Portfolio Choice with Jumps: A Closed Form Solution*

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Abstract

We analyze the consumption-portfolio selection problem of an investor facing both Brownian and jump risks. By adopting a factor structure for the asset returns and decomposing the two types of risks on a well chosen basis, we provide a new methodology for determining the optimal solution up to an implicitly defined constant, which in some cases can be reduced to a fully explicit closed form, irrespectively of the number of assets available to the investor. We show that the optimal policy is for the investor to focus on controlling his exposure to the jump risk, while exploiting differences in the asset returns diffusive characteristics in the orthogonal space. We also examine the solution to the portfolio problem as the number of assets increases and the impact of the jumps on the diversification of the optimal portfolio.

Keywords: Optimal portfolio; jumps; Merton problem; closed form solution.

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1. Introduction

Economists have long been aware of the potential benefits of international diversification, while at the same time noting that the portfolios held by actual investors typically suffer from a home bias effect (see e.g., Grubel (1968), Levy and Sarnat (1970), Solnik (1974), Grauer and Hakansson (1987)). One possible explanation is due to the risk of contagion across markets in times of crisis (see e.g., Claessens and Forbes (2001), Longin and Solnik (2001), Ang and Chen (2002), Bae et al. (2003) and Hartmann et al. (2004).) A natural way to capture contagion mathematically is by introducing jumps. Jumps of correlated sign will generate the type of asymmetric correlation across markets that is often used to justify the home bias exhibited by investors' portfolios. Namely, when a downward jump occurs, negative returns tend to be experienced simultaneously across most markets, which then results in a high positive correlation in bear markets. When no jump occurs, the only source of correlation is that generated by the driving Brownian motions and will typically be much lower.

Studying the impact of jumps on portfolio choice has a long history, going back to Merton (1971), who first studied a continuous-time consumption-portfolio problem. Many papers have considered the portfolio problem, either in the simple one-period Markowitz setting or in the more complex Merton setting, when asset returns are generated by jump processes, for instance Poisson processes, stable processes or more general Lévy processes. Early papers include Aase (1984), Jeanblanc-Picque and Pontier (1990) and Shirakawa (1990). More recently, see Han and Rachev (2000) for a study of the Markowitz one-period mean-variance problem when asset returns follow a stable-Paretian distribution; Kallsen (2000) for a study of the continuous-time utility maximization in a market where risky security prices follow Lévy processes, and a solution (up to integration) for power, logarithmic and exponential utility using the duality or martingale approach; Choulli and Hurd (2001) give solutions up to constants of the primal and dual Merton portfolio optimization problem for the exponential, power and logarithmic utility functions when a risk-free asset and a exponential Lévy stock are the investment assets; Liu et al. (2003) study the implications of jumps in both prices and volatility on investment strategies when a risk-free asset and a stochastic-volatility jump-diffusion stock are the available investment opportunities; Emmer and Klüppelberg (2004) study a continuous-time mean-variance problem with multiple assets; Madan (2004) derives the equilibrium prices in an economy with single period returns driven by exposure to explicit non-Gaussian systematic factors plus Gaussian idiosyncratic components. Cvitanic et al. (2005) propose a model where the asset returns have higher moments due to jumps and study the sensitivity of the investment in the risky asset to the higher moments, as well as the resulting utility loss from ignoring the presence of higher moments.

The potential role of jumps in generating contagion across markets, and hence limiting the benefits of diversification, has been investigated by Das and Uppal (2004), who evaluate the effect on portfolio choice of systemic risk, defined as the risk from infrequent events that are highly correlated across a large number of assets. They find that systemic risk reduces the gains from diversifying across a range of assets, and makes leveraged portfolios more susceptible to large losses. Upon calibrating their model to index returns, they find that the loss from the reduction in diversification is not substantial. Ang and Bekaert (2002) consider a two-regime model in a discrete-time setting, one with low correlations and low volatilities, and one with higher correlations, higher volatilities, and lower conditional means. They find that the existence of a high-volatility bear market regime does not negate the benefits of international diversification for an investor who dynamically rebalances his portfolio in response to regime switches.

In the presence of jumps, the portfolio choice problem had not been amenable to a closed-form solution. With n assets, one must solve numerically an $n-$dimensional nonlinear equation (as for instance in Das and Uppal (2004).) With more efficient global markets, capital flows and a considerably larger number of available assets to invest in, an investor has more investment opportunities than ever before. We would certainly like to be able to consider models with a large number of assets n, where the characteristic Lévy triple has specific forms of n dependence. This is difficult to do using existing methodologies.

Our contribution to the solution method is to show that by adopting a factor structure for the asset returns and selecting a *well-chosen basis* in the space spanned by the jump vector and the covariance matrix of returns, we can obtain the solution in closed-form, irrespectively of the number of assets. In our model, the structure of the Brownian volatility matrix is taken to reflect the existence of one or more economic sectors, each sector comprising a large number of related companies (or countries). The jump process takes the form of a "contagion risk factor" which generates highly correlated negative returns across the range of assets. We start with the case where the Brownian covariance matrix corresponds to only one economic sector, and then consider the more general case where the economy consists of m sectors or regions of the world (each consisting of k firms or countries).

The closed form solution allows us to do explicit comparative statics, and give a precise characterization of the optimal portfolio and resulting wealth dynamics. In particular, we are able to distinguish between the optimal portfolio positions in the space spanned by the jump risk (which the investor will attempt to limit) and those in the orthogonal space (where the investor will seek to exploit the opportunities arising from the traditional risk-return trade-off.)

Naturally, the closed form solution requires that one assumes specific jump distributions and utility functions. But in general, we can still reduce the starting $n-$dimensional problem to one of finding a constant in the one sector case, and to an $m-$dimensional constant vector in the $m-$sector case. This dimension reduction is particularly important when the number of assets available to the investors is large, that is when $n \to \infty$; in the $m-$sector case this corresponds to a fixed number of sectors or regions of the world (m fixed) and a growing number of firms in each sector or countries in each region ($k \to \infty$), with the total number of assets given by $n = mk$.

In general terms, the structure of the solution is as follows. If there is enough cross-sectional variability in the expected excess returns, then the investor will place a linearly increasing amount of wealth in the risky assets as the number of assets n grows. This, in turn, leads to increasing expected return and volatility of the portfolio value, both growing linearly in the number of assets. And the optimal policy is to control the exposure to jumps by keeping it bounded as the number of assets grows. As a result, the exposure to jumps is dwarfed by the exposure to diffusive risk asymptotically in n. Indeed, the additional investments in the risky assets are entirely in the direction that is orthogonal to the jump risk; they are all achieved with zero net additional exposure to the jump risk. In other words, the optimal investment policy is to control the overall exposure to jump risk, and then exploit, in the orthogonal space, any perceived differences in expected returns and diffusive variances. But in the special case where the expected excess returns have little variability in the orthogonal space, the opportunities for diversification effects are weak since controlling the exposure to jumps trumps other concerns (including the usual diversification policy.) The optimal portfolio in this case is not much better protected against those correlated jumps than a nondiversified portfolio.

The rest of the paper is organized as follows. In Section 2, we present our model of asset returns, and examine the investor's portfolio selection problem. In Section 3, we consider a one sector economy where the n risky assets have the same jump size and derive the optimal portfolio weights in closed form. In Section 4, we study the dependence of the optimal portfolio weights on the arrival intensity of the jumps, their magnitude, and the degree of risk aversion of the investor. In Section 5, we extend the model to an m-sector economy where sectors have different jump sizes and show how to solve the optimal portfolio problem in that case, again in closed form. In Section 6, we study the situation where the risky assets are subject to jumps but the investor assumes that the returns are driven exclusively by Brownian motions but with first and second moments that are adjusted to reflect the presence of the jumps; that is, the investor makes a partial adjustment to account for the jumps, by lumping them together with Brownian volatility, and compare that partial adjustment to the exact solution consisting in fully incorporating the effect of the jumps. Extensions, limitations of our theory, and conclusions are in Section 7.

2. The Portfolio Selection Model

2.1. Asset Return Dynamics

Like most of the above-mentioned literature, our paper focuses on Merton's problem of maximizing expected utility of terminal wealth by investing in a set of risky assets. That is, we select the amounts to be held in the n risky assets and the riskless asset at times $t \in [0, T]$. The available investment opportunities consist of a riskless asset with price $S_{0,t}$ and n risky assets with prices $\mathbf{S}_t = [S_{1,t}, \ldots, S_{n,t}]'$. These follow the exponential Lévy dynamics

$$\frac{dS_{0,t}}{S_{0,t}} = r dt, \tag{2.1}$$

$$\frac{dS_{i,t}}{S_{i,t-}} = (r + R_i)\, dt + \sum\nolimits_{j=1}^{n} \sigma_{i,j} dW_{j,t} + J_i Z_t dN_t,\ i = 1, \ldots, n \tag{2.2}$$

with a constant rate of interest $r \geq 0$. N_t is a scalar Poisson process with constant intensity λ, $\mathbf{W}_t = [W_{1,t}, \ldots, W_{n,t}]'$ is an $n-$dimensional standard Brownian motion, and $J_i Z_t$ is the random jump amplitude. Z_t is a scalar random variable with probability measure $\nu(dz)$ on $[0,1]$. The economy-wide jump amplitude Z_t is scaled on an asset-by-asset basis by the scaling factor $J_i \in [-1, 0]$. We assume that the individual Brownian motions, the Poisson jump and the random variables Z are mutually independent. The quantities R_i, σ_{ij} and jump scaling factors J_i are constant parameters. We write $\mathbf{R} = [R_1, \ldots, R_n]'$, $\mathbf{J} = [J_1, \ldots, J_n]'$, and assume that

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{pmatrix} \tag{2.3}$$

is a nonsingular matrix. Let $\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}'$.

The expected excess returns and the return covariance matrix over short time intervals are given by

$$\hat{R} = R + \lambda J \bar{Z} \tag{2.4}$$

$$\hat{\Sigma}_{ij} = (\sigma\sigma')_{ij} + \lambda J_i J_j \bar{Z}^2 \tag{2.5}$$

where $\bar{Z} = \int_0^1 z\nu(dz), \bar{Z}^2 = \int_0^1 z^2 \nu(dz)$.

2.2. Wealth Dynamics and Expected Utility

Let $\omega_{0,t}$ denote the percentage of wealth (or portfolio weight) invested at time t in the riskless asset and $\boldsymbol{\omega}_t = [\omega_{1,t}, \ldots, \omega_{n,t}]'$ denote the vector of portfolio weights in each of the n risky assets, assumed to be adapted cáglád processes. The portfolio weights satisfy

$$\omega_{0,t} + \sum\nolimits_{i=1}^{n} \omega_{i,t} = 1. \tag{2.6}$$

The investor consumes C_t at time t. In the absence of any income derived outside his investments in these assets, the investor's wealth, starting with the initial endowment X_0, follows the dynamics

$$\begin{aligned} dX_t &= -C_t dt + \omega_{0,t} X_t \frac{dS_{0,t}}{S_{0,t-}} + \sum\nolimits_{i=1}^{n} \omega_{i,t} X_t \frac{dS_{i,t}}{S_{i,t-}} \\ &= \left(rX_t + \boldsymbol{\omega}_t'\mathbf{R}X_t - C_t\right) dt + X_t \boldsymbol{\omega}_t' \boldsymbol{\sigma} d\mathbf{W}_t + X_t \boldsymbol{\omega}_t' \mathbf{J} Z_t dN_t. \end{aligned} \tag{2.7}$$

The investor's problem at time t is then to pick the consumption and portfolio weights $\{C_s, \boldsymbol{\omega}_s\}_{t\leq s\leq\infty}$ which maximize the infinite horizon, discounted at rate β, expected utility of consumption

$$V\left(X_t,t\right) = \max_{\{C_s,\boldsymbol{\omega}_s;\ t\leq s\leq\infty\}} E_t\left[\int_t^\infty e^{-\beta s}U(C_s)ds\right] \tag{2.8}$$

subject to the dynamics of his discounted wealth (2.7), and with X_t given. We will consider in detail in the rest of the paper the case where the investor has power utility, and in the Appendix the cases of exponential and log utilities, respectively.

