The portfolio selection problem via Hellinger processes

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Abstract

This paper addresses Merton's portfolio optimization problem in the setting of an exponential Lévy stock market. We investigate three canonical examples of utility functions, $-e^{-x}, x^p/p, \log x$ and in each case give the general solutions of the primal optimal problem and two versions of the dual problem. To study the second version of the dual problem, we apply a new method which involves the introduction of generalized Hellinger processes such that the solution of the dual problem is that martingale measure which minimizes the Hellinger process at each instant in time. When these solutions are compared to those derived via duality relations, we find that the true solution of the dual problem fails to be a martingale in certain cases in which there is a no borrowing/shortselling constraint which becomes important.

Keywords: Information theory, optimal portfolios, Merton's problem, Lévy processes, primal and dual problems, jump diffusions

1 Introduction.

In this paper we study Merton's problem of finding for a given utility function $U(x)$ a wealth process $\hat{X}_t$ which maximizes the expected utility of wealth at time $T > 0$, in other words it solves the "primal problem"

$$u(x) = \sup_{X \in \mathcal{A}(x)} E(U(X_T)) \tag{1}$$

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where $\mathcal{A}(x)$ is a class of admissible wealth portfolios with initial value x at $t = 0$. Various approaches to Merton's problem have been developed, beginning with Merton's original method [17],[18] involving dynamic programming and Hamilton–Jacobi–Bellman (HJB) equations applicable to the Markovian setting, followed later by the duality methodology of convex analysis combined with martingale techniques [8], [12], [3]. In recent years, [7] has extended this framework to address processes of independent increments while [14],[20] have proved fundamental results for general semimartingales. The main feature of the application of martingale techniques to portfolio optimization is the derivation of the equivalent "first dual problem"

$$v(y) = \inf_{Y \in \mathcal{A}^*(y)} E(V(Y_T)) \tag{2}$$

where $\mathcal{A}^*(y)$ is an appropriately defined "dual" to $\mathcal{A}(x)$. This duality is constructed so that the pairs (U, V) and (u, v) are related by the Legendre transform and the optimizers satisfy the relationship $\hat{Y}(u'(x) = U'(\hat{X}(x))$. In ideal cases, the solution $\hat{Y}$ defines a measure Q equivalent to the physical measure P which is interpreted as the "pricing" martingale measure, and which can be used for example in expectation pricing of derivative securities in the market.

Counterexamples in the general semimartingale theory given in [14] show that the solution of the dual problem is sometimes a supermartingale, not a martingale. When this happens, the standard financial interpretation of martingale measures becomes obscured. This raises the important question of when (2) may be replaced by the easier (and financially natural) "second dual problem"

$$v(y) = \inf_{Y \in \mathcal{M}^a(y)} E(V(Y_T)) \tag{3}$$

where $\mathcal{M}^a(y)$ denotes a space of positive martingales (which therefore yields equivalent martingale measures).

In the present paper we address this question by analyzing in complete detail three canonical utility functions $-e^{-x}, x^p/p$ and $\log x$ in a market of the jump–diffusion type modelled by an exponentiated Lévy process in which the log stock returns jumps may be unbounded. For each utility, we solve the primal problem, we solve the second dual problem (3), and then compare the results to give an explicit check on the dual correspondence which is the main result of [14].

Our findings show that $-e^{-x}$ leads to a picture free of any pathological counterexamples. However, for x^p/p and $\log x$, no borrowing from the bank account or shortselling of the stock will be admissible which leads to the consequence that for certain parameters the solution of (2) is a supermartingale not a martingale and thus (3) cannot give the correct solution. In exactly the same cases, we will also observe that the solution of (3) yields a martingale measure Q which is not equivalent to the physical measure P (in other words, Q assigns zero probability to some events with positive P–probability).

These two pathologies seem to be directly related each other and to the presence of no borrowing/shortselling constraints in the problem.

Our examples for $-e^{-x}$ are also consistent with the main result of [20] which shows that when U is finite on all of $\mathbb{R}$ and satisfies certain weak conditions, the solution of the dual problem (2) is a martingale, and hence can be found by solving (3). However, this result was proved under the condition that the stock exhibits bounded jumps only and therefore does not apply directly to our example.

Our solution of the dual problem (3) will make use of a technique introduced in [2], based on ideas from information theory (that part of probability theory which addresses the notion of distance between probability measures). The concepts of Hellinger process and integral arise from a dynamical approach to the Kakutani-Hellinger distance between two probability measures, see [10] and [15], and have been applied in mathematical finance by [11]. Very recently, [7] proposed the Hellinger martingale measure as an alternative to the minimal and optimal variance martingale measures for the case of processes with independent increments. In that paper, the pricing measure derived from a specific power–law utility (with exponent $p = -1$) is shown to be identical to the equivalent martingale measure which is nearest to the physical measure P in the sense of the ($q = 1/2$) Hellinger distance. Thus, the paper shows a link between information theory and portfolio theory for a single example of utility function: unfortunately it assumes a bounded jump condition which is unnatural in finance. In [2], we strengthened the link between information theory and portfolio theory by demonstrating that Grandits' example can be extended to more general utility functions. Working in an exponential Lévy process market model, we showed that for the most general power law utility, for the logarithmic utility, and for exponential utility, one can in each case define a process which possesses a number of properties similar to the Hellinger processes. Then the pricing measure in each case was shown to be identical to the equivalent martingale measure which minimizes the corresponding generalized Hellinger process. In the case of exponential utility, the corresponding pricing measure is the minimal entropy martingale measure which was introduced by Frittelli in [4],[5] (see also [1] for related works).

