal single trade which will minimize the estimated *risk-neutral* variance of the portfolio over the period until the next trading time.

The main theorem of this paper states that the risk-minimal portfolio for any trading time-interval is unique, as is the optimal single trade at any time.

We then discuss models such as the VG model [MCC99] for which the stock process is Markovian and has a closed form characteristic function. It is shown that the Fourier transform pricing method of [CM00] extends and allows for efficient computation of portfolio variances, and hence optimal portfolios and trades.

One important outcome of the present discussion is a clear operational definition of essential concepts of hedging which shows how these concepts can be refined and extended: delta hedging, gamma hedging, vega hedging, model recalibration. Furthermore, the inclusion of risk associated to “idiosyncratic errors” breaks the degeneracy (nonuniqueness) of the hedging decisions derived from naive delta-hedging (in practise by biasing trading toward near-the-money instruments). For a detailed discussion of hedging and derivatives from the trading perspective, see [Tal97]. For a discussion of model risk, see [Reb01]. For another approach to optimal investment in derivatives see [CJM01].

## 2 The model

### 2.1 The option pricing formula

The option pricing formula is assumed to arise via arbitrage pricing theory set in the risk-neutral filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbf{Q})$  over a time horizon  $[0, T]$ . For each fixed set of parameters  $\theta = (\theta^a)_{a=1}^N \in \mathbf{R}^N$ , the stock is a positive  $\mathbf{Q}$ -martingale (“cádlág”) (recall the interest rate is taken to be zero). We consider models in which  $S_t$  is one component of a multidimensional Markov process  $(S_t, Y_t)$  whose values are observable in the market: in our examples  $Y$  will have dimensions zero and one. Let  $\mathcal{Y}_t = \sigma\{S_\tau, Y_\tau, \tau \leq t\} \subset \mathcal{F}_t$  be the filtration of market observables.

The pricing formula giving the value at time  $t$  for any European style contingent claim with payoff  $F_T$  at date  $T$  is simply

$$F(t, S, Y, \theta) = E(F_T(S_T) | \mathcal{Y}_t, S_t = S, Y_t = Y, \theta \text{ fixed}), \quad t \in [0, T] \quad (1)$$

#### Examples:

1. Black–Scholes model: The single parameter can be taken as  $\theta = \log \sigma$  and the stock process is given by  $S_t = S_0 \exp[e^\theta W_t - e^{2\theta} t/2]$  (there are no extra  $Y$  variables). The pricing formula if  $F_T = (S_T - K)^+$  is simply the Black–Scholes call option formula.
2. Variance–Gamma model ([MCC99]) Here the log return process  $X_t = \log(S_t/S_0)$  is defined to be the Lévy process [Ber96],[IW89]:

$$X_t = \mu_\theta t + \int_0^t \int_{-\infty}^{\infty} x N^{(\nu_\theta)}(dx dt) \quad (2)$$

whose jump intensity measure is

$$\nu_\theta(x) = \frac{\alpha e^{-|x|/\eta_\pm}}{|x|}, \quad \pm x > 0 \quad (3)$$

for three positive parameters  $\theta = (\alpha, \eta_+, \eta_-)$ . The characteristic function turns out to be

$$\begin{aligned} \Phi_{X_t}(u) &= E(e^{iuX_t}) = [\Phi(u)]^t \\ \Phi(u) &= \left[ \frac{1}{(1 - i\eta_+ u)(1 + i\eta_- u)} \right]^\alpha \cdot e^{i\mu_\theta u} \end{aligned} \quad (4)$$

The martingale condition on  $S$  needs  $\eta_+ < 1$  and is then equivalent to  $\Phi(-i) = 1$ , so

$$e^{-\mu_\theta} = \left[ \frac{1}{(1 - \eta_+)(1 + \eta_-)} \right]^\alpha \quad (5)$$

With this condition satisfied, then the stock process  $S_t$  has the form

$$S_t = S_0 + \int_0^t \int_{-\infty}^{\infty} (e^x - 1) S_{\tau-} \tilde{N}^{(\nu_\theta)}(dx d\tau) \quad (6)$$

where  $\tilde{N}$  denotes the compensated (martingale) process

$$\tilde{N}^{(\nu_\theta)}(dx dt) = N^{(\nu_\theta)}(dx dt) - \nu_\theta(x) dx dt \quad (7)$$

3. Stochastic volatility models: Standard stochastic volatility models [Hes93],[PS00b],[PS00a],[DPS00] such as

$$\begin{aligned} dS_t &= \sqrt{v_t} S_t dW_t^1 \\ dv_t &= (a - bv_t) dt + \sigma \sqrt{v_t} dW_t^2 \end{aligned} \quad (8)$$

can be considered. This model is Markovian in  $(S_t, v_t)$ , not in  $S_t$  alone and the option pricing formula at time  $t$  depends on the values  $(S_t, v_t)$ . Our approach will be to follow [BZ01] and treat the stochastic squared volatility  $v_t$  as the “effective observable” defined by

$$v_t = \lim_{\Delta t \downarrow 0} \lim_{N \rightarrow \infty} N^{-1} \sum_{i=1}^N |\log(S_i/S_{i-1})|^2 \quad (9)$$

where  $S_i \equiv S_{t-(N-i)\Delta t/N}$ . This model has four parameters  $(a, b, \sigma, \rho)$  where  $d\langle W^1, W^2 \rangle = \rho dt$ .

