Finite Model Theory and CSPs

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Part I

FIRST-ORDER LOGIC, TYPES AND GAMES

Relational Structures vs. Functional Structures

Structures:

$$\mathbf{M} = (M, R_1^{\mathbf{M}}, R_2^{\mathbf{M}}, \ldots, f_1^{\mathbf{M}}, f_2^{\mathbf{M}}, \ldots)$$

I go relational:

relational structures
(≡ no functions)

Algebraists go functional:

algebras
(≡ no relations).

First-Order Logic: Syntax

Let $x_1, x_2, \ldots$ be a collection of first-order variables (intended to range over the points of the universe of a structure).

Definition

The collection of first-order formulas of σ (FO) is defined as:

- $x_{i_1} = x_{i_2}$ and $R_i(x_{i_1}, \ldots, x_{i_r})$ are formulas,
- $x_{i_1} \neq x_{i_2}$ and $\neg R_i(x_{i_1}, \ldots, x_{i_r})$ are formulas,
- if φ and ψ are formulas, so is $(\varphi \wedge \psi)$
- if φ and ψ are formulas, so is $(\varphi \vee \psi)$
- if φ is a formula, so is $(\exists x_i)(\varphi)$
- if φ is a formula, so is $(\forall x_i)(\varphi)$.

First-Order Logic: Semantics

Let $\varphi(\mathbf{x})$ be a formula with free variables $\mathbf{x} = (x_1, \ldots, x_r)$, let $\mathbf{A}$ be a structure, and let $\mathbf{a} = (a_1, \ldots, a_r) \in A^r$.

$$\mathbf{A} \models \varphi(x_1/a_1, \ldots, x_r/a_n)$$

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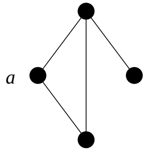
$$\mathbf{A} \models \varphi(x_1/a_1, \ldots, x_r/a_n)$$

Example

$$\varphi(x) := (\forall y)(\exists z)(E(x, z) \wedge E(y, z)).$$



$$\mathbf{G} \models \varphi(x/a)$$

Fragments:
full (FO)

- $x_{i_1} = x_{i_2}$ and $R_i(x_{i_1}, \ldots, x_{i_r})$ are formulas,
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Types

Definition

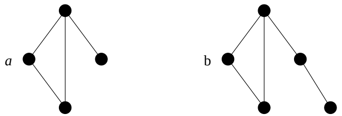
Let $\mathbf{A}$ be a structure, let $\mathbf{a} = (a_1, \ldots, a_r)$ an r-tuple in A^r, and let L be a collection of first-order formulas:

1. $\mathrm{tp}_L(\mathbf{A}, \mathbf{a}) = \{\varphi(x_1, \ldots, x_r) \in L : \mathbf{A} \models \varphi(x_1/a_1, \ldots, x_r/a_r)\}$
2. $\mathrm{tp}_L(\mathbf{A}) = \{\varphi \in L : \mathbf{A} \models \varphi\}$

Intuitively: if $\mathrm{tp}_L(\mathbf{A}, \mathbf{a}) \subseteq \mathrm{tp}_L(\mathbf{B}, \mathbf{b})$, then every L-expressible property satisfied by $\mathbf{a}$ in $\mathbf{A}$ is also satisfied by $\mathbf{b}$ in $\mathbf{B}$. We write

$$\mathbf{A}, \mathbf{a} \leq^L \mathbf{B}, \mathbf{b}$$

Examples of Types



$$\mathbf{G}, a \not\preceq^{\mathrm{FO}} \mathbf{H}, b$$

because

- In **G**, every point has a common neighbor with a
- In **H**, not every point has a common neighbor with b

$$\varphi(x) := (\forall y)(\exists z)(E(x,z) \wedge E(y,z))$$

Meaning of Types

What does $\mathbf{A}, \mathbf{a} \leq^L \mathbf{B}, \mathbf{b}$ mean?

- when $L = \{$all atomic formulas$\}$, it means

  the mapping $(a_i \mapsto b_i : i = 1, \ldots, r)$ *is a* **homomorphism** *between the substructures induced by* **a** *and* **b**

- when $L = \{$all atomic and negated atomic formulas$\}$, it means

  the mapping $(a_i \mapsto b_i : i = 1, \ldots, r)$ *is an* **isomorphism** *between the substructures induced by* **a** *and* **b**

Meaning of Types

What does $\mathbf{A}, \mathbf{a} \leq^L \mathbf{B}, \mathbf{b}$ mean?