Using stochastic dynamic programming and the appropriate form of Itô's lemma for semi-martingale processes, the Hamilton-Jacobi-Bellman equation characterizing the optimal solution to the investor's problem is:

$$\begin{aligned} 0 = \max_{\{C_t,\boldsymbol{\omega}_t\}} \Bigg\{ & e^{-\beta t}U(C_t) + \frac{\partial V\left(X_t,t\right)}{\partial t} + \frac{\partial V\left(X_t,t\right)}{\partial X}\left(rX_t + \boldsymbol{\omega}_t'\mathbf{R}X_t - C_t\right) \\ & + \frac{1}{2}\frac{\partial^2 V\left(X_t,t\right)}{\partial X^2}X_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t \\ & + \ \lambda\int_0^1\left[V\left(X_t + X_t\boldsymbol{\omega}_t'\mathbf{J}z,t\right) - V\left(X_t,t\right)\right]\nu\left(dz\right)\Bigg\} \end{aligned} \tag{2.9}$$

with the transversality condition $\lim_{t\to\infty} E\left[V\left(X_t,t\right)\right] = 0$ (see Merton (1969).)

Using the standard time-homogeneity argument for infinite horizon problems, we have

$$\begin{aligned} e^{\beta t}V\left(X_t,t\right) &= \max_{\{C_s,\boldsymbol{\omega}_s;\ t\leq s\leq\infty\}} E_t\left[\int_t^\infty e^{-\beta(s-t)}U(C_s)ds\right] \\ &= \max_{\{C_s,\boldsymbol{\omega}_s;\ t\leq s\leq\infty\}} E_t\left[\int_0^\infty e^{-\beta u}U(C_{t+u})du\right] \\ &= \max_{\{C_s,\boldsymbol{\omega}_s;\ 0\leq s\leq\infty\}} E_t\left[\int_0^\infty e^{-\beta u}U(C_u)du\right] \\ &\equiv L(X_t) \end{aligned}$$

is independent of time. Thus $V\left(X_t,t\right) = e^{-\beta t}L(X_t)$ and (2.9) reduces to the following equation for the time-homogeneous value function L :

$$\begin{aligned} 0 = \max_{\{C_t,\boldsymbol{\omega}_t\}} \Bigg\{ & U(C_t) - \beta L(X_t) + \frac{\partial L\left(X_t\right)}{\partial X}\left(rX_t + \boldsymbol{\omega}_t'\mathbf{R}X_t - C_t\right) + \frac{1}{2}\frac{\partial^2 L\left(X_t\right)}{\partial X^2}X_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t \\ & + \ \lambda\int_0^1\left[L\left(X_t + X_t\boldsymbol{\omega}_t'\mathbf{J}z\right) - L\left(X_t\right)\right]\nu\left(dz\right)\Bigg\} \end{aligned} \tag{2.10}$$

with the transversality condition

$$\lim_{t\to\infty} E\left[e^{-\beta t}L\left(X_t\right)\right]=0. \tag{2.11}$$

The maximization problem in (2.10) separates into one for C_t, with first order condition

$$\frac{\partial U\left(C_t\right)}{\partial C}=\frac{\partial L\left(X_t\right)}{\partial X}$$

and one for $\boldsymbol{\omega}_t$:

$$\begin{aligned}\max_{\{\boldsymbol{\omega}_t\}}\Bigg\{&\frac{\partial L\left(X_t\right)}{\partial X}\boldsymbol{\omega}_t'\mathbf{R}X_t+\frac{1}{2}\frac{\partial^2 L\left(X_t\right)}{\partial X^2}X_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t\\ &+\ \lambda\int_0^1\left[L\left(X_t+X_t\boldsymbol{\omega}_t'\mathbf{J}z\right)-L\left(X_t\right)\right]\nu\left(dz\right)\Bigg\}\end{aligned} \tag{2.12}$$

Given wealth X_t, the optimal consumption choice is therefore

$$C_t^*=\left[\frac{\partial U}{\partial C}\right]^{-1}\left(\frac{\partial L\left(X_t\right)}{\partial X}\right). \tag{2.13}$$

In order to determine the optimal portfolio weights, wealth and value function, we need to be more specific about the utility function U.

2.3. Power Utility

Consider an investor with power utility, $U\left(c\right)=c^{1-\gamma}/\left(1-\gamma\right)$ for $c>0$ and $U\left(c\right)=-\infty$ for $c\leq 0$ with CRRA coefficient $\gamma\in(0,1)\cup(1,\infty)$. (In the Appendix, we treat the exponential and log utility cases.) We will look for a solution to (2.10) in the form

$$L(x)=K^{-\gamma}x^{1-\gamma}/\left(1-\gamma\right) \tag{2.14}$$

where K is a constant, so that

$$\frac{\partial L\left(x\right)}{\partial x}=\left(1-\gamma\right)L(x)/x,\quad \frac{\partial^2 L\left(x\right)}{\partial x^2}=-\gamma\left(1-\gamma\right)L(x)/x^2. \tag{2.15}$$

Then (2.10) reduces to

$$\begin{aligned}0=\max_{\{C_t,\boldsymbol{\omega}_t\}}\Bigg(&U\left(C_t\right)-\beta L\left(X_t\right)+\left(1-\gamma\right)L(X_t)\left(\boldsymbol{\omega}_t'\mathbf{R}+r\right)-\left(1-\gamma\right)C_t\frac{L\left(X_t\right)}{X_t}\\ &-\frac{1}{2}\gamma\left(1-\gamma\right)L(X_t)\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t\\ &+\lambda\int_0^1\left[\left(1+\boldsymbol{\omega}_t'\mathbf{J}z\right)^{1-\gamma}L\left(X_t\right)-L\left(X_t\right)\right]\nu\left(dz\right)\Bigg)\end{aligned} \tag{2.16}$$

that is

$$0=\min_{\{C_t,\omega_t\}}\Bigg(-\frac{U\left(C_t\right)}{\left(1-\gamma\right)L\left(X_t\right)}+\frac{\beta}{\left(1-\gamma\right)}-\left(r+\boldsymbol{\omega}_t'\mathbf{R}\right)+\frac{C_t}{X_t}$$
$$+\frac{1}{2}\gamma\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t-\frac{\lambda}{\left(1-\gamma\right)}\int_0^1\left[\left(1+\boldsymbol{\omega}_t'\mathbf{J}z\right)^{1-\gamma}-1\right]\nu\left(dz\right)\Bigg) \tag{2.17}$$

after division by $-\left(1-\gamma\right)L(X_t)<0$, so that max becomes min.

2.4. Optimal Policies

The optimal policy for the portfolio weights $\boldsymbol{\omega}_t$ is

$$\boldsymbol{\omega}_t^*=\arg\min_{\{\boldsymbol{\omega}_t\}}g(\boldsymbol{\omega}_t). \tag{2.18}$$

where the functions

$$g(\boldsymbol{\omega})=-\boldsymbol{\omega}'\mathbf{R}+\frac{\gamma}{2}\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}+\lambda\psi\left(\boldsymbol{\omega}'\mathbf{J}\right) \tag{2.19}$$

and

$$\psi\left(\boldsymbol{\omega}'\mathbf{J}\right)=-\frac{1}{\left(1-\gamma\right)}\int_0^1\left[\left(1+\boldsymbol{\omega}'\mathbf{J}z\right)^{1-\gamma}-1\right]\nu\left(dz\right). \tag{2.20}$$

are both convex.

Since $(\gamma,\mathbf{R},\Sigma,\lambda,\mathbf{J})$ are constant, the objective function g is time independent, so it is clear that any optimal solution will be time independent. Furthermore, the objective function is state independent, so any optimal solution will also be state independent. In other words, any optimal $\boldsymbol{\omega}_t^*$ will be a constant $\boldsymbol{\omega}^*$ independent of time and state. Further, the objective function g is strictly convex, goes to $+\infty$ in all directions, and hence always has a unique minimizer. In the pure diffusive case, $\lambda=0$ and we obtain of course the familiar solution

$$\boldsymbol{\omega}^*=\frac{1}{\gamma}\Sigma^{-1}\mathbf{R}. \tag{2.21}$$

As to the optimal consumption policy, with $\left[\partial U/\partial C\right]^{-1}(y)=y^{-1/\gamma}$ and $\partial L\left(x\right)/\partial x$ $=K^{-\gamma}x^{-\gamma}$ in equation (2.13), we obtain

$$C_t^*=K\ X_t. \tag{2.22}$$

Next, we evaluate equation (2.17) at the optimal policies $(C_t^*, \boldsymbol{\omega}^*)$ to identify the constant K :

$$K = \frac{\beta}{\gamma} - \frac{(1-\gamma)}{\gamma}\left[\boldsymbol{\omega}^{*\prime}\mathbf{R} + r\right] + \frac{1}{2}(1-\gamma)\,\boldsymbol{\omega}^{*\prime}\Sigma\boldsymbol{\omega}^* + \lambda\frac{(1-\gamma)}{\gamma}\psi\left(\boldsymbol{\omega}^{*\prime}\mathbf{J}\right). \tag{2.23}$$

The constant K will be fully determined once we have solved below for the optimal portfolio weights, $\boldsymbol{\omega}^*$.

Finally, we have to check that the transversality condition is satisfied. By plugging the optimizers X^* and C_t^* into (2.8), and then taking expectations, one finds

$$E\left[V\left(X_t^*, t\right)\right] = E\left[\int_t^\infty e^{-\beta s}U(C_s^*)ds\right].$$

Now $e^{-\beta s}U\left(C_s^*\right) = KV\left(X_s^*, s\right)$ from which it follows that $E\left[V\left(X_t^*, t\right)\right]$ solves $df/dt = -Kf$ and hence decays exponentially to zero as $t \to \infty$, for any $K > 0$.

3. Optimal Portfolio in a One Sector Economy with Jumps

In order to fully solve the problem, we need to put more structure on the dynamics generating the asset returns.

3.1. Homogeneous Assets with a One-Factor Structure

To begin, we consider the simplest possible case, where the n risky assets have identical jump size and expected excess return (gross of the jump compensation) characteristics

$$\mathbf{J} = \bar{J}\mathbf{1} \tag{3.1}$$

$$\mathbf{R} = \bar{R}\mathbf{1} \tag{3.2}$$

with $\bar{J}$ and $\bar{R}$ scalars, and where $\mathbf{1}$ is the $n-$vector $\mathbf{1} = [1, ..., 1]'$. To fix ideas, let us assume that $\bar{J} < 0$ in order to capture the downward risk inherent in the types of jumps we are concerned about. As for Σ, we assume the one factor structure

$$\Sigma = v^2 \begin{pmatrix} 1 & \rho & \cdots \\ \rho & \ddots & \rho \\ \cdots & \rho & 1 \end{pmatrix} \tag{3.3}$$

where $v^2 > 0$ is the variance of the returns generated by the diffusive risk, and $-1/(n-1) < \rho < 1$ is their common correlation coefficient.

In terms of factor models, this model could be generated if we had one economy-wide common Brownian factor (say, dF_t) and n idiosyncratic Brownian shocks (say, $dE_{i,t}$, $i = 1, ..., n$), all independent, so that

$$\frac{dS_{i,t}}{S_{i,t-}} = \left(r + \bar{R}\right) dt + bdF_t + \sigma dE_{i,t} + \bar{J} Z_t dN_t \tag{3.4}$$

where b is the asset's common factor loading and σ its idiosyncratic volatility. With that, we have $v^2 = b^2 + \sigma^2$, $\rho = b^2/(b^2+\sigma^2)$ and given this factor structure, $\bar{R}$ is the assets' alpha.