Many models like the VG model and some stochastic volatility models have a closed formula for the characteristic function  $\Phi_{X_t}$ . In such cases [CM00] have shown how the Fast Fourier transform method provides a numerically efficient method for evaluating (1). For example, the European call payoff function with a strike  $K = e^k$  can be written as a complex contour integral along the shifted contour  $(-\infty - i\epsilon, \infty - i\epsilon)$  for any  $\epsilon > 0$ :

$$(S - e^k)^+ = \frac{1}{2\pi} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} S^{iu+1} e^{-iuk} Q(u) du \quad (10)$$

where

$$Q(u) = \frac{1}{iu - u^2} \quad (11)$$

To take the expectation to evaluate (1), plug in this formula and interchange the  $du$  and  $E(\cdot)$  integrals

$$\begin{aligned} F(t, S) &= \frac{1}{2\pi} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} E\left(S_T^{iu+1} e^{i(u-i)X_{T-t}} e^{-iuk} Q(u)\right) du \\ &= \frac{1}{2\pi} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} S_t^{iu+1} \Phi(u - i)^{T-t} e^{-iuk} Q(u) du \end{aligned} \quad (12)$$

The explicit integral can now be evaluated (approximately) very efficiently for a linearly spaced family of log strike values  $\{k_0 + n\delta k\}_{n=0, \pm 1, \pm 2, \dots}$  using the fast Fourier transform. Similar formulas hold for put options and the over-the-counter claim itself if its payoff has a known Fourier transform.

## 2.2 Varying market conditions

A careful trader using a pricing formula such as (1) will wish to correct for changing market conditions. It will be supposed that prior to its practical application (to hedging) the performance of the formula has been carefully benchmarked against real market data and a historical time series of mean square estimated values  $\hat{\theta}_t$  (as described in section §3.1) has been observed. When we allow the vector of parameters  $\theta$  to become a process, it is natural to pick a gaussian mean-reverting process and therefore we assume:

- A1** The time varying parameters  $\theta_t^i$  form an  $\mathbf{R}^N$ -valued Ornstein–Uhlenbeck Ito diffusion (gaussian, mean reverting). The noise process driving  $\theta_t$  is independent of the noise process driving  $S_t$ .

Let  $\mathcal{G}_t = \sigma\{\theta_s, s \leq t\} \subset \mathcal{F}_t$ .

A consistent pricing formula based on the filtration  $\mathcal{Y}_t \times \mathcal{G}_t$  is given by

$$\tilde{F}(t, S, Y, \theta) = E(F_T(S_T) | \mathcal{Y}_t \times \mathcal{G}_t, S_t = S, Y_t = Y, \theta_t = \theta), \quad t \in [0, T] \quad (13)$$

Note that  $\tilde{F}$  depends on the detailed modeling of the process (described of course by extra parameters), and hence  $\tilde{F} \neq F$ . For the present paper we regard use of  $\tilde{F}$  in place of  $F$  as a change in the underlying pricing model, and is hence “against the rules”.

Since our purpose is rather to use  $F$  to find a hedging strategy which will include low order approximations to account for variations in  $\theta$ , the point of view of varying parameters can be simplified. We adopt a natural compromise position which is to regard the parameters  $\theta$  as constant over the intertrading intervals  $[t_{i-1}, t_i)$ . At each time  $t_i$  the parameters are assumed to jump according to a *discrete*  $\mathbf{R}^N$ -valued OU process which is independent of the stock price  $S_{t_i}$ . Over the time period  $[t_{i-1}, t_i)$ , the stock price is assumed to evolve according to the model with fixed  $\theta = \theta_{t_{i-1}}$ . The distribution of  $\theta_{t_i}$  conditioned on  $\theta_{t_{i-1}}$  is gaussian. To keep the discussion simple, we will focus on the conditional covariance estimated by the quadratic variation statistic:

$$\hat{\Theta}^{ab} = \frac{1}{n} \sum_{k=1}^n [\hat{\theta}_{\tilde{t}_k}^a - \hat{\theta}_{\tilde{t}_{k-1}}^a][\hat{\theta}_{\tilde{t}_k}^b - \hat{\theta}_{\tilde{t}_{k-1}}^b], \quad a, b = 1, \dots, N \quad (14)$$

obtained from a series of times  $\tilde{t}_0, \tilde{t}_1, \dots, \tilde{t}_n$  (all prior to the trading times) with  $\tilde{t}_i - \tilde{t}_{i-1} = \delta t$ .