The organization of the paper is as follows. Section 2 introduces Merton's problem in the context of the exponential Lévy market model and reviews some preliminary results. Section 3 describes the method of [2] which connects the optimal martingale $\hat{Y}$ for (3) with optimal generalized Hellinger processes, for each of the three classes of utility discussed above. In sections 4,5 and 6 we give complete solutions for the primal problem (9), the first dual problem (2) and the second dual problem (3). In each case, we determine the solution $\hat{Y}$ of the dual problem directly from the primal solution by the duality relations of Kramkov and Schachermayer. When U is a power law or the logarithm, we see it is possible that $\hat{Y}$ is only a supermartingale not a martingale. In these sections we derive equations for solutions of the second dual problem, and prove existence and uniqueness of their solutions $\bar{Y}$. In the particular case of exponential utility, we find that $\bar{Y}$ agrees with $\hat{Y}$ obtained in section 3 whereas for the power and log utility, we find that $\bar{Y}$ gives a martingale measure which fails to be equivalent to P when $\hat{Y}$ fails to be a martingale. Finally, in section 7, we illustrate these results with the example of the variance–gamma market developed by Madan et al [16]. This example shows that explicit formulas are possible for special forms of the Lévy measure.

2 Merton's problem.

We start with a filtered probability space $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{0\le t\le T},\ P)$, a one–dimensional Brownian motion W and a one–dimensional Poisson random measure $N(dt, dy)$ with Lévy measure $\nu(dy)$. The filtration is supposed to satisfy the usual conditions by which we mean right continuity and completeness i.e. $\mathcal{F}_t = \bigcap_{s>t}\mathcal{F}_s$ and $\mathcal{N} \subset \mathcal{F}_0$ where $\mathcal{N}$ is the set of $\mathcal{F}$-measurable and P-negligible events.

The financial market consists of a risk-free asset (bank account) B given by $B_t = e^{rt}$ and a stock S (risky asset) assumed governed by the following stochastic differential equation (SDE)

$$S_t = S_0+\int_0^t bS_{s-}ds+\sigma\int_0^t S_{s-}dW_s+\int_0^t\int S_{s-}\ z\ I_{\{|z|\le 1\}}\widetilde{N}(ds,dz)+\int_0^t\int S_{s-}\ z\ I_{\{|z|>1\}}\ N(ds,dz). \tag{4}$$

Here $\sigma > 0, b$ are constants and $r > 0$ is a constant interest rate. $\widetilde{N}$ is the compensated Poisson random measure given by $\widetilde{N}(dt, dz) = N(dt, dz) - dt\ \nu(dz)$. For a detailed exposition of Poisson random measures, including the generalized Ito formula and any unexplained notation, we refer the reader to [9]. We make two assumptions on the Lévy measure:

$$1 \wedge |z| \text{ is } \nu\text{–integrable}; \tag{5}$$

$$\text{supp}(\nu) = [-1, \infty). \tag{6}$$

An ideal market is assumed in which transaction costs and liquidity effects are neglected. The discounted stock price $B_t^{-1}S_t$ can be written as the Doléans–Dade exponential $B^{-1}S = S_0\mathcal{E}(L)$ of the following Lévy process (stationary process with independent increments)

$$L_t = (b-r)t + \sigma W_t + \int_0^t\int z\ I_{\{|z|\le 1\}}\ \widetilde{N}(ds,dz) + \int_0^t\int z\ I_{\{|z|>1\}}\ N(ds,dz). \tag{7}$$

Here $\mathcal{E}(L)$ is the unique solution to the (SDE) $dK = K_- dL,\quad K_0 = 1$.

In what follows we specialize to the case $r = 0$. We can reduce the general problem of constant $r \neq 0$ by making changes of variables $\tilde{S}_t = B_t^{-1}S_t, \tilde{B}_t = 1$ in a consistent fashion throughout the development.

Consider an investor who wants to invest in their wealth in this market in an optimal way over the period $[0, T]$. Letting π_t be the wealth invested at time t in the stock and making the usual self–financing requirement (meaning no money is withdrawn from or added to the portfolio), then the wealth process $X_t^{\pi,x}$ which follows from an initial endowment $x = X_0$ is given by

$$X_t^{\pi,x} = x + \int_0^t b\pi_s ds + \int_0^t \pi_s \left[\sigma dW_s + \int (zI_{\{|z|\le 1\}}\widetilde{N}(ds,dz) + zI_{\{|z|>1\}}N(ds,dz))\right] \quad (8)$$

The Merton problem for a given initial wealth x is now to determine the strategy π^* to be implemented over the investment horizon $[0,T]$ which maximizes the expected utility of the terminal wealth $X_T^{\pi^*}$. Utility, the quantification of the investor's tolerance of risk, is a function $U : \mathbb{R} \to [-\infty, \infty)$ which is strictly increasing, strictly concave, twice continuously differentiable and such that $U'(\underline{x}) = \infty, \quad U'(\infty) = 0$ where $\underline{x} = \inf\{x : U(x) > -\infty\}$. Thus the Merton problem is to produce (if possible) the maximizer π^* amongst admissible strategies $\mathcal{A}(0,x)$ for the "primal problem"

$$u(x) = \sup_{\pi\in\mathcal{A}(0,x)} E\left(U(X_T^{\pi,x})\right). \quad (9)$$

Here, an "admissible trading strategy" $\pi \in \mathcal{A}(t,x)$ over the period $[t,T]$ is a predictable process π such that $X^{\pi,x} \ge \underline{x}$, $\quad P$–almost–surely.