- when $L = \{$all formulas with at most one quantifier$\}$, it means

  the substructures induced by $\mathbf{a}$ *and* $\mathbf{b}$ *are isomorphic and have the same types of extensions by one point*

- when $L = \{$all formulas with at most two quantifiers$\}$, it means

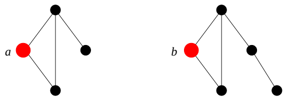
  the substructures induced by ...

- note that $\mathbf{A}, \mathbf{a} \leq^{\mathrm{FO}} \mathbf{B}, \mathbf{b}$ iff $\mathbf{B}, \mathbf{b} \leq^{\mathrm{FO}} \mathbf{A}, \mathbf{a}$. We write

$$\mathbf{A}, \mathbf{a} \equiv^L \mathbf{B}, \mathbf{b}$$

Ehrenfeucht-Fraïssé Games

Two players: Spoiler and Duplicator
Two structures: **A** and **B**
Unlimited pebbles: $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$
An initial position: $\mathbf{a} \in A^r$ and $\mathbf{b} \in B^r$
Rounds:



Referee: Spoiler wins if at any round the mapping $p_i \mapsto q_i$ is not a partial isomorphism. Otherwise, Duplicator wins.

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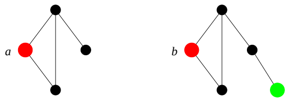

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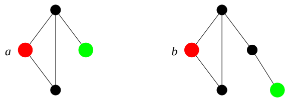

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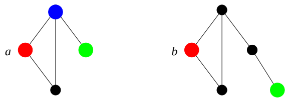

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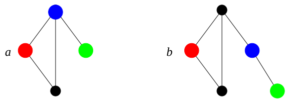

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Back-and-Forth Systems

Definition (Fraïssé)

An n-round winning strategy for the Duplicator on $\mathbf{A}, \mathbf{a}$ and $\mathbf{B}, \mathbf{b}$ is a sequence of non-empty sets of partial isomorphisms $(F_i : i < n)$ such that $(\mathbf{a} \mapsto \mathbf{b}) \in F_0$ and

1. Retract For every $i < n$, every $f \in F_i$ and every $g \subseteq f$, we have $g \in F_i$,
2. Forth For every $i < n - 1$, every $f \in F_i$, and every $a \in A$, there exists $g \in F_{i+1}$ with $a \in \mathrm{Dom}(g)$ and $f \subseteq g$.
3. Back For every $i < n - 1$, every $f \in F_i$, and every $b \in B$, there exists $g \in F_{i+1}$ with $b \in \mathrm{Rng}(g)$ and $f \subseteq g$.

$\mathbf{A}, \mathbf{a} \equiv^{\mathrm{EF}} \mathbf{B}, \mathbf{b}$: there is an n-round winning strategy for every n

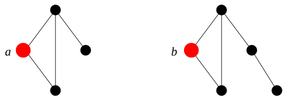
Indistinguishability vs Games

Theorem (Ehrenfeucht, Fraïssé)

$$\mathbf{A}, \mathbf{a} \equiv^{\mathrm{FO}} \mathbf{B}, \mathbf{b} \textit{ if and only if } \mathbf{A}, \mathbf{a} \equiv^{\mathrm{EF}} \mathbf{B}, \mathbf{b}$$

Ehrenfeucht-Fraïssé Games for Fragments

Two players: Spoiler and Duplicator
Two structures: **A** and **B**
Unlimited pebbles: $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$
Rounds:



Referee: Spoiler wins if at any round the mapping $p_i \mapsto q_i$ is not a partial **isomorphism** (resp. **homomorphism**).

$\equiv^{\mathrm{EF}}$, $\leq^{\exists\mathrm{EF}}$, $\leq^{\exists\mathrm{EF}^+}$: winning strategy for every n.