## 2.3 Idiosyncratic variations

Even if the model parameters  $\theta$  are confidently known at time  $t$ , the observed option prices  $D^\alpha$  will not fit the formula perfectly. There are certainly model errors which are idiosyncratic to each exchange traded security, and a careful trader must also account for this. As before, it is supposed that these idiosyncratic errors have been benchmarked by observing how the formula matches real data and that the process

$$\mathcal{E}_t^\alpha = D_t^\alpha - F^\alpha(t, S_t, \theta_t) \quad (15)$$

has been sampled over time. We assume

**A2** the errors  $\mathcal{E}_t$  form a multidimensional Ornstein–Uhlenbeck Ito diffusion, independent of the noise processes driving  $S_t$  and  $\theta_t$ . For simplicity, we assume that the covariance of  $\mathcal{E}_{t_i}$  conditioned on  $\mathcal{E}_{t_{i-1}}$  is diagonal.

Trading instincts suggest that the predicted variances will depend most strongly on two variables, namely moneyness  $\kappa = K/S$  and time to maturity  $\tau = T - t$ , both of which vary in time for a specific derivative  $\alpha$ . Therefore we adopt the following crude method for predicting the idiosyncratic variances: Let the variance of  $\mathcal{E}_{t+\delta t}^\alpha$  conditioned on  $\mathcal{E}_t^\alpha$  be

$$\hat{V}_t^\alpha = S_t^2 \hat{v}(K^\alpha/S_t, T^\alpha - t) \quad (16)$$

where  $\hat{v}$  is the following historically observed quantity which depends on moneyness and time to maturity:

$$\hat{v}(\kappa, \tau) = \frac{1}{n\delta t} \sum_{k=1}^n S_{\tilde{t}_k}^{-2} [\hat{\mathcal{E}}_{\tilde{t}_k+\delta t}^{\alpha(\tilde{t}_k, \kappa, \tau)} - \hat{\mathcal{E}}_{\tilde{t}_k}^{\alpha(\tilde{t}_k, \kappa, \tau)}]^2 \quad (17)$$

where  $\{\tilde{t}_k\}$  is a sequence of times prior to the trading times  $t_i$ . Here  $\alpha(t, \kappa, \tau)$  is defined to be that derivative whose characteristics  $(K^\alpha, T^\alpha)$  at time  $t$  most nearly match  $(\kappa, \tau)$ , i.e. the  $\alpha$  which minimizes

$$d(\alpha; (\kappa, \tau)) = (K^\alpha/S_t - \kappa)^2 + ((T^\alpha - t)/\tau - 1)^2 \quad (18)$$

### 3 Optimal trading

We now develop in detail the problem of optimal trading at time  $t_i$ . In what follows we use the shorthand notation  $S_i \equiv S_{t_i}$ , etc. for quantities evaluated at time  $t_i$ . The formulation we will adopt will take into account the following information at time  $t_i$ : the observed market prices  $D_i^\alpha$ , the empirical covariance  $\hat{\Theta}^{ab}$ , and the empirical variances  $\hat{V}_i^\alpha$ .

We assume that immediately following the previous trading time  $t_{i-1}$  the (self-financing) portfolio consists of  $\pi_{i-1}^\alpha$  units of each derivative, the liability for the claim  $D^0$  (so  $\pi^0 = -1$  always), and the remaining value  $B_{i-1}$  in the bank account. The portfolio value  $X$  is

$$X_{i-1} = \sum_{\alpha=0}^M \pi_{i-1}^\alpha D_{i-1}^\alpha + B_{i-1} \quad (19)$$

$$B_{i-1} = X_{i-1} - \sum_{\alpha} \pi_{i-1}^\alpha D_{i-1}^\alpha \quad (20)$$

#### 3.1 Calibration to observed prices

At time  $t_i$ , we compare the observed market prices  $D_i^\alpha, \alpha = 1, \dots, N$  to the model prices to update the market parameters  $\theta$ . Following [MCC99], the vector of estimated parameters  $\hat{\theta}_i$  is the minimizer of

$$\min_{\theta} \left\{ \sum_{\alpha} |\log(D_i^\alpha) - \log(F^\alpha(t_i, S_i, \theta))|^2 \right\} \quad (21)$$

Note that by definition  $D^0 = F^0$  and  $D^1 = F^1 = S$  so the sum is over  $\alpha = 2, \dots, N$ . Refining (21) by weighting by trading volume might be considered.