To study (9) it is useful to consider a dynamical version of the problem defined by

$$u(t,x) = \sup_{\pi\in\mathcal{A}(t,x)} E\left(U(X_T^{\pi}) \mid X_t^{\pi} = x\right). \quad (10)$$

Then in the markovian setting as we have here, the method of dynamic programming leads to the study of the HJB equation for $u(t,x)$:

$$\begin{cases} \frac{\partial u}{\partial t} + \sup_{\pi\in\mathbb{R}}\Big[\pi b u_x + \frac{1}{2}\pi^2\sigma^2 u_{xx} \\ \qquad + \int[u(x+\pi z) - u(x) - \pi z I_{\{|z|\le 1\}} u_x]\nu(dz)\Big] = 0 & t\in[0,T) \\ u(T,x) = U(x) & x \in \mathbb{R}^+ \end{cases} \quad (11)$$

As is now well known, the primal problem (9) can also be addressed by focusing on the Legendre transform V of U defined by

$$V(y) = \sup_{x>\underline{x}}[U(x) - xy]. \quad (12)$$

which is a strictly convex, twice differentiable function on $[0,\infty)$. Now one studies the "first dual problem"

$$v(y) = \inf_{Y\in\mathcal{A}^*(0,y)} E\left(V(Y_T)\right). \quad (13)$$

where $\mathcal{A}^*(0,y)$ is some space of processes which is dual to $\mathcal{A}(0,x)$ with $y = u'(x)$. Under some circumstances, [14],[20] have shown that the functions $u(x)$ and $v(y)$ can themselves be obtained from each other by using Legendre transform:

$$v(y) = \sup_{x \geq \underline{x}} [u(x) - xy], \quad u(x) = \inf_{y \geq 0} [v(y) + xy] \quad x, y \geq 0 \tag{14}$$

and the optimizers $\hat{X}(x), \hat{Y}(y)$ with $y = u'(x)$ are related by

$$\hat{X}(x) = -V'(\hat{Y}(y)), \qquad \hat{Y}(y) = U'(\hat{X}(x)) \tag{15}$$

If we replace $\mathcal{A}^*(0,y)$ by the space $\mathcal{M}^a(y)$ of local martingales Y such that SY is a local martingale, we obtain the "second dual problem"

$$v(y) = \inf_{Y \in \mathcal{M}^a(y)} E\left(V(Y_T)\right). \tag{16}$$

In some cases, the first and second dual problems have the same solution $\hat{Y}$ which can taken to be the conditional density $\hat{Y}_t = E(d\hat{Q}/dP|\mathcal{F}_t)$ of an absolutely continuous martingale measure $\hat{Q} \ll P$ interpreted as the martingale measure (pricing measure) which captures the risk preferences coded into the utility function U.

In this paper, we will treat the three important special cases of utility functions and their Legendre transforms:

$$U^{(q)}(x) = x^p/p, \; V^{(q)}(y) = -y^q/q, \; p = q/(q-1), \; p \in (-\infty, 0) \cup (0, 1) \tag{17}$$

$$U^L(x) = \log x, \quad V^L(y) = -\log y - 1 \tag{18}$$

$$U^E(x) = -e^{-x}, \quad V^E(y) = y(\log y - 1) \tag{19}$$

3 Hellinger processes and the dual problem.

Here we describe a method developed in [2] to solve the problem (16). We begin with the following proposition which gives a representation of $Y \in \mathcal{M}^a$, the density for the most general martingale measure, as the exponential of some (local) martingale with respect to the Brownian motion and the Poisson random measure.

Proposition 3.1 *Let Q be a probability measure absolutely continuous with respect to P with conditional density $Y_t = E(dQ/dP|\mathcal{F}_t)$. Then*

1. *Y can be written $\mathcal{E}(M)$ for a (local) martingale of the form*

$$M_t = \int_0^t \beta(\omega, s) dW_s + \int_0^t \int (Z(\omega, s, z) - 1) \widetilde{N}(ds, dz), \tag{20}$$

for a predictable W-integrable process β and non-negative $\mathcal{P} \times \mathcal{B}(I\!R)$-measurable function $Z(\omega, t, z)$ (we will omit ω in the notation of β, Z and simply denote them by $\beta_t, Z_t(z)$; recall $\mathcal{P} \subset \mathcal{F} \times \mathcal{B}(I\!R^+)$ is the predictable σ–algebra) which satisfies

$$\int_0^T \int |Z_t(z) - 1| \nu(dz) dt < \infty, \qquad P\text{-a.s.} \tag{21}$$

2. *The following two conditions are equivalent:*
   (a) $Y \in \mathcal{M}^a$
   (b) *β and Y satisfy*

$$\int_0^T \int |z||Z_t(z) - I_{\{|z| \le 1\}}| \nu(dz) dt < \infty \qquad P\text{-a.s.} \tag{22}$$

$$b + \sigma\beta_t + \int z \left[Z_t(z) - I_{\{|z| \le 1\}} \right] \nu(dz) = 0, \qquad dt\text{-a.e.} \tag{23}$$

Proof:

1. For details about the representation (20), we refer the reader to chapter III, section 4 of [10].
2. S is a Q-local martingale iff $YS = S_0\mathcal{E}(L + M + [L, M])$ is a P-local martingale iff $L + M + [L, M]$ is a P-local martingale and the result follows by direct calculation. □

The following theorem introduces the definition of three generalized Hellinger processes and provides some of their properties.