The key to characterizing the optimal portfolio solution in this simple situation (but also in the more complex ones we will consider later) is to exploit the spectral decomposing of the Σ matrix, and to look for the optimal portfolio solution $\boldsymbol{\omega}$ on the same basis (i.e., as a function of the same set of eigenvectors). In this case, this consists of the $n-$vector $\mathbf{1}$ and its orthogonal hyperplane. That is, the spectral decomposition of the Σ matrix is:

$$\Sigma \quad = \quad \underbrace{\kappa_1 \frac{1}{n} \mathbf{1}\mathbf{1}'}_{= \ \bar{\Sigma}} \ + \ \underbrace{\kappa_2 \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}'\right)}_{= \ \Sigma^{\perp}} \tag{3.5}$$

where $\mathbf{I}$ denotes the $n \times n$ identity matrix and

$$\kappa_1 = v^2 + v^2 (n-1) \rho \tag{3.6}$$

$$\kappa_2 = v^2 (1 - \rho) \tag{3.7}$$

are the two distinct eigenvalues of Σ, κ_1 with multiplicity 1 and eigenvector $\mathbf{1}$ and κ_2 with multiplicity $n - 1$.

Let us decompose the portfolio vector $\boldsymbol{\omega}$ on the same basis, namely

$$\boldsymbol{\omega} = \bar{\omega}\mathbf{1} + \boldsymbol{\omega}^{\perp}, \tag{3.8}$$

where $\bar{\omega}$ is scalar and $\boldsymbol{\omega}^{\perp}$ is an $n-$vector orthogonal to $\mathbf{1}$. Then, from (2.18), the optimal $\bar{\omega}^*$ and $\boldsymbol{\omega}^{\perp *}$ must satisfy

$$\begin{aligned} \left(\boldsymbol{\omega}^{\perp *}, \bar{\omega}^*\right) = \arg \min_{\{\boldsymbol{\omega}^{\perp}, \bar{\omega}\}} \Big\{ & -n\bar{\omega}\bar{R} + \frac{1}{2}\gamma n \bar{\omega}^2 \kappa_1 + \lambda \psi \left(n \bar{\omega} \bar{J}\right) \\ & + \ \frac{1}{2} \gamma \boldsymbol{\omega}^{\perp \prime} \Sigma^{\perp} \boldsymbol{\omega}^{\perp} \Big\}. \end{aligned} \tag{3.9}$$

And we now see that this separates into two separate optimization problems: one for $\boldsymbol{\omega}^{\perp}$ and one for the scalar $\bar{\omega}$:

$$\left(\boldsymbol{\omega}^{\perp *}, \bar{\omega}^{*}\right) = \arg \min_{\{\boldsymbol{\omega}^{\perp}, \bar{\omega}\}} \left\{ g^{\perp}(\boldsymbol{\omega}^{\perp}) + \bar{g}(\bar{\omega}) \right\} \tag{3.10}$$

where

$$g^{\perp}(\boldsymbol{\omega}^{\perp}) = \frac{1}{2}\gamma \boldsymbol{\omega}^{\perp\prime} \kappa_2 \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}' \right) \boldsymbol{\omega}^{\perp} \tag{3.11}$$

$$\bar{g}(\bar{\omega}) = -n\bar{\omega}\bar{R} + \frac{1}{2}\gamma n^2 \bar{\omega}^2 \kappa_1 \frac{1}{n} + \lambda \psi \left(n\bar{\omega}\bar{J} \right). \tag{3.12}$$

The optimal solution for $\boldsymbol{\omega}^{\perp}$ in this case is obviously

$$\boldsymbol{\omega}^{\perp *} = \mathbf{0}. \tag{3.13}$$

As for the optimal solution for $\bar{\omega}$, with the change of variable $\varpi_n = n\bar{\omega}$, we see that

$$\varpi_n^{*} = \arg \min_{\{\varpi_n\}} \left\{ -\varpi_n \bar{R} + \frac{1}{2}\gamma \varpi_n^2 \kappa_1 / n + \lambda \psi \left(\varpi_n \bar{J} \right) \right\}. \tag{3.14}$$

Letting $n \to \infty$, we have that $\kappa_1/n \to v^2 \rho$ and so $\varpi_n^{*} \to \varpi_{\infty}^{*}$ where

$$\varpi_{\infty}^{*} = \arg \min_{\{\varpi_{\infty}\}} \left\{ -\varpi_{\infty} \bar{R} + \frac{1}{2}\gamma \varpi_{\infty}^2 v^2 \rho + \lambda \psi \left(\varpi_{\infty} \bar{J} \right) \right\}. \tag{3.15}$$

Below, we will show how to determine the constant ϖ_n^{*} (or, in the asymptotic case, ϖ_{∞}^{*}) in closed form under further assumptions on the distribution of the jumps and the investor's utility function.

For now, we see that the optimal portfolio choice is characterized by

$$\boldsymbol{\omega}^{*} = \bar{\omega}^{*}\mathbf{1} + \boldsymbol{\omega}^{\perp *} = \bar{\omega}^{*}\mathbf{1} = \varpi_n^{*}\mathbf{1}/n$$

and an investor who selects this optimal portfolio will achieve a wealth process X_t^{*} which follows a geometric Lévy process[1] with characteristic triple (b, c, μ) given by

$$b = X_{t-}^{*} \boldsymbol{\omega}^{*\prime} \mathbf{R} = X_{t-}^{*} \varpi_n^{*} \mathbf{1}' \mathbf{R}/n = X_{t-}^{*} \left(\varpi_n^{*} \bar{R} + r - K \right) \tag{3.16}$$

$$c = X_{t-}^{*2} \boldsymbol{\omega}^{*\prime} \Sigma \boldsymbol{\omega}^{*} = X_{t-}^{*2} \varpi_n^{*2} \kappa_1 / n \tag{3.17}$$

$$\mu(dy) = \lambda \nu(dz) \quad \text{where} \quad y = X_{t-}^{*} \boldsymbol{\omega}^{*\prime} \mathbf{J} z = X_{t-}^{*} \varpi_n^{*} \bar{J} z \mathbf{1}'\mathbf{1}/n = X_{t-}^{*} \varpi_n^{*} \bar{J} z \tag{3.18}$$

[1] See the Appendix for basic definitions regarding Lévy processes.

with the equation (3.17) above following from $\Sigma = \bar{\Sigma} + \Sigma^{\perp}$ with $\mathbf{1}'\bar{\Sigma}\mathbf{1} = \kappa_1 \mathbf{1}'\mathbf{1}\mathbf{1}'\mathbf{1}/\mathbf{n} = \kappa_1 n$ and $\mathbf{1}'\Sigma^{\perp}\mathbf{1} = \mathbf{0}$.

All three of these quantities, b, c and μ, are $O(1)$ as $n \to \infty$, which means that the diversification effects are extremely weak. Moreover, since the Lévy measure is $O(1)$, the optimal portfolio in this case is not much better protected against those contagion jumps than a nondiversified portfolio.

3.2. Different Expected Excess Returns

The situation changes when we allow the n risky assets to have different expected excess returns while retaining the homogeneous covariance structure and jumps. Decomposing on the same basis as above, let

$$\mathbf{R} = \bar{R}\mathbf{1} + \mathbf{R}^{\perp} \tag{3.19}$$

with $\left\|\mathbf{R}^{\perp}\right\|^2 = \mathbf{R}^{\perp\prime}\mathbf{R}^{\perp} = O(n)$, while $\mathbf{J}^{\perp} = \mathbf{0}$, so that $\mathbf{J} = \bar{J}\mathbf{1}$.

The optimal portfolio solution, as above, can be decomposed as $\boldsymbol{\omega} = \bar{\omega}\mathbf{1} + \boldsymbol{\omega}^{\perp}$ so that the minimization problem again separates as in (3.10), where we now have

$$g^{\perp}(\boldsymbol{\omega}^{\perp}) = -\boldsymbol{\omega}^{\perp\prime}\mathbf{R}^{\perp} + \frac{1}{2}\gamma\boldsymbol{\omega}^{\perp\prime}\kappa_2\left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)\boldsymbol{\omega}^{\perp} \tag{3.20}$$

$$\bar{g}(\bar{\omega}) = -n\bar{\omega}\bar{R} + \frac{1}{2}\gamma n^2\bar{\omega}^2\kappa_1\frac{1}{n} + \lambda\psi\left(n\bar{\omega}\bar{J}\right). \tag{3.21}$$

The first order condition for minimizing (3.20) is

$$-\mathbf{R}^{\perp} + \gamma\kappa_2\left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)\boldsymbol{\omega}^{\perp *} = 0$$

whose solution is now

$$\boldsymbol{\omega}^{\perp *} = \frac{1}{\gamma\kappa_2}\mathbf{R}^{\perp} = \frac{1}{\gamma v^2(1-\rho)}\mathbf{R}^{\perp} \tag{3.22}$$

since $\mathbf{1}'\boldsymbol{\omega}^{\perp *} = 0$. Recalling that in our factor structure the elements of the $\mathbf{R}$ vector are the assets' alphas, (3.22) means that the investor is now going after the orthogonal component of the vector of alphas (subject to the usual continuous-risk provisions: the higher his risk aversion, the higher the continuous variance of the returns, and the more they are correlated, the less he will invest.)

Since $\bar{g}$ in (3.21) is unaffected by the presence of $\mathbf{R}^{\perp}$, the optimal solution for $\bar{\omega}$ is identical to that given above, namely $\bar{\omega}^* = \varpi_n^*/n$ where ϖ_n^* is given in (3.14). Also, as before, in the limit where $n \to \infty$, we have again $\varpi_n^* \to \varpi_\infty^*$ where ϖ_∞^* is given in (3.15).

Therefore, the optimal wealth process X_t follows a geometric Lévy process with the characteristic triple

$$b = X_{t-}^* \varpi_n^* \bar{R} + \frac{1}{\gamma v^2 (1-\rho)} X_{t-}^* \mathbf{R}^{\perp\prime} \mathbf{R}^{\perp} + X_{t-}^* (r - K) \tag{3.23}$$

$$c = X_{t-}^{*2} \varpi_n^{*2} \kappa_1 / n + \frac{1}{\gamma^2 v^2 (1-\rho)} X_{t-}^{*2} \mathbf{R}^{\perp\prime} \mathbf{R}^{\perp} \tag{3.24}$$

$$\mu(dy) = \lambda \nu(dz) \quad \text{where} \quad y = X_{t-}^* \varpi_n^* \bar{J} z. \tag{3.25}$$

Here, since $\mathbf{R}^{\perp\prime}\mathbf{R}^{\perp} = O(n)$ and $\kappa_1 = O(n)$ while $\kappa_2 = O(1)$, we have that b and c are $O(n)$, due to the second term in equations (5.18), and (3.24) respectively, while the Lévy measure remains $O(1)$ as $n \to \infty$. This means that nonhomogeneous expected excess returns $\mathbf{R}^{\perp}$ lead the investor to place a linearly increasing amount of wealth in the risky assets as n grows, which in turns leads to increasing expected returns b and variance c, both growing linearly in the number of assets. On the other hand, as n grows, the exposure to contagion jumps remains bounded, and is dwarfed by the exposure to diffusive risk.

Indeed, the additional investment in the risky assets due to the presence of $\mathbf{R}^{\perp}$ is entirely in the direction of $\boldsymbol{\omega}^{\perp}$, which is orthogonal to $\mathbf{J}$. So these additional amounts invested in the risky assets are all achieved with zero net additional exposure to the jump risk. Thus, compared to the case where $\mathbf{R}^{\perp} = \mathbf{0}$, the presence of $\mathbf{R}^{\perp}$ with $\mathbf{R}^{\perp\prime}\mathbf{R}^{\perp} = O(n)$ allows the investor to optimize in such a way as to increase their expected gains (at the expense of increased variance resulting from the diffusive risk), while keeping the exposure to jumps fixed.