### 3.2 Marking to market

Given the recalibrated market parameters  $\hat{\theta}_i$  and current market prices  $D_i^\alpha$ ,  $\alpha \geq 1$ , the updated portfolio value can be calculated for time  $t_i$ . The best estimate of the current value of the contingent claim  $D^0$  is the expected value at  $t_i$  of the (discounted) final claim:

$$D_i^0 = F^0(t_i, S_i, \hat{\theta}_i) \quad (22)$$

(this explains why  $\mathcal{E}^0 = 0$  always). Therefore the updated portfolio value, just prior to trading, can be written

$$X_i = \sum_{\alpha \geq 0} \pi_{i-1}^\alpha F^\alpha(t_i, S_i, \hat{\theta}_i) + \sum_{\alpha \geq 2} \pi_{i-1}^\alpha \mathcal{E}_i^\alpha + B_{i-1} \quad (23)$$

### 3.3 Finding the optimal trade

To determine the optimal trade (i.e. to find the best new portfolio weights  $\pi_i^\alpha, B_i$ ), we project

$$X_i = \sum_{\alpha \geq 0} \pi_i^\alpha F^\alpha(t_i, S_i, \hat{\theta}_i) + \sum_{\alpha \geq 2} \pi_i^\alpha \mathcal{E}_i^\alpha + B_i \quad (24)$$

forward to the next trading time  $t_{i+1}$  and consider the (stochastic) value  $X_{i+1}$ . This is characterized by the values  $\pi_i^\alpha, B_i$ , the new market prices  $D_{i+1}^\alpha$ , and the new parameter values  $\hat{\theta}_{i+1}$ . Consider the difference  $\Delta X \equiv X_{i+1} - X_i = \sum_{\alpha \geq 0} \pi_i^\alpha \Delta X^\alpha$  where

$$\begin{aligned} \Delta X^\alpha &= [F^\alpha(t_{i+1}, S_{i+1}, \hat{\theta}_i) - F^\alpha(t_i, S_i, \hat{\theta}_i)] \\ &\quad + [F^\alpha(t_{i+1}, S_{i+1}, \hat{\theta}_{i+1}) - F^\alpha(t_{i+1}, S_{i+1}, \hat{\theta}_i)] + [\mathcal{E}_{i+1}^\alpha - \mathcal{E}_i^\alpha] \\ &\equiv \Delta X_1^\alpha + \Delta X_2^\alpha + \Delta X_3^\alpha \end{aligned}$$

The middle term  $\Delta X_2$  will become a bit awkward, so we reorganize it by Taylor expanding in powers of  $\widehat{\Delta \theta}_i = \hat{\theta}_{i+1} - \hat{\theta}_i$ :

$$F^\alpha(t_{i+1}, S_{i+1}, \hat{\theta}_{i+1}) - F^\alpha(t_{i+1}, S_{i+1}, \hat{\theta}_i) = \sum_{n \geq 1} (\partial_{\hat{\theta}}^n F^\alpha)(t_{i+1}, S_{i+1}, \hat{\theta}_i) \widehat{\Delta \theta}_i^n \quad (25)$$

We have adopted a condensed multiindex notation so that  $\partial_{\hat{\theta}}^n \equiv \partial_{\hat{\theta}^1}^{n_1} \dots \partial_{\hat{\theta}^N}^{n_N}$  and  $\widehat{\Delta \theta}^n \equiv (\hat{\theta}^1)^{n_1} \dots (\hat{\theta}^N)^{n_N}$  with  $n = (n_1, \dots, n_N)$ .  $n \geq 1$  is shorthand for  $\sum_i n_i \geq 1$ .

In what follows, let  $E_i(\cdot)$  denote conditional expectations  $E_Q(\cdot | \mathcal{F}_{t_i})$ . We consider first the mean of  $\Delta X$ . Note  $E_i(\Delta X_1^\alpha) = 0$  since  $F^\alpha$  is a martingale. In general, however,  $E_i(\Delta X_2^\alpha) \neq 0, E_i(\Delta X_3^\alpha) \neq 0$  since conditional means of OU processes are never identically zero. Statistically, mean returns are small, difficult to measure and very unstable in time. From a trading point-of-view, seeking mean returns over short time intervals is the job of the speculator, the day-trader and the arbitrageur. For our hedger, it makes no sense to trade on expected returns. Therefore, from both the statistical and trading points of view it makes sense to introduce an extra assumption

**A3** The trader will ignore the mean returns of  $\Delta X_2, \Delta X_3$ , and will trade as if  $E_i(\Delta X_2^\alpha) = 0$  and  $E_i(\Delta X_3^\alpha) = 0$ .

Thus we assume that when conditioned on  $\mathcal{F}_i, \widehat{\Delta\theta}_i$  and  $\mathcal{E}_{i+1}^\alpha - \mathcal{E}_i^\alpha$  are taken to be mean zero gaussians, with conditional covariances  $\widehat{\Theta}^{ab}$  and  $\widehat{V}_i^\alpha$ .