Theorem 3.2 *Let M be a local martingale such that $1 + \Delta M > 0$ P-almost surely and let $Y = \mathcal{E}(M)$. In the following, $h^{(q)}, h^L, h^E$ are predictable increasing processes and*

1. *for $0 < q < 1$, the process $V^{(q)} = -Y^q/q$ is a negative local submartingale which can be written*

$$\begin{aligned} V_t^{(q)} &= -\int_0^t V_{s-}^{(q)} \, dh_s^{(q)} + \text{local martingale} \qquad (24) \\ h_t^{(q)}(\beta, Z) &= \frac{1}{2} q(1-q) \int_0^t \beta_s^2 ds - \int_0^t \int \left[Z_s(z)^q - 1 - q\left(Z_s(z) - 1\right)\right] \nu(dz) ds; \end{aligned}$$

2. *for $q < 0$, the process $V^{(q)} = -Y^q/q$ is a positive local submartingale which can be written*

$$\begin{aligned} V_t^{(q)} &= \int_0^t V_{s-}^{(q)} \, dh_s^{(q)} + \text{local martingale} \qquad (25) \\ h_t^{(q)}(\beta, Z) &= \frac{1}{2} q(q-1) \int_0^t \beta_s^2 ds + \int_0^t \int \left[Z_s(z)^q - 1 - q\left(Z_s(z) - 1\right)\right] \nu(dz) ds; \end{aligned}$$

3. The process $V^L = -\log(Y) - 1$ is a local submartingale which can be written

$$\begin{aligned} V_t^L &= h_t^L + \text{local martingale} \qquad (26) \\ h_t^L(\beta, Z) &= \frac{1}{2}\int_0^t \beta_s^2 ds + \int_0^t \int \left[-\log(Z_s(z)) + Z_s(z) - 1\right] \nu(dz)ds; \end{aligned}$$

4. The process $V^E = Y(\log(Y) - 1)$ is a local submartingale which can be written

$$\begin{aligned} V_t^E &= \int_0^t Y_{s-}\; dh_s^E + \text{local martingale} \qquad (27) \\ h_t^E(\beta, Z) &= \frac{1}{2}\int_0^t \beta_s^2 ds + \int_0^t \int \left[Z_s(z)\log(Z_s(z)) - Z_s(z) + 1\right] \nu(dz)ds. \end{aligned}$$

Proof. Proof in each case that V is a local submartingale is a straightforward consequence of the convexity of V and Jensen's inequality. Then the decompositions (25), (26), (27) and (28) follow as an immediate consequence of the Doob-Meyer decomposition and the generalized Ito formula. □

Remarks:

1. We call $h_t^{(q)}$, h_t^L and h_t^E the generalized Hellinger processes associated to the power, logarithmic and exponential utility functions respectively. For $0 < q < 1$ we obtain the Hellinger processes defined in [10] and the case $q < 0$ is a straightforward extension. The processes (27) and (28) were introduced in [13] where they were called Kullback-Leibler processes. We are grateful to A. Gushchin and Y. Kabanov informing us of this last paper.

2. When Y is defined by (20) with $\beta = \beta(t), Z = Z(t, z)$ deterministic functions, we write $Y \in \mathcal{M}^{a,\det}$, and in this case the processes $h = h(t)$ are themselves deterministic functions of t.

For the next result, we define Hellinger–like integrals for the above functions V by

$$H_t = E(V(Y_t)) \qquad (28)$$

for each $t \geq 0$. We also denote by $\mathcal{K}_t$ the space of $\mathcal{F}_{t^-}$ random variables (β, Y), $Y \geq 0$ which satisfy (23).

Theorem 3.3 *Consider one of the four utilities identified in Theorem 3.2. Let (β_t^*, Z_t^*) solve the problem*

$$\inf_{(\beta,Z)\in\mathcal{K}_t} \frac{dh_t(\beta, Z)}{dt} \qquad (29)$$

for all $t \in [0, T]$. Then (β_t^*, Z_t^*) can be taken as a deterministic process and $Y^* = y\mathcal{E}(M(\beta^*, Z^*))$ solves the second dual problem

$$\inf_{Y\in\mathcal{M}^a(y)} E(V(Y_T)) \qquad (30)$$

Proof: Clearly the solution (β_t^*, Z_t^*) to (29) can always be taken to be deterministic. We consider the case of $V = V^{(q)}, q > 0$, with the other three cases being similar. Let $Y \in \mathcal{M}^a(y)$ be arbitrary. Then

$$\begin{aligned} H_T(Y) = E(V(Y_T)) &= V(Y_0) - E\left(\int_0^T V(Y_\tau)\, dh_\tau\right) \\ &\geq V(Y_0) - \int_0^T H_\tau(Y) dh_\tau(\beta^*, Z^*) \end{aligned} \tag{31}$$