3.3. Fully Explicit Portfolio Weights

To fully compute the optimal policy, we need to specify the utility function and the distribution $\nu(dz)$ driving the common jumps; then we can compute the integral in (3.15). Some special cases lead to a closed form solution for the last remaining constant, ϖ_n^* for any given number n of assets, or in the asymptotic case, ϖ_∞^*, yielding fully closed form solutions for the optimal portfolio weights.

Consider for instance the case of a power utility investor with CRRA coefficient $\gamma = 2$ and jump measure satisfying a power law, $\nu(dz) = dz/z$. Equation (3.15) specializes to

$$\varpi_\infty^* = \arg\min_{\{\varpi\}} \; f_\infty(\varpi) \tag{3.26}$$

where

$$\begin{aligned} f_\infty(\varpi) &= -\varpi\bar{R} + \varpi^2 v^2 \rho + \lambda \int_0^1 \left[\left(1+\varpi\bar{J}z\right)^{-1} - 1\right] dz/z \\ &= -\varpi\bar{R} + \varpi^2 v^2 \rho - \lambda \log\left(1+\varpi\bar{J}\right). \end{aligned} \tag{3.27}$$

The first order condition (FOC) for ϖ is given by

$$-\bar{R} + 2\varpi v^2 \rho - \lambda\bar{J}\left(1+\varpi\bar{J}\right)^{-1} = 0. \tag{3.28}$$

The optimal solution must satisfy

$$\varpi_\infty^* < -1/\bar{J} \tag{3.29}$$

otherwise, there is a positive probability of wealth Y_t becoming negative, which is inadmissible in the power utility case. Equation (3.28) admits a unique root ϖ_∞^* satisfying the solvency constraint (3.29), and that solution is given by

$$\varpi_\infty^* = \frac{-2\rho\nu^2 + \bar{J}\bar{R} + \sqrt{\left(2\nu^2\rho - \bar{J}\bar{R}\right)^2 + 8\bar{J}(\bar{R}+\bar{J}\lambda)\nu^2\rho}}{4\bar{J}\nu^2\rho}. \tag{3.30}$$

It is also worth nothing that

$$\varpi_\infty^* < \frac{\bar{R}}{2v^2\rho} \tag{3.31}$$

so that the optimal investment in the risky assets is always less than what it would be in the absence of jumps. This is natural since $\bar{J} < 0$. Visual inspection of (3.30) also reveals that $\bar{J}$ and λ do not have a symmetric effect on the optimal portfolio weights (more on that below.) Figure 1 plots ϖ_∞^* as a function of $\bar{J}$ and λ.

In the exact small sample case, the optimal solution to (3.14) under the solvency constraint is

$$\varpi_n^* = \frac{-2\kappa_1/n + \bar{J}\bar{R} + \sqrt{\left(2\kappa_1/n - \bar{J}\bar{R}\right)^2 + 8\bar{J}(\bar{R}+\bar{J}\lambda)\kappa_1/n}}{4\bar{J}\kappa_1/n}. \tag{3.32}$$

Figure 2 plots the objective function, $f_n(\varpi) = \bar{g}(\varpi/n)$ and shows its convergence to $f_\infty(\varpi)$ as $n \to \infty$, along with $\arg\min f_n(\varpi) = \varpi_n^*$, converging to ϖ_∞^*.

Other cases that lead to a closed form solution for ϖ_∞^* and ϖ_n^* include power utility with $\gamma = 3$ and either power law jumps $\nu(dz) = dz/z$, or uniform jumps $\nu(dz) = dz$. In either case, the FOC is then a cubic equation, solvable in closed form using standard methods. Another case is one where the investor has log utility with jumps of a fixed size, $\nu(dz) = \delta(z = \bar{z})\, dz$, for some $\bar{z} \in [0, 1]$. Then the FOC leads to a quadratic equation, as (3.28).

4. Comparative Statics

Based on the explicit solution for the portfolio weights, we can investigate how the optimal portfolio responds to different jump intensities, jump sizes and degrees of risk aversion.

4.1. Response to Jumps of Different Arrival Intensity

We have

$$\varpi_\infty^* \to -\infty \text{ as } \lambda \to \infty \tag{4.1}$$

$$\varpi_\infty^* \to \min\left(\frac{\bar{R}}{2v^2\rho}, -\frac{1}{\bar{J}}\right) \text{ as } \lambda \to 0. \tag{4.2}$$

The first limit means that the investor will go short to an unbounded extent on all the risky assets if the arrival rate of the jumps goes to infinity. This is to be expected, since $\bar{J} < 0$ and we impose no short sale constraints.

Further, ϖ_∞^* tends to $-\infty$ when $\lambda \to \infty$ at the following rate

$$\varpi_\infty^* = -\frac{\sqrt{\lambda}}{\sqrt{2v^2\rho}}(1 + o(1)).$$

If, on the other hand, the jumps become less and less frequent, then ϖ_∞^* tends to a finite limit driven by the diffusive characteristics of the assets. In particular, the higher the variance of the assets and/or the more heavily correlated they are, the smaller the investment in each one of them. And the higher the expected excess return of the assets $\bar{R}$, the higher the amount invested. For a small perceived jump risk (λ small), the optimal solution behaves like

$$\varpi_\infty^* = \min\left(\frac{\bar{R}}{2v^2\rho}, -\frac{1}{\bar{J}}\right) + \frac{\bar{J}\lambda}{\left|\bar{R}+\bar{J}\lambda\right|} + o(\lambda).$$

Of course, the first correction term is always negative so that the optimal policy is always within (3.29) and (3.31)

In the exponential utility case, there is no solvency constraint on ϖ_∞^* so the limit as $\lambda \to 0$ is simply $\bar{R}/(\gamma v^2 \rho)$.

4.1.1. Jumps vs. Expected Return Trade-off

The weights ϖ_∞^* are monotonic in λ, with

$$\frac{\partial \varpi_\infty^*}{\partial \lambda} = \frac{\bar{J}}{\sqrt{\left(2\nu^2\rho - \bar{J}\bar{R}\right)^2 + 8\bar{J}(\bar{R}+\bar{J}\lambda)\nu^2\rho}} < 0.$$

If $\bar{R} > 0$, there exists a critical value $\tilde{\lambda}$ such that

$$\varpi_\infty^* > 0 \text{ for } \lambda < \tilde{\lambda} \tag{4.3}$$

$$\varpi_\infty^* \leq 0 \text{ for } \lambda \geq \tilde{\lambda}. \tag{4.4}$$

That is, as long as jumps do not occur too frequently ($\lambda < \tilde{\lambda}$), the investor will go long on the assets in order to capture their expected return, even though that involves taking on the (negative) risk of the jumps. When the jumps occur frequently enough ($\lambda \geq \tilde{\lambda}$), then the investor decides to forgo the expected return of the assets and focuses on canceling his exposure to the jump risk by going short these assets.

The critical value $\tilde{\lambda}$ is given by

$$\tilde{\lambda} = -\frac{\bar{R}}{\bar{J}\int_0^1 z\nu\,(dz)} \tag{4.5}$$

Clearly, the higher $\bar{R}$ relative to $(-\bar{J})$, the higher $\tilde{\lambda}$. And the smaller the expected value of Z, the bigger $\tilde{\lambda}$. With $\nu\,(dz) = dz/z$, we get

$$\tilde{\lambda} = -\frac{\bar{R}}{\bar{J}}. \tag{4.6}$$

Now, if $\bar{R} \leq 0$, then $\varpi^*_\infty \leq 0$ for every $\lambda \geq 0$. In that case, there is no point in ever going long those assets since both the expected return and the jump components negatively impact the investor's rate of return.

4.1.2. Flight to Quality

The solution above can explain a well documented empirical phenomenon. Starting from a situation where $\lambda < \tilde{\lambda}$, if the perception of the jump risk increases ($\lambda \uparrow \tilde{\lambda}$), then the optimal policy for the investor is to flee-to-quality, by reducing his exposure to the risky assets ($\varpi^*_\infty \downarrow 0$) and investing the proceeds in the riskless asset. If the perception of the jump risk exceeds the critical value $\tilde{\lambda}$ given in (4.5), then the investor should go even further and start short–selling the risky assets.

Because the jump risk affects all the assets, the perception of an increase in the intensity of the jumps leads the investor to dump all the risky assets indiscriminately.

4.2. Response to Jumps of Different Magnitudes

If we now concentrate on the effect of an increase in the jump size instead of the jump magnitude, then

$$\frac{\partial \varpi^*_\infty}{\partial \bar{J}} = \frac{1}{2\bar{J}^2}\left(1 - \frac{2\rho\nu^2 + \bar{J}\bar{R}}{\sqrt{\left(2\nu^2\rho - \bar{J}\bar{R}\right)^2 + 8\bar{J}(\bar{R} + \bar{J}\lambda)\nu^2\rho}}\right) > 0$$

for, as usual, $\bar{J} < 0$. The monotonicity implies that as the jump size gets closer to zero ($\bar{J} \uparrow 0$), the investor increases his holdings in the risky assets and conversely as $\bar{J} \downarrow (-1)$.

As to the sign of ϖ^*_∞, we have

$$\varpi^*_\infty > 0 \text{ for } -\bar{R}/\lambda < \bar{J} < 0 \tag{4.7}$$

$$\varpi^*_\infty < 0 \text{ for } -1 < \bar{J} < -\bar{R}/\lambda. \tag{4.8}$$

as long as $\bar{R}/\lambda < 1$. If $\bar{R}/\lambda > 1$, the expected return is high enough relative to the jump intensity that the investor will always maintain a positive investment $\varpi^*_\infty > 0$ in the different assets, no matter how large the jump size (within the constraint $\bar{J} > -1$, of course.)

4.3. Sensitivity to Risk Aversion

Here we consider the effect of the CRRA coefficient γ on ϖ_∞^*. For a CRRA investor, the first order condition of equation (3.15) is given by

$$-\bar{R} + \gamma \varpi_\infty v^2 \rho - \lambda \int_0^1 \bar{J} z \left(1 + \varpi_\infty \bar{J} z\right)^{-\gamma} \nu(dz) = 0, \tag{4.9}$$

then, making use of the implicit function theorem, we get

$$\frac{\partial \varpi_\infty^*}{\partial \gamma} = -\frac{\varpi_\infty^* v^2 \rho + \lambda \int_0^1 \bar{J} z \left(1 + \varpi_\infty^* \bar{J} z\right)^{-\gamma} \ln\left(1 + \varpi_\infty^* \bar{J} z\right) \nu(dz)}{\gamma \left(v^2 \rho + \lambda \int_0^1 \bar{J}^2 z^2 \left(1 + \varpi_\infty^* \bar{J} z\right)^{-\gamma-1} \nu(dz)\right)}. \tag{4.10}$$

The denominator is always positive but the numerator could be negative, zero or positive depending on the sign of ϖ_∞^*. That is,

$$\frac{\partial \varpi_\infty^*}{\partial \gamma} = \begin{cases} > 0 & \text{if } \varpi_\infty < 0 \\ = 0 & \text{if } \varpi_\infty^* = 0 \\ < 0 & \text{if } \varpi_\infty^* > 0 \end{cases}. \tag{4.11}$$

This, in turn, implies that the higher the CRRA coefficient of an investor the smaller will be his ϖ_∞^* in absolute value. In the limit where γ increases to ∞, $|\varpi_\infty^*|$ decreases to zero.

5. Optimal Portfolio in a Multi–Sector Economy with Jumps

We now generalize our previous results by studying the more realistic portfolio selection problem in an economy composed of m sectors (or regions of the world), each containing k firms (or countries). The total number of assets available to the investor is $n = mk$. Asymptotically, in terms of the impact of jumps on diversification, we are primarily interested in the situation where m is fixed and k goes to infinity with n. But we provide the full solution, including the special case where all the assets are fundamentally different, that is $k = 1$ and $m = n$.