Under these assumptions, the variance or total quadratic risk

$$R \equiv \text{var}_i(\Delta X) = \sum_{\alpha\beta} \pi_i^\alpha \pi_i^\beta R^{\alpha\beta}$$

decomposes into three terms

$$\begin{aligned} R^{\alpha\beta} &\equiv \text{Cov}_i(\Delta X_1^\alpha; \Delta X_1^\beta) + \text{Cov}_i(\Delta X_2^\alpha; \Delta X_2^\beta) \\ &\quad + \text{Cov}_i(\Delta X_3^\alpha; \Delta X_3^\beta) \\ &= R_1^{\alpha\beta} + R_2^{\alpha\beta} + R_3^{\alpha\beta} \end{aligned}$$

- $R_1$  is the risk associated with changes in the underlier  $S$ ;
- $R_2$  is the risk associated with changes in the market reflected in evolving parameter values;
- $R_3$  is the risk associated with deviations between observed prices and model prices.

This type of decomposition is quite general.

We have now succeeded in expressing the quadratic portfolio risk in terms of the pricing formulas  $F^\alpha$ , the conditional covariance matrix  $\widehat{\Theta}^{ab}$  and the variances  $\widehat{V}^\alpha$ .

**Proposition 1** *Consider a pricing model of the above type under assumptions **A1,A2,A3**. Then*

1. *there is a unique portfolio  $\pi_i^*$  with  $\pi_i^0 = -1$  which minimizes the quadratic risk*

$$\pi_i^* = \arg \min_{\pi: \pi^0 = -1} \sum_{\alpha, \beta \geq 0} \pi^\alpha \pi^\beta [R_1^{\alpha\beta} + R_2^{\alpha\beta} + R_3^{\alpha\beta}] \quad (26)$$

*over the period  $[t_i, t_{i+1}]$  given by the solution of*

$$R^{0\alpha} = \sum_{\beta > 0} R^{\alpha\beta} \pi_i^{*\beta}, \alpha > 0 \quad (27)$$

2. *for any portfolio  $\pi_{i-1}$  with  $\pi^0 = -1$ , there is a unique optimal single trade consisting of  $\lambda^*$  units of the  $\gamma^*$ th derivative,  $\gamma^* \geq 1$  where:*

$$(\lambda^*, \gamma^*) = \arg \min_{\lambda, \gamma \geq 1} \sum_{\alpha, \beta \geq 0} (\pi_{i-1}^\alpha + \lambda \delta_\gamma^\alpha) (\pi_{i-1}^\beta + \lambda \delta_\gamma^\beta) [R_1^{\alpha\beta} + R_2^{\alpha\beta} + R_3^{\alpha\beta}] \quad (28)$$

*Here arg min denotes the solution of the associated minimization problem.*

**Proof:** We need only note that  $R_1$  is positive definite while  $R_2, R_3$  are positive semidefinite matrices.  $\square$

Selecting the optimal single trade given  $\pi_{i-1}$  is a simple matter given the matrix  $R$ . The optimal amount  $\lambda^{*,\gamma}$  to trade of a given derivative  $\gamma$  is

$$\lambda^{*,\gamma} = -\frac{\sum_{\beta} R^{\gamma\beta} \pi_{i-1}^{\beta}}{R^{\gamma\gamma}} \quad (29)$$

Such a trade decreases the unimproved portfolio risk  $\sum_{\alpha,\beta} \pi_{i-1}^{\alpha} \pi_{i-1}^{\beta} R^{\alpha\beta}$  by the amount  $(\sum_{\beta} R^{\gamma\beta} \pi_{i-1}^{\beta})^2 / R^{\gamma\gamma}$ ; the optimal asset to trade is simply that  $\gamma^*$  which shows the maximal improvement.

Similar arguments give one the optimal double, triple trade etc.

## 4 Implementing the method

Implementation of the method for a particular trading formula depends on efficient determination of the risk matrix  $R^{\alpha\beta}$  before every trade. We demonstrate how this can be done by generalizing the Fast Fourier Transform (FFT) method of [CM00] when the log-return process  $X_t = \log(S_t/S_0)$  has a known characteristic function  $\Phi(u)$ . We suppose that for each  $\alpha = 0, 1, \dots, M$  the payoff function  $F^{\alpha}(S_T)$  has a Fourier formula similar to (10) but with the function  $Q(u)$  replaced by  $Q^{\alpha}(u)$ .

First note that  $R_3$  is always simply the diagonal matrix

$$R_3^{\alpha\beta} = \delta^{\alpha\beta} \hat{V}_i^{\alpha} = \delta^{\alpha\beta} S_i^2 \hat{v}(K^{\alpha}/S_i, T^{\alpha} - t_i) \quad (30)$$

$R_1, R_2$  are more complicated. We note that they are symmetric matrices which fall naturally into block submatrices

$$R = \begin{bmatrix} R_{00} & R_{01} & R_{0c} & R_{0p} \\ R_{10} & R_{11} & R_{1c} & R_{1p} \\ R_{c0} & R_{c1} & R_{cc} & R_{cp} \\ R_{p0} & R_{p1} & R_{pc} & R_{pp} \end{bmatrix} \quad (31)$$

where  $R_{0c}$  is  $1 \times M_c, R_{cp}$  is  $M_c \times M_p$  etc. where  $M_c, M_p$  are the number of traded calls and puts. Put–call parity leads to a number of relations amongst the components:

$$R_{cc} - R_{cp} = R_{c1}, \quad R_{cp} - R_{pp} = R_{1p}, \quad \text{etc.} \quad (32)$$

We indicate here how to compute the parts of  $R_{cc}$  with fixed maturities  $T^{\alpha}, T^{\beta}$  and any log strikes  $k^{\alpha}, k^{\beta}$ .