Indistinguishability vs Games

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$$\mathbf{A}, \mathbf{a} \leq^{\exists\mathrm{FO}^+} \mathbf{B}, \mathbf{b} \text{ if and only if } \mathbf{A}, \mathbf{a} \leq^{\exists\mathrm{EF}^+} \mathbf{B}, \mathbf{b}$$

Indistinguishability is too Strong for Finite Structures

For finite structures, these concepts add nothing:

- $\mathbf{A} \equiv^{\mathrm{FO}} \mathbf{B} \iff \mathbf{A} \cong \mathbf{B}$
- $\mathbf{A} \leq^{\exists\mathrm{FO}^+} \mathbf{B} \iff \mathbf{A} \rightarrow \mathbf{B}$

where

- $\mathbf{A} \cong \mathbf{B}$: there is an isomorphism between $\mathbf{A}$ and $\mathbf{B}$
- $\mathbf{A} \rightarrow \mathbf{B}$: there is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$

Indistinguishability is too Strong for Finite Structures

Why? Canonical formulas:

- For every finite $\mathbf{A}$, there exists an FO-sentence $\varphi_{\mathbf{A}}$ such that

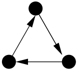
$$\mathbf{B} \models \varphi_{\mathbf{A}} \iff \mathbf{A} \cong \mathbf{B}$$

- For every finite $\mathbf{A}$, there exists an $\exists\mathrm{FO}^{+}$-sentence $\psi_{\mathbf{A}}$ such that

$$\mathbf{B} \models \psi_{\mathbf{A}} \iff \mathbf{A} \to \mathbf{B}.$$

Second has a name: the canonical conjunctive query of $\mathbf{A}$ [Chandra and Merlin]

Examples of Canonical Formulas

Let **A** be: 

$$\begin{aligned}\varphi_{\mathbf{A}} \quad = \quad & (\exists x)(\exists y)(\exists z) \\ & \quad (x \neq y \wedge y \neq z \wedge x \neq z \wedge \\ & \quad (\forall u)(u = x \vee u = y \vee u = z) \wedge \\ & \quad E(x,y) \wedge E(y,z) \wedge E(z,x) \wedge \\ & \quad \neg E(y,x) \wedge \neg E(z,y) \wedge \neg E(x,z) \wedge \\ & \quad \neg E(x,x) \wedge \neg E(y,y) \wedge \neg E(z,z)).\end{aligned}$$

$$\begin{aligned}\psi_{\mathbf{A}} \quad = \quad & (\exists x)(\exists y)(\exists z) \\ & \quad (E(x,y) \wedge E(y,z) \wedge E(z,x))\end{aligned}$$

First-Order Logic: k-Variable Fragments

Let us limit the set of first-order variables to $x_1, \ldots, x_k$.

- FO^k: k-variable fragment of FO
- $\exists\mathrm{FO}^k$: k-variable fragment of $\exists\mathrm{FO}$
- $\exists\mathrm{FO}^{+,k}$: k-variable fragment of $\exists\mathrm{FO}^+$

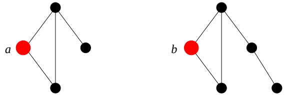
Note: Variables may be reused!

Example:

$$\begin{aligned}\mathrm{path}_5(x, y) \quad := \quad & (\exists z)(E(x, z) \wedge \\ & \quad (\exists x)(E(z, x) \wedge \\ & \quad\quad (\exists z)(E(x, z) \wedge \\ & \quad\quad\quad (\exists x)(E(z, x) \wedge E(x, y))))).\end{aligned}$$

k-Pebble EF-Games

Two players: Spoiler and Duplicator
Two structures: $\mathbf{A}$ and $\mathbf{B}$
Limited pebbles: $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$
An initial position: $\mathbf{a} \in A^r$ and $\mathbf{b} \in B^r$
Rounds:



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$\equiv^{\mathrm{EF}^k}$, $\leq^{\exists\mathrm{EF}^k}$, $\leq^{\exists\mathrm{EF}^{+,k}}$: a strategy for every n.

Indistinguishability vs k-Pebbles Games

Theorem (Barwise, Immerman, Kolaitis and Vardi)

$$\mathbf{A}, \mathbf{a} \equiv^{\mathrm{FO}^k} \mathbf{B}, \mathbf{b} \textit{ if and only if } \mathbf{A}, \mathbf{a} \equiv^{\mathrm{EF}^k} \mathbf{B}, \mathbf{b}$$

$$\mathbf{A}, \mathbf{a} \leq^{\exists\mathrm{EF}^k} \mathbf{B}, \mathbf{b} \textit{ if and only if } \mathbf{A}, \mathbf{a} \leq^{\exists\mathrm{FO}^k} \mathbf{B}, \mathbf{b}$$

$$\mathbf{A}, \mathbf{a} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B}, \mathbf{b} \textit{ if and only if } \mathbf{A}, \mathbf{a} \leq^{\exists\mathrm{EF}^{+,k}} \mathbf{B}, \mathbf{b}$$