In an asset pricing framework, we are now considering a multifactor model for the returns process. Specifically, we consider a situation where we have one economy-wide common Brownian factor (say, dF_t), m sector-specific common Brownian factors (say, $dF_{l,t}$,

$l = 1, ..., m$), and n idiosyncratic Brownian shocks (say, $dE_{i,t}$, $i = 1, ..., n$), all independent, so that the model (2.2) can be rewritten in the form

$$\frac{dS_{i,t}}{S_{i,t-}} = (r + R_i)\, dt + bdF_t + \sum\nolimits_{l=1}^{m} b_l \delta_{i,l} dF_{l,t} + \sigma dE_{i,t} + J_i Z_t dN_t \tag{5.1}$$

where $\delta_{i,l} = 1$ if asset i belongs to sector l, and 0 otherwise. The $b's$ are the respective factor loadings, and σ is the asset's idiosyncratic volatility. It follows that the continuous (or Brownian) part of the covariance between the returns of assets i and j is

$$\mathrm{Cov}\,(dS_{i,t}/S_{i,t-}, dS_{j,t}/S_{j,t-}) = \begin{cases} (b^2 + b_l^2)dt & \text{if assets } i \text{ and } j \text{ are in the same sector } l \\ b^2 dt & \text{otherwise} \end{cases}$$

while the continuous part of the variance is $\mathrm{Var}\,(dS_{i,t}/S_{i,t-}) = (b^2 + b_l^2 + \sigma^2)dt$ when asset i belongs to sector l.

The above implies the following structure for the variance-covariance matrix of returns:

$$\underset{n\times n}{\Sigma} = \begin{pmatrix} \Sigma_1 & v^2\rho_0 & \cdots \\ v^2\rho_0 & \ddots & v^2\rho_0 \\ \cdots & v^2\rho_0 & \Sigma_m \end{pmatrix} \tag{5.2}$$

is block diagonal with sector blocks

$$\underset{k\times k}{\Sigma_l} = v_l^2 \begin{pmatrix} 1 & \rho_l & \cdots \\ \rho_l & \ddots & \rho_l \\ \cdots & \rho_l & 1 \end{pmatrix} \tag{5.3}$$

where $1 > \rho_l > \rho_0 \geq 0$ (in full generality, we can start directly from (5.2)-(5.3) and only require $\rho_l \geq -1/(k-1)$ and $\rho_0 \geq -1/(n-1)$).

5.1. No Cross–Sectorial Diffusive Correlation

We start with the situation where the diffusive risk generates correlated returns within sectors, but not across sectors, that is $\rho_0 = 0$ in (5.2). Then the only source of cross-sectorial correlation is due to the jumps.

The spectral decomposition of the resulting Σ matrix is

$$\Sigma \;=\; \underbrace{\sum\nolimits_{l=1}^{m} \kappa_{1l} \frac{1}{k} \mathbf{1}_l \mathbf{1}_l'}_{=\;\bar{\Sigma}} \;+\; \underbrace{\sum\nolimits_{l=1}^{m} \kappa_{2l} \left(\mathbf{M}_l - \frac{1}{k} \mathbf{1}_l \mathbf{1}_l' \right)}_{=\;\Sigma^{\perp}} \tag{5.4}$$

where

$$\kappa_{1l} = v_l^2 + v_l^2 (k-1) \rho_l \tag{5.5}$$

$$\kappa_{2l} = v_l^2 (1 - \rho_l) \tag{5.6}$$

are the $2m$ distinct eigenvalues of Σ. The multiplicity of each κ_{1l} is 1, and the multiplicity of each κ_{2l} is $k-1$. The eigenvector for κ_{1l} is $\mathbf{1}_l$, the $n-$vector with ones placed in the rows corresponding to the $l-$block and zeros everywhere else, that is

$$\mathbf{1}_l = [0, \ldots, 0, \underbrace{1, \ldots, 1}_{\text{sector } l}, 0, \ldots, 0]', \tag{5.7}$$

where the first 1 is located in the $k(l-1)+1$ coordinate. $\mathbf{M}_l$ is an $n \times n$ block diagonal matrix with a $k \times k$ identity matrix $\mathbf{I}_k$ placed in the $l-$block and zeros everywhere else:

$$\underset{n \times n}{\mathbf{M}_l} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \mathbf{I}_k & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \tag{5.8}$$

Corresponding to the above spectral structure, we have the orthogonal decomposition $\mathbb{R}^n = \bar{V} \oplus V^{\perp}$ where $\bar{V}$ is the span of $\{\mathbf{1}_l\}_{l=1,..,m}$.

As for the jump vector $\mathbf{J}$ in our m–sector economy, we assume that $\mathbf{J} \in \bar{V}$:

$$\mathbf{J} = \sum\nolimits_{l=1}^{m} j_l \mathbf{1}_l = [\underbrace{j_1, \ldots, j_1}_{\text{sector } 1}, \underbrace{j_2, \ldots, j_2}_{\text{sector } 2}, \cdots, \underbrace{j_m, \ldots, j_m}_{\text{sector } m}]' \tag{5.9}$$

meaning that firms within a given sector have the same response to the arrival of a Poisson jump, i.e., to a change in N_t. But the proportional response j_l of firms of different sectors to the arrival of a jump can be different.

Finally, we assume that the vector of expected excess returns has the form

$$\mathbf{R} = \sum\nolimits_{l=1}^{m} r_l \mathbf{1}_l + \mathbf{R}^{\perp} = \bar{\mathbf{R}} + \mathbf{R}^{\perp}. \tag{5.10}$$

Here, we allow the expected excess returns to differ both within and across sectors, by allowing $\mathbf{R}^{\perp} \neq \mathbf{0}$. As in the one factor case, the components of $\mathbf{R}$ play the role of the assets' alphas. The general $\mathbf{R}^{\perp}$ is orthogonal to each $\mathbf{1}_l$ and has the form

$$\mathbf{R}^{\perp} = [\mathbf{R}_1^{\perp\prime}, ..., \mathbf{R}_m^{\perp\prime}]'$$

where each of the k-vectors $\mathbf{R}_l^{\perp}$ is orthogonal to the k-vector $\mathbf{1}$. As in section 3.2, we may suppose that each component of $\mathbf{R}^{\perp}$ is $O(1)$.

With this structure, we will be looking for a vector of optimal portfolio weights of the form

$$\boldsymbol{\omega} \;=\; \sum\nolimits_{l=1}^{m} \bar{\omega}_l \mathbf{1}_l + \boldsymbol{\omega}^{\perp} = \bar{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\perp}. \tag{5.11}$$

The minimization problem again separates as

$$\left(\boldsymbol{\omega}^{\perp *}, \bar{\boldsymbol{\omega}}^{*}\right) = \arg \min_{\{\boldsymbol{\omega}^{\perp}, \bar{\boldsymbol{\omega}}\}} \left\{ g^{\perp}(\boldsymbol{\omega}^{\perp}) + \bar{g}(\bar{\omega}) \right\} \tag{5.12}$$

where

$$g^{\perp}(\boldsymbol{\omega}^{\perp}) = -\boldsymbol{\omega}^{\perp\prime}\mathbf{R}^{\perp} + \frac{1}{2}\gamma \boldsymbol{\omega}^{\perp\prime}\Sigma^{\perp}\boldsymbol{\omega}^{\perp} \tag{5.13}$$

$$\bar{g}(\bar{\boldsymbol{\omega}}) = -k \sum\nolimits_{l=1}^{m} \bar{\omega}_l r_l + \frac{1}{2}\gamma k \sum\nolimits_{l=1}^{m} \bar{\omega}_l^2 \kappa_{1l} + \lambda\psi\left(k \sum\nolimits_{l=1}^{m} \bar{\omega}_l j_l\right) \tag{5.14}$$

The first order condition for minimizing $g^{\perp}(\boldsymbol{\omega}^{\perp})$ leads to the optimal solution

$$\boldsymbol{\omega}^{\perp *} = \frac{1}{\gamma}\Sigma^{-1}\mathbf{R}^{\perp} \tag{5.15}$$

where, by the block diagonal form of Σ in (5.2), can be written in the form $\boldsymbol{\omega}^{\perp *} = [\boldsymbol{\omega}_1^{\perp *\prime}, ..., \boldsymbol{\omega}_m^{\perp *\prime}]'$ with

$$\boldsymbol{\omega}_l^{\perp *} = \frac{1}{\gamma \kappa_{2l}}\mathbf{R}_l^{\perp}$$

for $l = 1, \ldots, m$.

The problem of minimizing $\bar{g}(\bar{\boldsymbol{\omega}})$ is an analogue of what happens with one sector, as in Section 3, but in dimension m, and similarly its solution has a limit as k goes to infinity with n (the number of sectors m being fixed). With the change of variable $\varpi_n = k\bar{\omega}$ we see that

$$\varpi_n^* = \arg \min_{\{\varpi_n\}} \left\{ -\sum\nolimits_{l=1}^{m} \varpi_{ln} r_l + \frac{1}{2}\gamma \sum\nolimits_{l=1}^{m} \varpi_{ln}^2 \kappa_{1l}/k + \lambda\psi\left(\sum\nolimits_{l=1}^{m} \varpi_{ln} j_l\right)\right\} \tag{5.16}$$

Letting $k \to \infty$, we get $\varpi_n^* \to \varpi_\infty^*$ where

$$\varpi_\infty^* = \arg \min_{\{\varpi_\infty\}} \left\{ -\sum\nolimits_{l=1}^{m} \varpi_{l\infty} r_l + \frac{1}{2}\gamma \sum\nolimits_{l=1}^{m} \varpi_{l\infty}^2 v_l^2 \rho_l + \lambda\psi\left(\sum\nolimits_{l=1}^{m} \varpi_{l\infty} j_l\right)\right\}, \tag{5.17}$$

which, compared to (3.15), is an $m-$dimensional minimization problem, instead of a one-dimensional one. But the convexity of the objective function implies the existence of the minimizer. As in the one-factor case, we will determine below ϖ_n^* (and in the asymmptotic case, ϖ_∞^*) in closed form.

The wealth process X_t^* of the optimizing investor will thus have geometric Lévy dynamics for a CRRA with $\boldsymbol{\omega}^*$ given by $\bar{\omega}^* + \boldsymbol{\omega}^{\perp *}$. The characteristic triple of X_t^* for a CRRA investor is then

$$b = X_{t-}^* \sum_{l=1}^m \varpi_{ln}^* r_l + X_{t-}^* \sum_{l=1}^m \frac{1}{\gamma v_l^2 (1-\rho_l)} \mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} \tag{5.18}$$
$$+ X_{t-}^* (r - K)$$
$$c = X_{t-}^{*2} \sum_{l=1}^m \varpi_{ln}^{*2} \kappa_{1l}/k + X_{t-}^{*2} \sum_{l=1}^m \frac{1}{\gamma^2 v_l^2 (1-\rho_l)} \mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} \tag{5.19}$$
$$\mu(dy) = \lambda \nu(dz) \quad \text{where} \quad y = X_{t-}^* \sum_{l=1}^m \varpi_{ln}^* j_l z. \tag{5.20}$$

Under the natural condition that $\mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} = O(k)$, we are lead to conclusions similar to those we drew in section 3.2. As k, the number of stocks per sector, increases, the optimal portfolio can achieve expected gains at the expense of variance which both grow approximately linearly with k, while keeping the exposure to jumps bounded. Essentially this result is achieved by the investor apportioning an increasing fraction of assets in the subspace orthogonal to the vectors $\mathbf{1}_l$.