For  $R_1$  terms note that

$$\begin{aligned} R_{1,cc}^{\alpha\beta} &= E_i \left( [F_{i+1}^{\alpha} - F_i^{\alpha}] [F_{i+1}^{\beta} - F_i^{\beta}] \right) \\ &= E_i \left( F_{i+1}^{\alpha} F_{i+1}^{\beta} \right) - F_i^{\alpha} F_i^{\beta} \end{aligned} \quad (33)$$

The usual approach is to Taylor expand in powers of  $\Delta t$  which leads to generalized delta–gamma hedging. However with the FFT method, we can perform the one remaining expectation leading to an explicit double Fourier integral

$$\begin{aligned} R_{1,cc}^{\alpha\beta} &= \frac{1}{(2\pi)^2} \int \int S_i^{i(u_1+u_2)+2} e^{-iu_1k^\alpha - iu_2k^\beta} Q(u_1)Q(u_2) \\ &\quad \times \left[ \Phi(u_1 + u_2 - 2i)^{\Delta t} - \Phi(u_1 - i)^{\Delta t} \Phi(u_2 - i)^{\Delta t} \right] \\ &\quad \times \Phi(u_1 - i)^{T^\alpha - t_{i+1}} \Phi(u_2 - i)^{T^\beta - t_{i+1}} du_1 du_2 \end{aligned} \quad (34)$$

A single application of the two dimensional FFT solves this problem for a family of calls with linearly spaced log strikes but with fixed dates  $T^\alpha, T^\beta$ .

Now we seek a similar formula for

$$\begin{aligned} R_{2,cc}^{\alpha\beta} &= E_i \left[ \left( F_{i+1}^\alpha(\hat{\theta}_{i+1}) - F_{i+1}^\alpha(\hat{\theta}_i) \right) \left( F_{i+1}^\beta(\hat{\theta}_{i+1}) - F_{i+1}^\beta(\hat{\theta}_i) \right) \right] \\ &\quad - E_i \left[ F_{i+1}^\alpha(\hat{\theta}_{i+1}) - F_{i+1}^\alpha(\hat{\theta}_i) \right] E_i \left[ F_{i+1}^\beta(\hat{\theta}_{i+1}) - F_{i+1}^\beta(\hat{\theta}_i) \right] \end{aligned} \quad (35)$$

When we plug in the Fourier integral, we highlight the  $\hat{\theta}$ –dependence of  $\Phi$  by writing  $\Phi(u; \hat{\theta})$ . Then using the independence of  $S$  and  $\theta$  expectations leads to

$$\begin{aligned} R_{2,cc}^{\alpha\beta} &= \frac{1}{(2\pi)^2} \int \int e^{-iu_1k^\alpha - iu_2k^\beta} Q(u_1)Q(u_2) \\ &\quad \times E_i \left( S_{i+1}^{i(u_1+u_2)+2} \left[ \Phi(u_1 - i; \hat{\theta}_{i+1})^{T^\alpha - t_{i+1}} - \Phi(u_1 - i; \hat{\theta}_i)^{T^\alpha - t_{i+1}} \right] \right. \\ &\quad \times \left. \left[ \Phi(u_2 - i; \hat{\theta}_{i+1})^{T^\beta - t_{i+1}} - \Phi(u_2 - i; \hat{\theta}_i)^{T^\beta - t_{i+1}} \right] \right) du_1 du_2 \\ &\quad - \frac{1}{(2\pi)^2} \int \int e^{-iu_1k^\alpha - iu_2k^\beta} Q(u_1)Q(u_2) \\ &\quad \times E_i \left( S_{i+1}^{iu_1+1} \left[ \Phi(u_1 - i; \hat{\theta}_{i+1})^{T^\alpha - t_{i+1}} - \Phi(u_1 - i; \hat{\theta}_i)^{T^\alpha - t_{i+1}} \right] \right) \\ &\quad \times E_i \left( S_{i+1}^{iu_2+1} \left[ \Phi(u_2 - i; \hat{\theta}_{i+1})^{T^\beta - t_{i+1}} - \Phi(u_2 - i; \hat{\theta}_i)^{T^\beta - t_{i+1}} \right] \right) du_1 du_2 \end{aligned} \quad (36)$$