Fundamental Questions

Obviously,

- $\mathbf{A} \cong \mathbf{B} \Longrightarrow \mathbf{A} \equiv^{\mathrm{FO}^k} \mathbf{B}$
- $\mathbf{A} \to \mathbf{B} \Longrightarrow \mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B}$

k-**Width Problem**: For what **A**'s do we have

- $\mathbf{A} \equiv^{\mathrm{FO}^k} \mathbf{B} \Longrightarrow \mathbf{A} \cong \mathbf{B}$
- $\mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B} \Longrightarrow \mathbf{A} \to \mathbf{B}$

Width-*k* Problem: For what **B**'s do we have

- $\mathbf{A} \equiv^{\mathrm{FO}^k} \mathbf{B} \Longrightarrow \mathbf{A} \cong \mathbf{B}$
- $\mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B} \Longrightarrow \mathbf{A} \to \mathbf{B}$

Why Are These Questions Relevant for Us?

Theorem (Kolaitis and Vardi)

For finite **A** *and* **B**, *the following are equivalent:*

- **A** $\leq^{\exists\mathrm{FO}^{k,+}}$ **B**
- *the (strong) k-consistency algorithm run on the CSP instance given by the scopes in* **A** *and the constraint relations in* **B** *does not detect a contradiction.*

Note: The k-consistency algorithm runs in polynomial time for every fixed k.

Why Are These Questions Relevant for Us?

The k-width/width-k problems (for homomorphisms) aim for a classification of the scopes/templates that are solvable by a widely used algorithm.

Part II

ON THE k-WIDTH PROBLEM

Some Sufficient Conditions for FO^k

Theorem (Lindell)

If $\mathbf{G}$ *is a tree of degree* d*, then for all* $\mathbf{H}$ *we have*

$$\mathbf{G} \equiv^{\mathrm{FO}^{d+2}} \mathbf{H} \Longrightarrow \mathbf{G} \cong \mathbf{H}$$

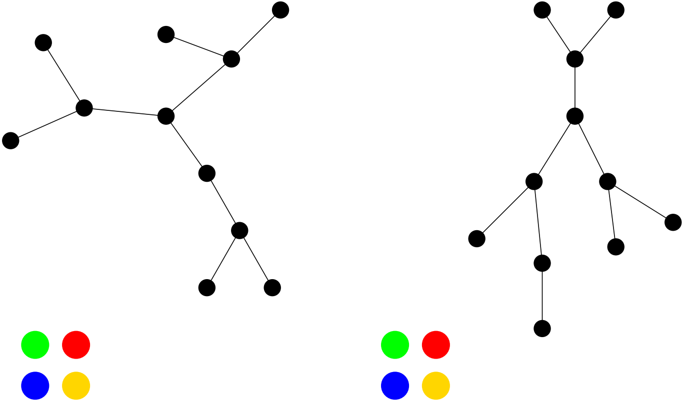
Theorem (Grohe)

If $\mathbf{G}$ *is a 3-connected planar graph, then for all* $\mathbf{H}$ *we have*

$$\mathbf{G} \equiv^{\mathrm{FO}^{30}} \mathbf{H} \Longrightarrow \mathbf{G} \cong \mathbf{H}$$

Proof of Lindell's Theorem

Proof by Example:



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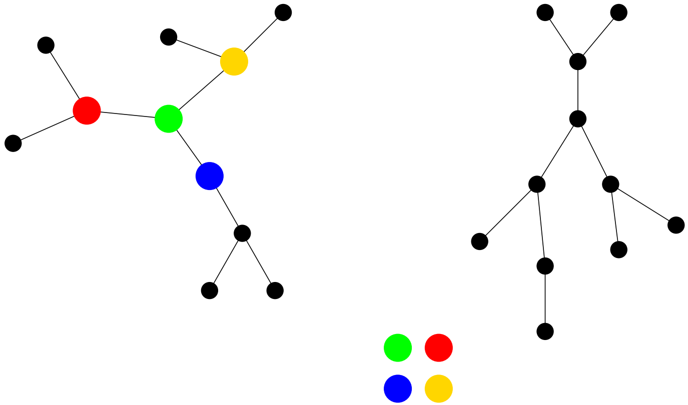