For the example of $\nu(dz) = dz/z$ of section 3.3 we get the following objective functions for a power utility investor with CRRA coefficient $\gamma = 2$ in the m-sector case:

$$f_n(\varpi) = -\sum_{l=1}^m \varpi_{ln} r_l + \sum_{l=1}^m \varpi_{ln}^2 \kappa_{1l}/k$$
$$- \lambda \log \left(1 + \sum_{l=1}^m \varpi_{ln} j_l\right). \tag{5.21}$$

The first order conditions for ϖ_n are given by

$$-r_l + 2\varpi_{ln} \kappa_{1l}/k - \lambda j_l \left(1 + \sum_{l=1}^m \varpi_{ln} j_l\right)^{-1} = 0 \text{ for } l = 1, \ldots, m. \tag{5.22}$$

These first order conditions form a system of m quadratic equations. Equations (5.22) admit a unique solution ϖ_n satisfying the solvency constraint $\sum_{l=1}^m \varpi_{ln} j_l > -1$. These are solvable in closed form. For example, in the special case of a two-sector economy ($m = 2$), the solution is

$$\varpi_{1n}^{*}=\frac{-\left(2\kappa_{12}/k+r_{2}\bar{J}_{2}\right)\bar{J}_{1}\kappa_{11}/k+r_{1}\left(\bar{J}_{1}^{2}\kappa_{12}/k+2\bar{J}_{2}^{2}\kappa_{11}/k\right)-\bar{J}_{1}A}{4\left(\bar{J}_{1}^{2}\kappa_{12}/k+\bar{J}_{2}^{2}\kappa_{11}/k\right)\kappa_{11}/k} \tag{5.23}$$

$$\varpi_{2n}^{*}=\frac{-\left(2\kappa_{11}/k+r_{1}\bar{J}_{1}\right)\bar{J}_{2}\kappa_{12}/k+r_{2}\left(2\bar{J}_{1}^{2}\kappa_{12}/k+\bar{J}_{2}^{2}\kappa_{11}/k\right)-\bar{J}_{2}A}{4\left(\bar{J}_{1}^{2}\kappa_{12}/k+\bar{J}_{2}^{2}\kappa_{11}/k\right)\kappa_{12}/k} \tag{5.24}$$

where

$$A=\sqrt{\left(\frac{r_{2}\bar{J}_{2}\kappa_{11}}{k}+\frac{r_{1}\bar{J}_{1}\kappa_{12}}{k}+2\frac{\kappa_{11}\kappa_{12}}{k^{2}}\right)^{2}+8\lambda\left(\frac{\bar{J}_{1}^{2}\kappa_{12}}{k}+\frac{\bar{J}_{2}^{2}\kappa_{11}}{k}\right)\frac{\kappa_{11}\kappa_{12}}{k^{2}}}. \tag{5.25}$$

Figure 3 plots the objective function, $f_n(\varpi) = \bar{g}(\varpi/k)$ that we obtain in a two-sector economy, that is the function (5.21) with $m = 2$.

5.2. Cross–Sectorial Diffusive Correlation

We now allow the diffusive risk to generate correlated returns within sectors, as well as across sectors. In addition, the jumps generate cross-sectorial correlation. That is, we allow for a non-zero ρ_0 in (5.2). where $1 > \rho > \rho_0 \geq -1/(n-1)$ and, as before, $n = mk$. For simplicity, we make the within-sector correlation coefficient $\rho_l = \rho$, and $v_l^2 = v^2$ identical across sectors.

This Σ matrix then has the three distinct eigenvalues

$$\begin{aligned}
\kappa_1 &= v^2 + v^2 k(m-1)\rho_0 + v^2(k-1)\rho \\
\kappa_2 &= v^2 - v^2 k\rho_0 + v^2(k-1)\rho \\
\kappa_3 &= v^2(1-\rho)
\end{aligned}$$

with multiplicity 1, $m-1$ and $(k-1)m$ respectively. In parallel with the previous section, we focus on the orthogonal decomposition $\mathbb{R}^n = \bar{V} \oplus V^{\perp}$ where $V^{\perp}$, the κ_3-eigenspace, consists of vectors orthogonal to each $\mathbf{1}_l$ and $\bar{V}$ is the m-dimensional subspace spanned by the vectors $\{\mathbf{1}_l\}_{l=1,\ldots,m}$.

We again assume that $\mathbf{J} = \sum_{l=1}^{m} j_l \mathbf{1}_l \in \bar{V}$, in other words, that firms within the same sector have the same response to the arrival of a Poisson jump, but not necessarily for firms across sectors. The vector of expected excess returns again has the form

$$\mathbf{R} = \sum\nolimits_{l=1}^{m} r_l \mathbf{1}_l + \mathbf{R}^{\perp} = \bar{\mathbf{R}} + \mathbf{R}^{\perp}. \tag{5.26}$$

where the general $\mathbf{R}^{\perp}$ is orthogonal to each $\mathbf{1}_l$ and has the form

$$\mathbf{R}^{\perp} = [\mathbf{R}_1^{\perp\prime}, ..., \mathbf{R}_m^{\perp\prime}]'$$

As in section 3.2, we may suppose that each component of $\mathbf{R}^{\perp}$ is $O(1)$.

The vector of optimal portfolio weights has the form

$$\boldsymbol{\omega} \;=\; \sum\nolimits_{l=1}^{m} \bar{\omega}_l \mathbf{1}_l + \boldsymbol{\omega}^{\perp} = \bar{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\perp}. \tag{5.27}$$

The minimization problem again separates as

$$\left(\boldsymbol{\omega}^{\perp *}, \bar{\boldsymbol{\omega}}^*\right) = \arg \min_{\{\boldsymbol{\omega}^{\perp}, \bar{\boldsymbol{\omega}}\}} \left\{ g^{\perp}(\boldsymbol{\omega}^{\perp}) + \bar{g}(\bar{\omega}) \right\} \tag{5.28}$$

where now

$$g^{\perp}(\boldsymbol{\omega}^{\perp}) = -\boldsymbol{\omega}'^{\perp}\mathbf{R}^{\perp} + \frac{\gamma}{2}\kappa_3 \boldsymbol{\omega}^{\perp\prime}\boldsymbol{\omega}^{\perp} \tag{5.29}$$

$$\begin{aligned} \bar{g}(\bar{\boldsymbol{\omega}}) &= -k \sum\nolimits_{l=1}^{m} \bar{\omega}_l r_l + \frac{\gamma}{2}\frac{k\kappa_1}{m}\left(\sum\nolimits_{l=1}^{m} \bar{\omega}_l\right)^2 \\ &\quad + \frac{\gamma}{2}k\kappa_2 \left[\sum\nolimits_{l=1}^{m} \bar{\omega}_l^2 - \frac{1}{m}\left(\sum\nolimits_{l=1}^{m} \bar{\omega}_l\right)^2\right] \\ &\quad + \lambda\psi\left(k \sum\nolimits_{l=1}^{m} \bar{\omega}_l j_l\right). \end{aligned} \tag{5.30}$$

The essential structure of the solution can be deducted from that of the previous section. First, the minimizer of $g^{\perp}(\boldsymbol{\omega}^{\perp})$ can be written in the form $\boldsymbol{\omega}^{\perp *} = [\boldsymbol{\omega}_1^{\perp *\prime}, ..., \boldsymbol{\omega}_m^{\perp *\prime}]'$ with

$$\boldsymbol{\omega}_l^{\perp *} = \frac{1}{\gamma\kappa_3}\mathbf{R}_l^{\perp}$$

for $l = 1, \ldots, m$. The m-dimensional minimization of $\bar{g}(\bar{\boldsymbol{\omega}})$ can always be solved, and the resulting solution has n dependence similar to before. With the change of variable $\varpi_n = k\bar{\omega}$ we see that the minimizer $\varpi_n^* = \arg\min_{\{\varpi_n\}} \bar{g}(\boldsymbol{\varpi}_n/k)$ will have the limit $\varpi_n^* \to \varpi_\infty^*$ as $k \to \infty$, where

$$\begin{aligned} \varpi_\infty^* = \arg \min_{\{\varpi_\infty\}} \Bigg\{ &- \sum\nolimits_{l=1}^{m} \varpi_{l\infty} r_l + \frac{\gamma v^2}{2} \Bigg\{ \frac{(m-1)\rho_0 + \rho}{m} \left(\sum\nolimits_{l=1}^{m} \varpi_{l\infty}\right)^2 \\ &+ (\rho - \rho_0)\left[\sum\nolimits_{l=1}^{m} \varpi_{l\infty}^2 - \frac{1}{m}\left(\sum\nolimits_{l=1}^{m} \varpi_{l\infty}\right)^2\right]\Bigg\} \\ &+ \lambda\psi\left(\sum\nolimits_{l=1}^{m} \varpi_{l\infty} j_l\right)\Bigg\}, \end{aligned} \tag{5.31}$$

The wealth process X_t^* of the optimizing investor will thus have geometric Lévy dynamics for a CRRA investor with $\boldsymbol{\omega}^*$ given by $\bar{\omega}^* + \boldsymbol{\omega}^{\perp *}$. The characteristic triple of X_t^* for a CRRA investor is then

$$b = X_{t-}^* \sum\nolimits_{l=1}^m \varpi_{ln}^* r_l + \frac{1}{\gamma v^2 (1-\rho)} X_{t-}^* \sum\nolimits_{l=1}^m \mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} + X_{t-}^* (r - K) \tag{5.32}$$

$$c = \frac{\kappa_1 - \kappa_2}{n} X_{t-}^{*2} \left(\sum\nolimits_{l=1}^m \varpi_{ln}^* \right)^2 + \frac{\kappa_2}{k} X_{t-}^{*2} \sum\nolimits_{l=1}^m \varpi_{ln}^{*2}$$
$$+ \frac{1}{\gamma v^2 (1-\rho)} X_{t-}^{*2} \sum\nolimits_{l=1}^m \mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} \tag{5.33}$$

$$\mu(dy) = \lambda \nu(dz) \quad \text{where} \quad y = X_{t-}^* \sum\nolimits_{l=1}^m \varpi_{ln}^* j_l z. \tag{5.34}$$

Under the natural condition that $\mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} = O(k)$ the conclusions are the same as those of the previous section: As k, the number of stocks per sector, increases, the optimal portfolio can achieve expected gains at the expense of variance which both grow approximately linearly with k, while keeping the exposure to jumps bounded.

6. Partial Response to the Jumps

Suppose now that the true model is as before but the investor (mistakenly) thinks instead that the risky assets are driven by Brownian motions only, with no jumps:

$$\frac{dS_{i,t}}{S_{i,t-}} = \left(r + \hat{R}_i \right) dt + \sum_{j=1}^n \hat{\sigma}_{i,j} dW_{j,t}, \ i = 1, \ldots, n. \tag{6.1}$$

While this investor ignores the Poisson process and assumes that returns are driven only by Brownian motions, he still accounts correctly for the additional variance and covariances generated by the jumps.