But $H_t(Y^*)$ satisfies the same integral relation with equality. Therefore

$$H_T(Y) \geq H_T(Y^*) \tag{32}$$

and the result is proved. □

In the examples of the next three sections, we will find optimizers $Y \in \mathcal{M}^{a,\text{det}}$, for which the following formulas are true:

$$H_t^{(q)} = -1/q + \int_0^t |H_s^{(q)}| dh_s^{(q)}, \qquad q \in (-\infty, 0) \cup (0, 1) \tag{33}$$

$$H_t^L = -1 + \int_0^t dh_s^L \tag{34}$$

$$H_t^E = -1 + \int_0^t dh_s^E \tag{35}$$

We also observe that for $Y \in \mathcal{M}^{a,\text{det}}$

$$E(V(yY_t)) = \begin{cases} y^q H_t^{(q)} & V = V^{(q)} \\ -\log y + H_t^L & V = V^L \\ y \log y + y H_t^E & V = V^E \end{cases} \tag{36}$$

4 Exponential utility.

In the next three sections, we give complete solutions of the primal and dual problems for our three families of utilities.

Theorem 4.1 *Let the utility function be $U(x) = -e^{-x}$.*

1. *The solution of the primal problem is the pair $(u, \hat{X}_T(x))$ where*

$$u(t, x) = -e^{K^E(T-t)-x} \tag{37}$$

and $\hat{X}_T(x)$ is given by (8) with the constant trading strategy $\pi_t = \pi^E$ which is the unique minimizer of the convex function

$$G(\pi) = -b\pi + \frac{1}{2}\sigma^2\pi^2 + \int [e^{-\pi z} - 1 + \pi z I_{\{|z| \leq 1\}}]\nu(dz) \tag{38}$$

and $K^E = G(\pi^E)$.

2. The solution of the second dual problem is the pair $(v, \bar{Y}_T(y))$ *where*

$$v(t, y) = y(\log y - 1 + K^E(T - t)). \tag{39}$$

and $\bar{Y}_T(y) = y\mathcal{E}(M)_T$ *where*

$$M_t := -\int_0^t \pi^E \sigma dW_s + \int_0^t \int [e^{-\pi^E z} - 1]\widetilde{N}(ds, dz) \tag{40}$$

3. The solutions to the two dual problems (13) and (16) coincide.

Remark: The quantity $E(Z_T \log Z_T)$ is the entropy of P relative to Q and we see that the solution of the resulting optimal problem gives the "minimal entropy martingale measure" put forward by Frittelli [4].

Proof:

1. Direct substitution of u into the HJB equation (11) leads to the minimization problem

$$K^E = \min_{\pi \in \mathbb{R}} G(\pi) \tag{41}$$

For any π

$$\int_{-1}^{1} |z|(e^{-\pi z} - 1)\nu(dz) \le C \int (1 \wedge z^2)\nu(dz) < \infty \tag{42}$$

Now let $\underline{\pi} = \inf\{\pi : \int_1^\infty z e^{-\pi z}\nu(dz) < \infty\}$. Then

$$G'(\pi) = -b + \sigma^2\pi + \int [-ze^{-\pi z} + zI_{\le}]\nu(dz) \tag{43}$$

is a continuous increasing function on $(\underline{\pi}, \infty)$ with $\lim_{\pi \to \underline{\pi}} G'(\pi) = -\infty$ and $\lim_{\pi \to \infty} G'(\pi) = \infty$. The unique root $\pi^E \in (\underline{\pi}, \infty)$ leads to the desired solution.

2. We apply the method outlined in section 3 to obtain a direct solution of the second dual problem. By (29), we need to minimize a concave function subject to a convex constraint (23) and by the Saddle Point Theorem [19] it is enough to consider the Lagrangian

$$L(Z, \beta, \lambda) \equiv \frac{dh_t(\beta,\ Z)}{dt} - \lambda \left[b + \sigma\beta + \int z[Z(z) - I_{\{|z| \le 1\}}]\nu(dz) \right]. \tag{44}$$

where λ is a Lagrange multiplier for the constraint. Solutions to (29) are given by solutions to the Euler-Lagrange equation $DL = 0$ where D is the Fréchet differential operator on functions of (Z, β, λ):

$$\beta = \lambda\sigma \tag{45}$$

$$Z(z) = e^{\lambda z} \tag{46}$$

$$0 = b + \sigma\beta + \int z[Z(z) - I_{\{|z| \le 1\}}]\nu(dz) \tag{47}$$

By inserting (45), (46) into (47), we derive an equation for λ:

$$0 = b + \sigma^2\lambda + \int z[e^{\lambda z} - I_{\leq}(z)]\nu(dz). \tag{48}$$

which by comparison with (43) is seen to have a unique solution $\lambda = -\pi^E$. One can easily check that the bounds (21),(22) hold for $Z = e^{-\pi^E z}$.