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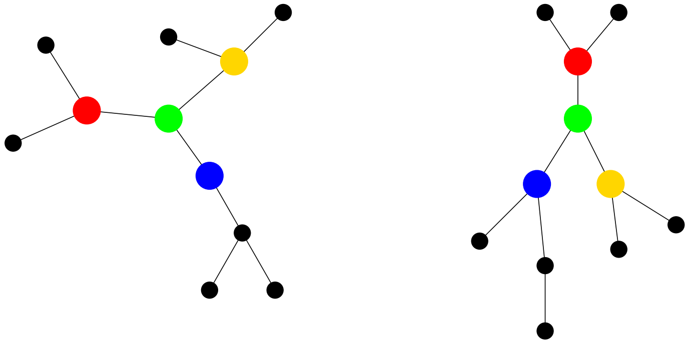

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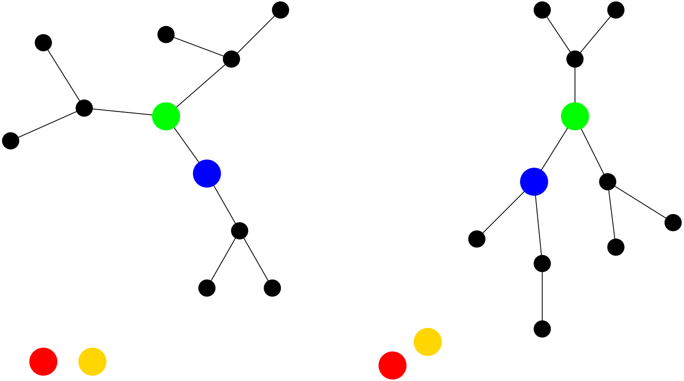

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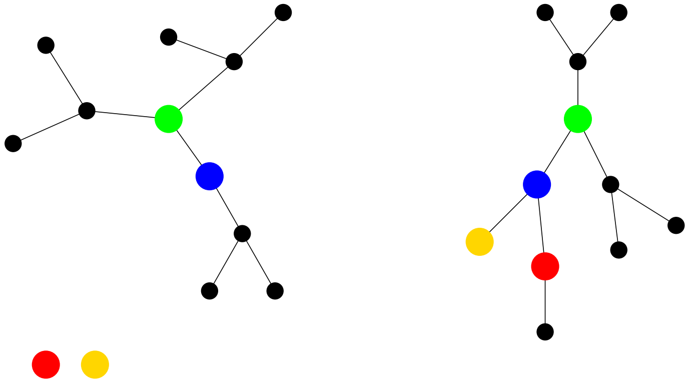

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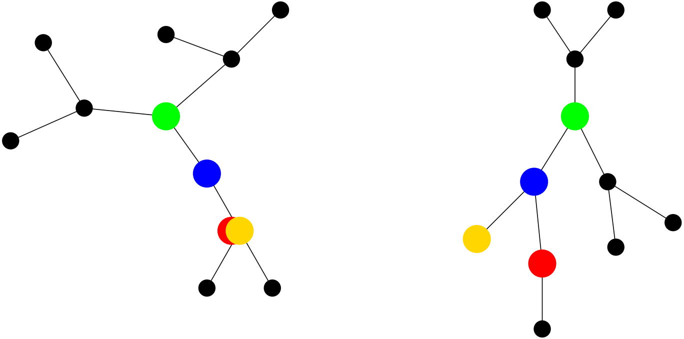
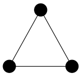


Treewidth

Definition

- $\mathbf{K}_{k+1}$ is a k-tree,
- if $\mathbf{G}$ is a k-tree, then adding a vertex connected to all vertices of a $\mathbf{K}_k$-subgraph of $\mathbf{G}$ is a k-tree.



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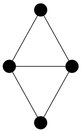

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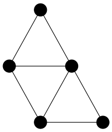

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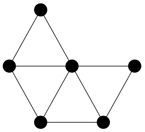

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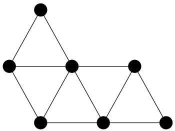

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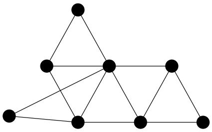

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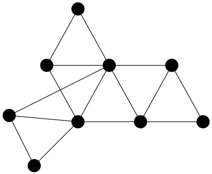

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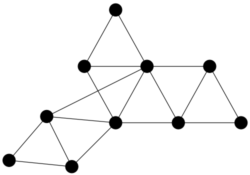

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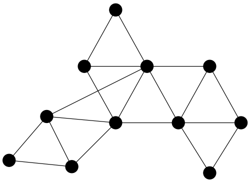

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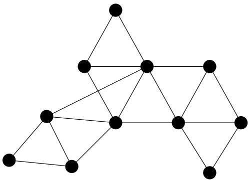

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A graph has treewidth at most k if it is the subgraph of a k-tree.