which becomes

$$\begin{aligned} &= \frac{1}{(2\pi)^2} \int \int S_i^{i(u_1+u_2)+2} e^{-iu_1k^\alpha - iu_2k^\beta} Q(u_1)Q(u_2) \\ &\quad \left( \Phi(u_1 + u_2 - 2i; \hat{\theta}_i)^{\Delta t} E_i \left[ \left[ \Phi(u_1 - i; \hat{\theta}_{i+1})^{T^\alpha - t_{i+1}} - \Phi(u_1 - i; \hat{\theta}_i)^{T^\alpha - t_{i+1}} \right] \right. \right. \\ &\quad \times \left. \left[ \Phi(u_2 - i; \hat{\theta}_{i+1})^{T^\beta - t_{i+1}} - \Phi(u_2 - i; \hat{\theta}_i)^{T^\beta - t_{i+1}} \right] \right] \\ &\quad - \Phi(u_1 - i; \hat{\theta}_i)^{\Delta t} \Phi(u_2 - i; \hat{\theta}_i)^{\Delta t} E_i \left[ \Phi(u_1 - i; \hat{\theta}_{i+1})^{T^\alpha - t_{i+1}} - \Phi(u_1 - i; \hat{\theta}_i)^{T^\alpha - t_{i+1}} \right] \\ &\quad \times E_i \left[ \Phi(u_2 - i; \hat{\theta}_{i+1})^{T^\beta - t_{i+1}} - \Phi(u_2 - i; \hat{\theta}_i)^{T^\beta - t_{i+1}} \right] \left. \right) du_1 du_2 \end{aligned} \quad (37)$$

The remaining  $\theta$ -expectations will be quite complicated since they involve differencing the function  $\Phi$ . A pragmatic approach is simply to Taylor expand in powers of  $\widehat{\Delta\theta}$ . Then the leading term is

$$\frac{1}{(2\pi)^2} \int \int S_i^{i(u_1+u_2)+2} e^{-iu_1k^\alpha - iu_2k^\beta} Q(u_1)Q(u_2) \Phi(u_1 + u_2 - 2i; \hat{\theta}_i)^{\Delta t} \times \sum_{a,b=1}^N \left[ \partial_{\theta^a} (\Phi(u_1 - i; \hat{\theta}_i)^{T^\alpha - t_{i+1}}) \partial_{\theta^b} (\Phi(u_2 - i; \hat{\theta}_i)^{T^\beta - t_{i+1}}) \hat{\Theta}^{ab} \right] du_1 du_2 \quad (38)$$

with higher order terms given by more complicated integrals. These integrals are efficiently calculated by FFT as before.

## 4.1 Example: the Black–Scholes model

The Fourier technique above applies to this model, since the characteristic function is simply

$$\Phi(u) = \exp[-e^{2\theta}(u^2 + iu)/2] \quad (39)$$

The matrices  $R_1, R_2, R_3$  can be interpreted in the standard language of hedging. For example, by taking the second order Taylor approximation to  $R_1$  in powers of  $\Delta t$  and setting  $\sum_{\alpha,\beta} R_1^{\alpha\beta} \pi^\alpha \pi^\beta = 0$ , we obtain the conditions for a delta–gamma–hedged portfolio. Note that in our present formulation, the optimal solutions make this term small but not zero. Similarly, setting the lowest order  $\Delta t$  term of  $R_2$  equal to zero yields a vega–hedged portfolio which shows vanishing first–order sensitivity to changes in the estimated volatility. Again our optimal portfolios will be approximately but not perfectly vega–hedged. Finally, making  $R_3$  small amounts to choosing preferentially amongst the more liquid derivatives, since their idiosyncratic risks are generally relatively smallest. In practise, this means favoring investment in near–the–money assets to far in/out–of–the–money derivatives.

We illustrate the method with an example which shows the optimal hedge for a Euro–call contract with strike 1 and maturity  $T = 3$  (all times in years) whose total Black–Scholes value is \$1. In a pure Black–Scholes market where the current stock price is \$1, with puts and calls with maturity 1/2 and 11 log strikes  $k = (-0.5, -0.6, \dots, 1.5)$ , we calculated the optimal hedge (excluding the underlier  $S$ ) for the period  $\delta t = 1/12$ . For illustrative purposes, we take the other parameters to be  $\hat{\Theta} = 10^{-2}$ ,  $\hat{v}_c = \hat{v}_p = 10^{-3}(6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6)$ . The graph shows the value bought of each call option (+) and the value (×) sold of each put. The ratio of quadratic hedged risk to unhedged risk is 0.0319, the hedged to unhedged delta ratio is  $-0.0441$  and the hedged to unhedged gamma ratio is  $-0.7411$ . With optimized choices for FFT parameters the computation required 21.3 megaflops.