6.1. A Misspecified Model that Matches the First Two Moments

In this case the total expected excess returns are

$$\hat{R}_i dt = E_t \left[\frac{dS_{i,t}}{S_{i,t-}} \right] - r dt = \left(R_i + J_i \lambda \bar{Z} \right) dt \tag{6.2}$$

and the total variance-covariance matrix $\hat{\Sigma} = \hat{\sigma} \hat{\sigma}'$ is given by

$$\hat{\Sigma} dt = E_t \left[\left(\frac{d\mathbf{S}_t}{\mathbf{S}_{t-}} \right) \left(\frac{d\mathbf{S}_t}{\mathbf{S}_{t-}} \right)' \right] = \left(\Sigma + \lambda \mathbf{J} \mathbf{J}' \bar{Z}^2 \right) dt \tag{6.3}$$

Now the investor thinks that $\boldsymbol{\omega}^{NJ*}$ has to satisfy:

$$\boldsymbol{\omega}^{NJ*} = \arg\min_{\{\boldsymbol{\omega}^{NJ}\}} g(\boldsymbol{\omega}^{NJ}) \tag{6.4}$$

where

$$g(\boldsymbol{\omega}^{NJ}) = -\left(\boldsymbol{\omega}^{NJ}\right)' \hat{\mathbf{R}} + \frac{1}{2}\gamma \left(\boldsymbol{\omega}^{NJ}\right)' \hat{\Sigma}\boldsymbol{\omega}^{NJ} \tag{6.5}$$

We again assume a one sector economy with different expected excess returns, then

$$\begin{aligned}
\hat{\mathbf{R}} &= \left(\bar{R} + \lambda \bar{J}\bar{Z}\right)\mathbf{1} + \mathbf{R}^{\perp}, \\
\hat{\Sigma} &= \left(\bar{\Sigma} + \lambda \bar{J}^2 \bar{Z}^2\right)\mathbf{1}\mathbf{1}' + \Sigma^{\perp}, \\
\boldsymbol{\omega}^{NJ} &= \bar{\omega}^{NJ}\mathbf{1} + \boldsymbol{\omega}^{\perp},
\end{aligned}$$

thus, the minimization problem becomes

$$\left(\boldsymbol{\omega}^{\perp *}, \bar{\omega}^{NJ*}\right) = \arg\min_{\left\{\omega^{\perp}, \bar{\omega}^{NJ}\right\}} \left\{g^{\perp}(\boldsymbol{\omega}^{\perp}) + \bar{g}(\bar{\omega}^{NJ})\right\} \tag{6.6}$$

where

$$g^{\perp}(\boldsymbol{\omega}^{\perp}) = -\boldsymbol{\omega}^{\perp\prime}\mathbf{R}^{\perp} + \frac{1}{2}\gamma\boldsymbol{\omega}^{\perp\prime}\Sigma^{\perp}\boldsymbol{\omega}^{\perp}, \tag{6.7}$$

$$\bar{g}(\bar{\omega}^{NJ}) = -n\bar{\omega}^{NJ}\left(\bar{R} + \lambda\bar{J}\bar{Z}\right) + \frac{1}{2}\gamma n^2 \left(\bar{\omega}^{NJ}\right)^2 \left(\bar{\Sigma} + \lambda\bar{J}^2\bar{Z}^2\right) \tag{6.8}$$

The optimal solution for $\boldsymbol{\omega}^{\perp}$ in this case is

$$\boldsymbol{\omega}^{\perp *} = \frac{1}{\gamma v^2 (1-\rho)}\mathbf{R}^{\perp}. \tag{6.9}$$

As for the optimal solution for $\bar{\omega}^{NJ}$, with the change of variable $\varpi_n^{NJ} = n\bar{\omega}^{NJ}$, we see

$$\varpi_n^{NJ*} = \frac{1}{\gamma}\left(\frac{\kappa_1}{n} + \lambda\bar{J}^2\bar{Z}^2\right)^{-1}\left(\bar{R} + \lambda\bar{J}\bar{Z}\right). \tag{6.10}$$

Letting $n \to \infty$, we have that $\varpi_n^{NJ*} \to \varpi_\infty^{NJ*}$ where

$$\varpi_\infty^{NJ*} = \frac{1}{\gamma}\left(v^2\rho + \lambda\bar{J}^2\bar{Z}^2\right)^{-1}\left(\bar{R} + \lambda\bar{J}\bar{Z}\right). \tag{6.11}$$

It is important to notice that $\boldsymbol{\omega}^{\perp *}$ is the same whether the investor recognizes the presence of jumps or not.

6.2. Higher Moment Effect

On the other hand, matching the first two moments is not sufficient to fully deliver the optimal solution. We can compare ϖ_∞^{NJ*} to

$$\varpi_\infty^* = \arg\min_{\{\varpi_\infty\}} \left\{ -\varpi_\infty \bar{R} + \frac{1}{2}\gamma\varpi_\infty^2 v^2 \rho + \lambda\psi\left(\varpi_\infty \bar{J} z\right) \right\}$$

Taking the third order Taylor's expansion of $\psi\left(\varpi_\infty \bar{J} z\right)$ with respect to ϖ_∞, we get

$$\begin{aligned} \psi\left(\varpi_\infty \bar{J} z\right) = & -\bar{J}\varpi_\infty \bar{Z} + \frac{1}{2}\gamma \bar{J}^2 \varpi_\infty^2 \bar{Z}^2 \\ & + \frac{1}{3!}\bar{J}^3 \varpi_\infty^3 \bar{Z}^3 \psi'''(0) + o\left(\varpi_\infty^3\right), \end{aligned} \tag{6.12}$$

then

$$\begin{aligned} \varpi_\infty^* = \arg\min_{\{\varpi_\infty\}} & \left\{ -\varpi_\infty \left(\bar{R} + \lambda \bar{J}\bar{Z}\right) + \frac{1}{2}\gamma\varpi_\infty^2 \left(\bar{\Sigma} + \lambda \bar{J}^2 \bar{Z}^2\right) \right. \\ & \left. + \frac{1}{3!}\lambda \bar{J}^3 \varpi_\infty^3 \bar{Z}^3 \psi'''(0) + o\left(\varpi_\infty^3\right) \right\}. \end{aligned}$$

where $\bar{Z}^3 = \int_0^1 z^3 \nu(dz)$. Hence, if $\bar{J} < 0$ then we have

$$\varpi_\infty^* < \varpi_\infty^{NJ*}. \tag{6.13}$$

This result suggests that the investor who recognizes the presence of jumps will invest more money in the riskless asset. This is an effect driven by differences in the higher moments of the two processes, the one that is correctly specified (with jumps) and the one that is misspecified (no jumps but still a partial adjustment consisting of matching the first two moments correctly.) In the same spirit, Cvitanic et al. (2005) recently proposed a model where their single risky asset can jump, thereby leading to higher moments in the distrtibution of that asset's returns. They found that ignoring higher moments can lead to significant overinvestment in the risky asset.

7. Extensions, Limitations and Conclusions

We have proposed a new approach to characterize in closed form the portfolio selection problem for an investor concerned with the possibility of market contagion effects, and who seeks to control this risk by diversification or other means. The framework is illustrated by certain families of asset return models of increasing complexity, where each family allows models for an unbounded number of assets. We can address certain questions. How exactly does increasing the number of available assets improve the investor's exposure to both diffusive and contagion risk? How does the portfolio of an investor who fears contagion differ from the portfolio of one who does not? Is there a simple form for the optimal portfolio which is achieved asymptotically as the number of assets grows to infinity? Is there a systematic way to add complexity to the market model while retaining computational tractability?

In this paper, the standard multi–asset geometric Brownian motion models is extended to an exponential Lévy models by the inclusion of correlation effects through a one dimensional jump distribution. For the general exponential Lévy model, the portfolio selection problem for n assets reduces to the minimization of a convex function in n dimensions. However, the crucial assumption of a special relation between the diffusive correlation matrix and the jump distribution, $\mathbf{J}^{\perp} = 0$, enables a further reduction of the problem to a convex optimization in the dimension of the number of sectors m, which we can take to be small while the total number of assets n is typically large. As an empirical strategy, one could imagine determining the number of sectors through spectral analysis or similar techniques, in order to determine endogenously the shape of the Σ matrix.

Our analysis allows us to draw the following three conclusions. First, our examples show that when the asset returns are sufficiently nonhomogeneous the total amount in the optimal portfolio invested in risky assets, hence the expected return and volatility, all grow linearly with n, while the exposure to the jumps remains bounded. Moreover, the optimal portfolio is asymptotically normally distributed as n gets large. Finally, the investor who correctly accounts for jumps always invests less in the risky assets than the investor who fails to include these jumps.

One can ask if these conclusions continue to hold and the approach remains valid when extensions and generalizations of this work are considered:

1. The first extension to consider is to allow the jump dimension to grow to the number of sectors m, while retaining the condition $\mathbf{J}^{\perp} = 0$. In that case, the selection problem again reduces to an m dimensional optimization, and hence our main conclusions

remain intact.

2. Another extension to consider is to generalize the cross sectorial correlation structure: it is clear that much more general correlation structures will preserve the condition $\mathbf{J}^{\perp} = 0$ and hence the dimensional reduction, leading to similar conclusions. In fact, one could consider modelling the spectral decomposition of the Σ matrix directly, instead of parametrizing the matrix itself and then determining its spectral decomposition. Of course, the form of the matrix is easier to interpret or derive from an economic model than its spectral decomposition, which argues for the indirect (or structural) approach. But the reduced form approach has the advantage of greater generality, since we are no longer constrained to being able to derive the spectral decomposition explicitly from the assumed form of the Σ matrix.

3. Stochastic volatility of the type considered in Liu et al. (2003) requires solving our nonlinear equations for weight vectors stepwise in time, in parallel with ordinary differential equations (which themselves depend on the current portfolio weights). This does not appear doable in closed form.

4. Portfolio constraints such as short-selling constraints are sometimes introduced into portfolio theory but, when generic constraints are imposed on the optimal portfolio, we cannot expect the dimensional reduction to be preserved or our conclusions to hold. However, a utility function such as power utility which becomes $-\infty$ for wealth below a finite threshold, sometimes automatically implies certain constraints: it appears that in this case, much of our analysis remains intact.

5. One further extension to consider is to allow $\mathbf{J}^{\perp} \neq 0$. In this case, the reduction in dimension breaks down: One is left with a problem of full complexity, and there is little of interest we can prove. While the condition $\mathbf{J}^{\perp} = 0$ is without economic justification, breaking it seems to be the last thing a mathematician would want to do because the extra generality does not justify the additional computational complexity of the solution. Since in any real life application there can never be enough market data to calibrate the jumps, the specification of the jumps will always be largely subjective. The main purpose of adding jumps must be to stress test or correct a proposed optimal portfolio, and in this case imposing $\mathbf{J}^{\perp} = 0$ on the jumps is justified by mathematical elegance rather than economics.

6. Finally, we would like to be able to better capture contagion, in the form not just of simultaneous jumps (as we are currently able to model) but rather in the form of a jump in one sector causing an increase in the likelihood that a different jump will occur in another sector. Self-exciting jump processes seem a promising approach which we intend to investigate in future work.

Appendix

A. Lévy Processes

We give here a definition of Lévy processes and our notation. An n dimensional Lévy process $\mathbf{L}_t$ is specified by its "characteristic triple" $(\mathbf{b}, \mathbf{c}, \mu)$ where $\mathbf{b} \in \mathbb{R}^n$ is the drift or mean return vector, $\mathbf{c} \in \mathbb{R}^{n \times n}$ is the diffusion matrix, or local variance of the continuous part of $\mathbf{L}_t$, and μ is the jump or Lévy measure on $\mathbb{R}^n$, which satisfies

$$\int_{\mathbb{R}^n} \left(1 \wedge \|\mathbf{x}\|^2\right) \mu(d\mathbf{x}) < \infty.$$

The characteristic function of $\mathbf{L}_t$ is given by the Lévy-Khintchine formula

$$E(e^{i\mathbf{u}'\mathbf{L}_t}) = \exp\left(t\left(i\mathbf{u}'\mathbf{b} - \frac{\mathbf{1}}{2}\mathbf{u}'\mathbf{c}\mathbf{u} + \int_{\mathbb{R}^n\setminus\{0\}} \mu(d\mathbf{x})\left(e^{i\mathbf{u}'\mathbf{x}} - 1 - i\mathbf{u}'\mathbf{h}(\mathbf{x})\right)\right)\right) \tag{A.1}$$

for $\mathbf{u} \in \mathbb{R}^n$and $\mathbf{h}(\mathbf{x})$ is a truncation function which, because we are dealing with finite intensity measures, can be set to zero (see e.g., Chapter II.2 in Jacod and Shiryaev (2003).) The stochastic differential equation for $\mathbf{L}_t$ written in terms of its characteristics is

$$d\mathbf{L}_t = \mathbf{b}dt \quad + \mathbf{c}^{1/2}d\mathbf{W}_t + \int_{\mathbb{R}^n} \mathbf{x}\ N^{(\mu)}(d\mathbf{x}, dt) \tag{A.2}$$

where $\mathbf{c}^{1/2}$ is a matrix square root satisfying $\mathbf{c}^{1/2}(\mathbf{c}^{1/2})' = \mathbf{c}$, and $N^{(\mu)}$ is called the Poisson random measure associated with the Lévy measure μ.