The minimal value of dh_t^E/dt is

$$\tilde{K}^E = \frac{(\pi^E)^2\sigma^2}{2} + \int [(-\pi^E z - 1)e^{-\pi^E z} + 1]\nu(dz) \tag{49}$$

which is a positive constant independent of t. Therefore integration of (35) leads to $H_t = -1 + \tilde{K}^E t$, and hence the dual value function is

$$v^E(y) = y(\log y - 1 + \tilde{K}^E T). \tag{50}$$

3. We need to verify that $\bar{Y}_T(y) = U'(\hat{X}_T(x))$ when $y = u'(x)$. It is enough to show $F_t = e^{-K^E t - \hat{X}_t(0)}$ satisfies the SDE $dF_t = F_{t-}dM_t$ which implies $\bar{Y}_t(1) = \mathcal{E}(M)_t = F_t$. By the generalized Ito formula with the SDE for $\hat{X}$,

$$\begin{aligned} dF_t &= F_{t-}\Big[\Big(-K^E - b\pi^E + \pi^{E^2}\sigma^2/2 + \int [e^{-\pi^E z} - 1 + \pi^E z I_{\{|z|\leq 1\}}\nu(dz)\Big)dt \\ &\quad -\pi^E\sigma dW_t + \int [e^{-\pi^E z} - 1]\widetilde{N}(dt, dz)\Big] \end{aligned} \tag{51}$$

The martingale terms agree with dM_t while the dt terms are zero since $K^E - G(\pi^E) = 0$.

One can also check that $\tilde{K}^E = -K^E + \pi^E G'(\pi^E) = -K^E$ and hence that (50) is the Legendre transform of $u(0, x)$. □

5 Power utility.

Theorem 5.1 *Let the utility function be $U(x) = x^p/p$ for $p \in (-\infty, 0) \cup (0, 1)$ and let $q = p/(p-1)$.*

1. *The solution of the primal problem is the pair $(u, \hat{X}_T(x))$ where*

$$u(t, x) = e^{K^{(p)}(T-t)}\ x^p/p \tag{52}$$

and $\hat{X}_T(x)$ is given by (8) with the trading strategy $\pi_t = \phi^{(p)}\hat{X}_t(x)$. The constants $K^{(p)}, \phi^{(p)}$ are determined by the concave function

$$F(\phi) = b\phi + (p-1)\sigma^2\phi^2/2 + p^{-1}\int [(1+\phi z)^p - 1 - p\phi z I_{\{|z|\leq 1\}}]\nu(dz) \tag{53}$$

$K^{(p)} = pF(\phi^{(p)})$ where

(a) *If* $F'(0) < 0$, $\phi^{(p)} = 0$;

(b) *If* $F'(1) > 0$, $\phi^{(p)} = 1$;

(c) *If* $F'(1) \le 0 \le F'(0)$, $\phi^{(p)} \in [0,1]$ *is the unique root of* $F'(\phi) = 0$.

2. *The solution of the first dual problem is the pair* $(v, \hat{Y}_T(y))$ *where*

$$v(t,y) = -e^{-K^{(p)}(T-t)(q-1)}\ y^q/q \tag{54}$$

and $\hat{Y}_T(y) = y\mathcal{E}(M)_T$ *where*

$$M_t := \int_0^t (p-1)\phi^{(p)}\sigma dW_s + \int_0^t \int [(1+\phi^{(p)}z)^{p-1} - 1]\widetilde{N}(ds,dz) - \int_0^t \phi^{(p)} F'(\phi^{(p)})ds \tag{55}$$

3. *The solution of the second dual problem (16) is the pair* $(\tilde{v}, \bar{Y}_T(y))$ *where*

$$\tilde{v}(t,y) = -e^{K^{(q)}(t-T)(q-1)}\ y^q/q \tag{56}$$

and $\bar{Y}_T(y)) = y\mathcal{E}(\bar{M})_T$ *with*

$$\bar{M}_t := \int_0^t (p-1)\tilde{\phi}^{(p)}\sigma dW_s + \int_0^t \int [\max(1+\tilde{\phi}^{(p)}z, 0)^{p-1} - 1]\widetilde{N}(dtdz). \tag{57}$$

Here $\tilde{\phi}^{(p)}$ *is the unique root of the equation*

$$b + (p-1)\sigma^2\phi + \int [z(\max(1+\phi z, 0))^{p-1} - zI_{\le}]\nu(dz) = 0 \tag{58}$$

4. *The solutions to the two dual problems (13) and (16) coincide if and only if* $F'(1) \le 0 \le F'(0)$.

Remarks:

1. If $F'(0) \le 0$ the optimal strategy is the risk free strategy $\hat{X}_t = B_t$ because the mean rate of return of the stock is lower than the risk–free rate. In the presence of unbounded jumps short–selling the stock involves the risk of a negative portfolio value, and thus the optimal solution has zero investment in the risky asset.

2. If $F'(1) \ge 0$, then $(\phi^{(p)}, K^{(p)}) = (1, F(1))$, and the solution is $\hat{X}_t = S_t$, the maximally risky strategy which can be tolerated without violating the no–borrowing constraint.

3. We see from the equations for $(\phi^{(p)}, K^{(p)})$ that $\hat{Y}$ is a P–martingale if and only if $\phi^{(p)}F'(\phi^{(p)}) = 0$ (i.e. $F'(1) \le 0$). If $F'(1) > 0$, then $\hat{Y}$ is a supermartingale. One can also check that $S\hat{Y}$ is a P–martingale if and only if $F'(0) \ge 0$. If $F'(0) < 0$ then $S\hat{Y}$ is only a supermartingale.

Proof:

1. Substitution for u into the HJB equation (11) and implementing the constraint $X_t \geq 0$ leads to
$$\sup_{0\leq\phi\leq 1}\left[-uK^{(p)} + puF(\phi)\right] = 0 \tag{59}$$
Since the function F is continuous and concave on $[0,1]$ the stated result follows.