## 4.2 Example: the VG model

We begin by noting that for this model the parameterization  $\hat{\theta} = (\alpha, \eta_+, \eta_-)$  is not a linear space, but can be associated with a natural riemannian manifold with metric given by the Fisher information metric. Therefore, we replace assumption **A1** by the natural family of Ornstein-Uhlenbeck processes which constitute a preferred class of mean-reverting processes on a general riemannian manifold. However, we expect this nontrivial geometry to affect only the higher order  $\widehat{\Delta\theta}$  terms in the Taylor expansion for  $R_2$  and for a preliminary implementation we would work in the tangent space and keep only first order Taylor terms. Thus we would use (38) with the estimated  $\widehat{\Theta}$  matrix as before.

Now, since the characteristic function for the VG model has the simple analytic expression (4), the important double integrals can be efficiently calculated via the FFT.

It is now known that the VG model does not work well in pricing derivatives across maturities, and so our framework cannot be expected to perform adequately. Work is in progress to extend the VG model to include nontrivial time correlations.

## 4.3 Example: the affine stochastic volatility model

In the model given by equation (8) the characteristic function  $\Phi(t, u) = E(e^{iX_t u})$  for  $X_t = \log(S_t/S_0)$  depends on the current level of squared volatility  $v_0$ :

$$\Phi(t, u) = \exp[C(t, u) + D(t, u)v_0] \quad (40)$$

where

$$\begin{aligned} C(t, u) &= \frac{a}{\sigma^2} \left[ (b - i\rho\sigma u + d)t - 2 \log \left( \frac{1 - ge^{dt}}{1 - g} \right) \right] \\ D(t, u) &= \frac{b - i\rho\sigma u + d}{\sigma^2} \left[ \frac{1 - e^{dt}}{1 - ge^{dt}} \right] \\ g &= \frac{b - i\rho\sigma u + d}{b - i\rho\sigma u - d} \\ d &= \sqrt{(i\rho\sigma u - b)^2 - \sigma^2(-iu - u^2)} \end{aligned} \quad (41)$$

where  $\rho$  is the correlation between  $W^1, W^2$ . Our general method still applies, however the dependence on  $v_0$  leads to more complicated integrals for  $R_1, R_2$  which are deserving of further study.

## 5 Conclusions

The general picture is that a finite parameter pricing model can be augmented by an assumption that parameters vary stochastically (hopefully slowly) in time, independently of other sources of randomness. Furthermore, one need not assume idiosyncratic errors

are zero: these too can be modeled by a simple independent random process. These assumptions can be built naturally into hedging strategies which minimize risk over a given time interval between trades. This risk naturally decomposes into a sum of three terms: a term involving changes in the underlier, a term involving changes in the parameters and a term involving the idiosyncratic errors. Expectations involving changes in the underlier are with respect to the risk-neutral measure, and thus incorporate a non-zero price of risk, whereas the remaining probabilities are with respect to the physical (historical) probabilities. Decision making for the resulting hedging strategy at a given trading time requires knowing how to calculate model prices and generalized greeks for fixed values of the parameters (using the analytical formulas, Monte Carlo or some other method) and knowing the following data:

1. the current prices of all exchange traded securities;
2. historically estimated unconditional variances for the changes in parameters and idiosyncratic pricing errors.

The strategy which results has a number of practical advantages:

- It provides a systematic treatment of model errors, hence leading to smaller absolute hedging errors;
- Since decision making depends only on current market prices and benchmark statistics which are relatively robust and stable in time, we expect hedging errors over successive trading intervals to exhibit only small statistical dependence. This means that the probability distribution of the hedging error over longer terms can be estimated by the central limit theorem;
- The method can be implemented efficiently in a wide variety of models, both simple and sophisticated, which include pure diffusions, jump diffusions and pure jump models.

It should be observed once more that we are introducing new parameters (here  $\hat{V}, \hat{\Theta}$ ) which describe idiosyncratic errors and the changes in model parameters. Of course these themselves are best estimated dynamically in time. Thus the finicky practitioner might be lead to introduce further parameters which describe the changes in the parameters which describe the changes in model parameters, etc! The logic of the paper still applies.

Further work is suggested along a number of lines:

- A comparison of the performance of our method relative to the standard discrete time hedging strategies;
- The statistical dependence of successive hedging errors should be studied and compared to that of other strategies;

- A study of whether the strategy is consistent with observed trading patterns in the market or whether on the contrary traders use different criteria in decision making;
- In the current paper we avoid discussing transaction costs by assuming a fixed schedule of single trades. The work should be extended to include transaction costs, and which will allow questions of when it is optimal to trade, and when double or higher order trades are preferable over single trades;
- The method should also be applied to models for other markets, such as the commodity, bond and foreign exchange markets.

To conclude, the current paper is not intended to present a mathematically consistent model. On the contrary, it addresses the question of how any mathematically consistent pricing model can be extended for use by a rational and prudent practitioner who cares about the inevitable errors made by the model.

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