We can identify the right hand side of equation (2.2) as the dynamics of a Lévy process with triple $(r + \mathbf{R}, \boldsymbol{\sigma}\boldsymbol{\sigma}', \mu)$ where $\mu(d\mathbf{l}) = \lambda\nu(dz)$ with $\mathbf{l} = \mathbf{J}z$ a measure on a line segment in the direction of $\mathbf{J}$ in $\mathbb{R}^n$. We say that $\mathbf{S}_t$ itself has geometric Lévy dynamics, meaning that each component satisfies $dS_{i,t}/S_{i,t-} = dL_{i,t}$ where $\mathbf{L}_t = [L_{1,t}, \ldots, L_{n,t}]'$ follows an SDE of the type (A.2).

B. Exponential Utility

For Merton's problem with CRRA utility function the optimal wealth process X_t^* achieved by picking the constant portfolio fractions $\boldsymbol{\omega}^*$ is itself a one dimensional geometric Lévy process whose characteristic triple is $(X_{t-}^{*}\left(\boldsymbol{\omega}^{*\prime}\mathbf{R}+r-K\right), X_{t-}^{*2}\boldsymbol{\omega}^{*\prime}\boldsymbol{\sigma}\boldsymbol{\sigma}'\boldsymbol{\omega}^{*}, \mu)$ where $\mu(dy) = \lambda\nu(dz)$ with $y = X_{t-}^{*}\boldsymbol{\omega}'\mathbf{J}z$. In this case, the investor keeps constant fractions of wealth in each risky asset, and the constant remaining fraction $1-\sum_j \omega_j$ in the riskless asset.

Now, consider an investor with exponential utility, $U(C) = -\frac{1}{q}\exp(-qC)$ with CARA coefficient $q > 0$. We can look for a solution to (2.10) in the form

$$L(x) = -\frac{K}{q}e^{-rqx} \tag{B.1}$$

so that

$$\frac{\partial L(x)}{\partial x} = -rqL(x), \quad \frac{\partial^2 L(x)}{\partial x^2} = r^2q^2L(x). \tag{B.2}$$

Then (2.10) reduces to

$$\begin{aligned} 0 = \max_{\{C_t,\boldsymbol{\omega}_t\}} & \Big\{U(C_t) - \beta L(X_t) - rqL(X_t)\left(X_t r + X_t\boldsymbol{\omega}_t'\mathbf{R} - C_t\right) \\ & + \frac{1}{2}r^2q^2L(X_t)X_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t + \lambda\int_0^1 \left[e^{-rqX_t\boldsymbol{\omega}_t'\mathbf{J}z}L(X_t) - L(X_t)\right]\nu(dz)\Big\} \end{aligned} \tag{B.3}$$

that is

$$\begin{aligned} 0 = \min_{\{C_t,\boldsymbol{\omega}_t\}} & \left\{\frac{U(C_t)}{rqL(X_t)} - \frac{\beta}{qr} - \left(X_t r + X_t\boldsymbol{\omega}_t'\mathbf{R} - C_t\right)\right. \\ & \left. + \frac{1}{2}rqX_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t + \frac{\lambda}{rq}\int_0^1 \left[e^{-qX_t\boldsymbol{\omega}_t'\mathbf{J}z} - 1\right]\nu(dz)\right\} \end{aligned} \tag{B.4}$$

after division by $qL(X_t)$ (note that max becomes min as a result of $qL(X_t) < 0$).

The optimal policy of $\boldsymbol{\omega} = X_t\boldsymbol{\omega}_t$ is given by the objective function

$$\min_{\{\omega\}}\left(-\boldsymbol{\omega}'\mathbf{R} + \frac{1}{2}rq\boldsymbol{\omega}'\Sigma\boldsymbol{\omega} + \frac{\lambda}{rq}\int_0^1 \left[e^{-rq\boldsymbol{\omega}'\mathbf{J}z} - 1\right]\nu(dz)\right), \tag{B.5}$$

and the optimal consumption choice is therefore

$$C_t^* = rX_t - \frac{1}{q}\log(rK). \tag{B.6}$$

Finally, we evaluate equation (B.4) at C^* and $\boldsymbol{\omega}^*$ to identify K,

$$K = \frac{1}{r}\exp\left(1 - \frac{\beta}{r} - q\boldsymbol{\omega}^{*\prime}\mathbf{R} + \frac{1}{2}rq^2\boldsymbol{\omega}^{*\prime}\Sigma\boldsymbol{\omega}^{*} + \frac{\lambda}{r}\int_0^1 \left[e^{-q\boldsymbol{\omega}^{*\prime}\mathbf{J}z} - 1\right]\nu(dz)\right). \tag{B.7}$$

C. Log Utility

Finally, consider an investor with log utility, $U(x) = \log(x)$. We can look for a solution to (2.10) in the form

$$L(x) = K_1^{-1} \log(x) + K_2 \tag{C.1}$$

where K_1 and K_2 are constant, so that

$$\frac{\partial L(x)}{\partial x} = K_1 x^{-1}, \quad \frac{\partial^2 L(x)}{\partial x^2} = -K_1 x^{-2}. \tag{C.2}$$

Then (2.10) reduces to

$$\begin{aligned} 0 = \max_{\{C_t, \boldsymbol{\omega}_t\}} & \Big\{ \log(C_t) - \beta K_1^{-1} \log(X_t) - \beta K_2 + K_1^{-1} X_t^{-1} \left(r X_t + \boldsymbol{\omega}_t' \mathbf{R} X_t - C_t \right) \\ & - \frac{1}{2} K_1^{-1} \boldsymbol{\omega}_t' \Sigma \boldsymbol{\omega}_t + \lambda K_1^{-1} \int_0^1 \log\left(1 + \boldsymbol{\omega}_t' \mathbf{J} z\right) \nu(dz) \Big\} \end{aligned} \tag{C.3}$$

that is

$$\begin{aligned} 0 = \min_{\{C_t, \boldsymbol{\omega}_t\}} & \Big\{ -\ln(C_t) + \beta K_1^{-1} \ln(X_t) + \beta K_2 - K_1^{-1} r - K_1^{-1} \boldsymbol{\omega}_t' \mathbf{R} + K_1^{-1} X_t^{-1} C_t \\ & + K_1^{-1} \frac{1}{2} \boldsymbol{\omega}_t' \Sigma \boldsymbol{\omega}_t - \lambda K_1^{-1} \int_0^1 \ln\left(1 + \boldsymbol{\omega}_t' \mathbf{J} z\right) \nu(dz) \Big\} \end{aligned} \tag{C.4}$$

The optimal policy of $\boldsymbol{\omega}_t$ is given by the objective function,

$$\min_{\{\boldsymbol{\omega}_t\}} \left(-\boldsymbol{\omega}_t' \mathbf{R} + \frac{1}{2} \boldsymbol{\omega}_t' \Sigma \boldsymbol{\omega}_t - \lambda \int_0^1 \log\left(1 + \boldsymbol{\omega}_t' \mathbf{J} z\right) \nu(dz) \right), \tag{C.5}$$

and the optimal consumption choice is therefore

$$C_t^* = K_1 X_t. \tag{C.6}$$

To identify K_1 and K_2, we evaluate equation (C.4) at C^* and $\boldsymbol{\omega}^*$,

$$K_2 = \frac{1}{\beta} \left\{ \log(\beta) + \frac{r}{\beta} + \frac{1}{\beta} \boldsymbol{\omega}^{*\prime} \mathbf{R} - 1 - \frac{1}{2\beta} \boldsymbol{\theta}^{*\prime} \Sigma \boldsymbol{\theta}^* + \frac{\lambda}{\beta} \int_0^1 \log\left(1 + \boldsymbol{\theta}^{*\prime} \mathbf{J} z\right) \nu(dz) \right\} \tag{C.7}$$

$$K_1 = \beta. \tag{C.8}$$

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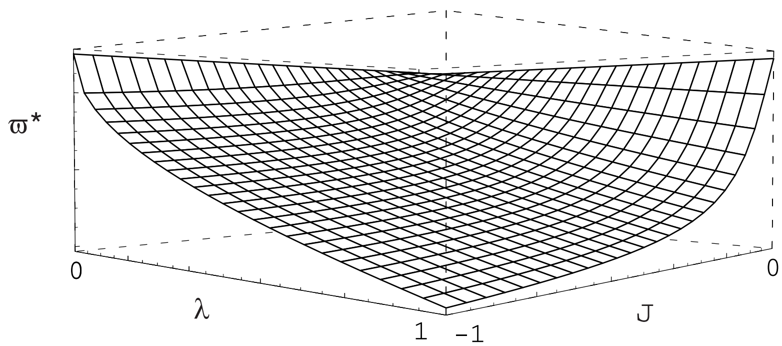

Figure 1: Optimal portfolio weight ϖ^*_∞ as a function of $\bar{J}$ and λ.

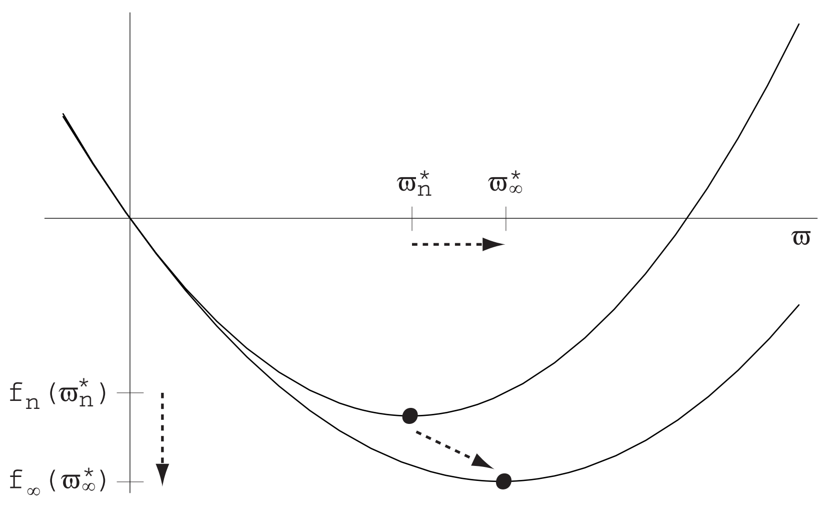

Figure 2: Scalar objective function used to determine the optimal portfolio weight ϖ^*_n and its large asset asymptotic limit, ϖ^*_∞.

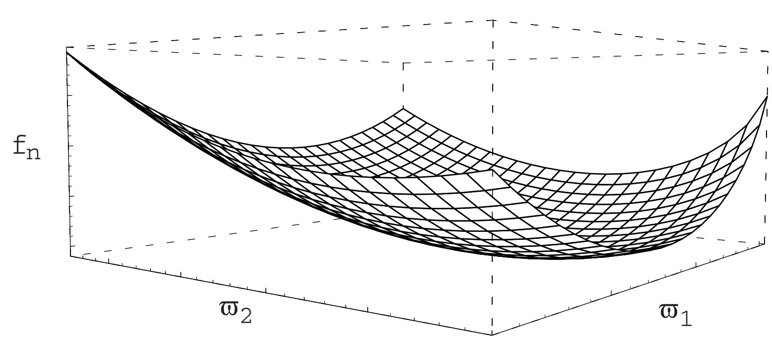

Figure 3: Bivariate objective function in a two-sector economy.