2. By (15) the solution of the first dual problem is given by:
$$\hat{Y}_T(y) = ye^{-K^{(p)}T}\ \hat{X}_T(1)^{p-1} \tag{60}$$
where $\hat{X}_T(1)$ is the optimal wealth portfolio with initial value 1. Application of the generalized Ito formula applied to $Y_t = ye^{-K^{(p)}t}\ \hat{X}_t(1)^{p-1}$ using (8) leads to $\hat{Y}_T = y\mathcal{E}(M_T)$ with
$$\begin{aligned} M_t &= \int_0^t (p-1)\phi^{(p)}\sigma dW_s + \int_0^t\int [(1+\phi^{(p)}z)^{p-1} - 1]\widetilde{N}(ds,dz) \\ &\quad + \int_0^t \Big[-K^{(p)} + (p-1)[b\phi^{(p)} + (p-2)\phi^{(p)2}\sigma^2/2] \\ &\quad + \int [(1+\phi^{(p)}z)^{p-1} - 1 - (p-1)\phi^{(p)}zI_{\leq}]\nu(dz)\Big]dt \end{aligned} \tag{61}$$
The ds term simplifies to $-\int_0^t \phi^{(p)}F'(\phi^{(p)})ds$.

3. We follow the solution to the second dual problem in §4, with the difference that here we need to introduce an extra lagrange term $\int_{-1}^{\infty} l(z)Z(z)\nu(dz)$ to (44) to ensure the constraint $Z(z)\geq 0$. The Euler–Lagrange equation corresponding to this problem gives
$$0 = |q|(1-q)\beta - \lambda\sigma \tag{62}$$
$$0 = |q|Z(z)^{q-1} - |q| - l(z) + \lambda z \tag{63}$$
$$0 = b + \sigma\beta + \int z(Z - I_{\leq})\nu(dz) \tag{64}$$
together with the conditions
$$l(z)Z(z) = 0; \quad l(z)\geq 0; \quad Z(z)\geq 0 \tag{65}$$
For each value of λ one can solve (62),(63),(65) for β, Z and plug into equation (64), leading to the equation $G(\lambda) = 0$. We divide the discussion into three cases
   (a) $\lambda \leq -|q|$: Here $l(z) = 0$ for $z \in [-|q/\lambda|, \infty)$ and $Z(z) = 0$ for $z \leq -|q/\lambda|$.
   (b) $\lambda \in [-|q|, 0]$: Here $l(z) = 0$ for all z.

(c) $\lambda \geq 0$: Here $l(z) = 0$ for $z \in [-1, |q/\lambda|]$ and $Z(z) = 0$ for $z \geq |q/\lambda|$.

In all three cases, the function G can be written

$$G(\lambda) = b + \frac{\lambda\sigma^2}{|q|(1-q)} + \int z \left[\max(1 - \lambda/|q|z, 0)^{1/(q-1)} - 1\right] \nu(dz) \tag{66}$$

So defined, G is continuous, strictly increasing, satisfies

$$\lim_{\lambda\to-\infty} G(\lambda) = -\infty, \lim_{\lambda\to\infty} G(\lambda) = \infty \tag{67}$$

and thus has a unique root which by comparison with (58) is $\lambda^* = -|q|\tilde{\phi}^{(p)}$.

The minimal value of $dh_t^{(q)}/dt$ is the positive constant

$$K^{(q)} = \frac{|q|(\tilde{\phi}^{(p)})^2\sigma^2}{2(1-q)} + \tag{68}$$
$$\int \left[\max(1 + \tilde{\phi}^{(p)}z, 0)^{q/(q-1)} - 1 - q\left(\max(1 + \tilde{\phi}^{(p)}z, 0)^{1/(q-1)} - 1\right)\right] \nu(dz)$$

Simple manipulation shows that $K^{(q)} = p\tilde{F}(\tilde{\phi}^{(p)})$ where

$$\tilde{F}(\phi) = b\phi + (p-1)\sigma^2\phi^2/2 + p^{-1}\int[\max(1 + \phi z, 0)^p - 1 - p\phi z I_{\leq}]\nu(dz)$$

Therefore integration of (33) leads to the dual value function

$$\tilde{v}(t, y) = -e^{K^{(q)}(t-T)}y^q/q \tag{69}$$

4. A direct comparison of $\bar{Y}$ to $\hat{Y}$ shows they coincide when $F'(1) \leq 0 \leq F'(0)$. Precisely when either $\hat{Y}$ or $S\hat{Y}$ fails to be a martingale (if $F'(0) < 0$ or $F'(1) > 0$) we see that $\bar{Y}$ vanishes on sets of positive P–measure, and thus leads to a martingale measure which is absolutely continuous but not equivalent to P. □

6 Logarithmic utility.

The proof of the following case parallels the proof of the power law result. These results are similar to results of [6].

Theorem 6.1 *Let the utility function be $U(x) = \log x$.*

1. *The solution of the primal problem is the pair* $(u, \hat{X}_T(x))$ *where*

$$u(t,x) = \log x + K^L(T-t) \tag{70}$$

and $\hat{X}_T(x)$ *is given by (8) with the trading strategy* $\pi_t = \phi^L X_t$*. The constants* K^L, ϕ^L *are determined by the concave function*

$$F(\phi) = b\phi - \sigma^2\phi^2/2 + \int [\log(1+\phi z) - \phi z I_{\leq}]\nu(dz) \tag{71}$$

$K^L = F(\phi^L)$ *where*

   (a) *If* $F'(0) < 0$*,* $\phi^L = 0$*;*

   (b) *If* $F'(1) > 0$*,* $\phi^L = 1$*;*

   (c) *If* $F'(1) \leq 0 \leq F'(0)$*,* $\phi^L \in [0,1]$ *is the unique root of* $F'(\phi) = 0$*.*

2. *The solution of the first dual problem is the pair* $(v, \hat{Y}_T(y))$ *where*

$$v(t,y) = -\log y - 1 + K^L(T-t) \tag{72}$$

and $\hat{Y}_T(y) = y\mathcal{E}(M_T)$ *with* M_t *given by*

$$-\int_0^t \phi^L \sigma dW_s + \int_0^t \int [(1+\phi^L z)^{-1} - 1]\widetilde{N}(ds,dz) - \int_0^t \phi^L F'(\phi^L) ds \tag{73}$$

3. *The solution of the second dual problem is the pair* $(\tilde{v}, \bar{Y}_T(y))$ *where*

$$\tilde{v}(t,y) = -\log y - 1 + \tilde{K}^L(T-t) \tag{74}$$

and $\bar{Y}_T(y)) = y\mathcal{E}(M)_T$ *where*

$$M_t := -\int_0^t \tilde{\phi}^L \sigma dW_s + \int_0^t \int [\max(1+\tilde{\phi}^L z, 0)^{-1} - 1]\widetilde{N}(ds,dz) \tag{75}$$

Here $\tilde{\phi}^L$ *is the unique root of*

$$b - \sigma^2\phi + \int [z(\max(1+\phi z, 0))^{-1} - zI_{\leq}]\nu(dz) = 0 \tag{76}$$

4. *The solutions to the two dual problems (13) and (16) respectively coincide if and only if* $F'(1) \leq 0 \leq F'(0)$*.*

7 Example: the Variance–Gamma model.

This market model, analysed in [16], has the stock process S_t defined to be the exponential pure–jump Lévy process (4) with $\sigma = 0$ and Lévy measure

$$\nu(dz) = \tilde{\nu}(dx) = \frac{\gamma e^{-|x|/\eta_\pm}}{|x|} dx, \quad \pm x > 0 \tag{77}$$

under the change of variables $z = e^x - 1$, for three positive parameters γ, η_+, η_-. In this section we will calculate closed formulas for the quantities $F'(0), F'(1)$ for power utilities with $p \in (-\infty, 0) \cup (0, 1)$, and show that all of the distinct possibilities of Theorem 5.1 may arise in this model. First,

$$F'(0) = b + \int_1^\infty z\nu(dz) = b + \gamma \int_{\log 2}^\infty (e^x - 1)e^{-\eta_+ x} \frac{dx}{x} \tag{78}$$

may be interpreted as the excess expected return, and may clearly be negative leading to the optimal portfolio $\hat{X}_t = B_t$. We now evaluate

$$\begin{aligned} F'(1) - F'(0) &= \gamma \int_{-1}^\infty z \left[(1+z)^{p-1} - 1\right] \nu(dz) = \gamma \int_{-\infty}^\infty (e^x - 1)(e^{-\rho x} - 1)\tilde{\nu}(dx) \\ &= \gamma \int_0^\infty \Big[e^{(1-\rho-\eta_+)x} - e^{(-\rho-\eta_+)x} - e^{(1-\eta_+)x} + e^{-\eta_+ x} + \\ &\qquad e^{(-1+\rho-\eta_-)x} - e^{(\rho-\eta_-)x} - e^{(-1-\eta_-)x} + e^{-\eta_- x}\Big] \frac{dx}{x} \end{aligned} \tag{79}$$

which is always negative (here $\rho = (1-p) > 0$). All these integrals can be evaluated in closed form using

$$\int_0^\infty [e^{-ux} - e^{-vx}] \frac{dx}{x} = \log(v/u) \tag{80}$$

leading to the general formula

$$F'(1) - F'(0) = \gamma \log \left[\frac{(\eta_+ - \rho)(\eta_- + \rho)(\eta_+ - 1)(\eta_- + 1)}{\eta_+ \eta_- (\eta_+ - \rho - 1)(\eta_- + \rho + 1)} \right] \tag{81}$$

Certainly it is possible for $F'(1) > 0$ and hence the optimal portfolio may turn out to be $\hat{X}_t = S_t$.

8 Conclusion.

We have given a complete solution of the primal and dual optimal problems for three classic utility functions $-e^{-x}, x^p/p, \log x$ in exponential Lévy markets. These examples clearly exhibit a range of possibilities which cannot arise in continuous market models. For the utilities defined on $\mathbb{R}^+$,

1. The optimal portfolio may vary in only a piecewise smooth way with the parameters;
2. the optimal martingale measure (the solution to (16)) may fail to be equivalent to the physical measure;
3. either $S\hat{Y}$ or the dual optimizer $\hat{Y}$ may fail to be a martingale.

Moreover, these three features are all strongly related to the positive wealth constraint. Finally, we see clearly that no such pathologies arise in the case of $-e^{-x}$ which is finite on $\mathbb{R}$.
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