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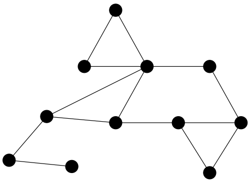

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Sufficient Condition for $\exists\mathrm{FO}^{+,k}$

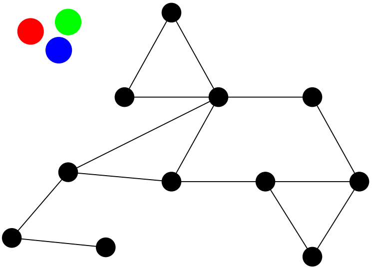
Theorem (Dalmau, Kolaitis, and Vardi)

If the treewidth of the Gaifman graph of the core of **A** *is less than* k*, then for all* **B** *be have*

$$\mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B} \Longrightarrow \mathbf{A} \to \mathbf{B}$$

Proof of DKV's Theorem

Proof by Example:



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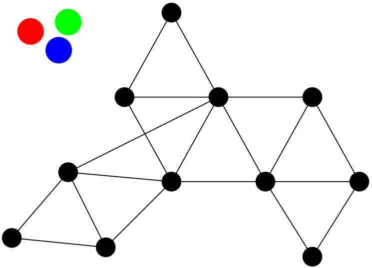

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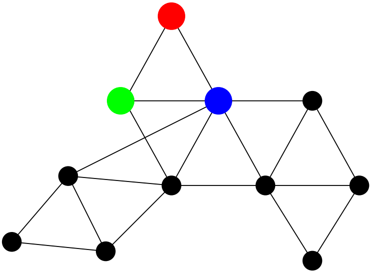

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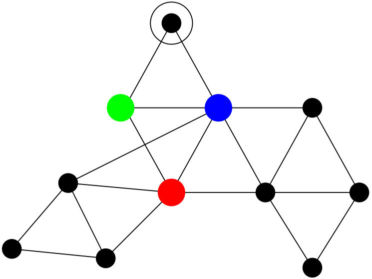

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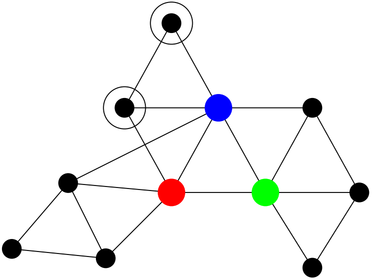

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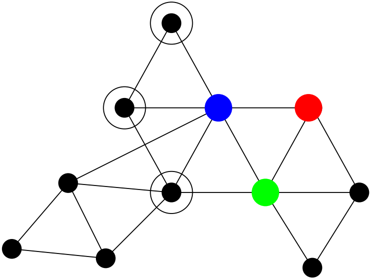

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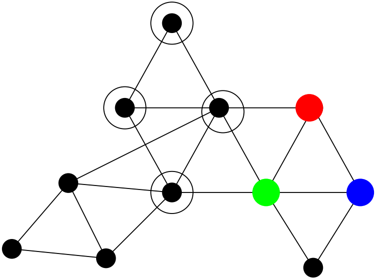

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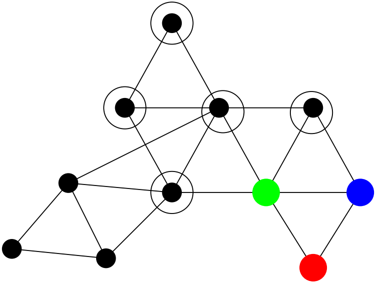

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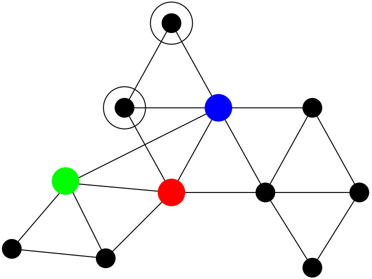

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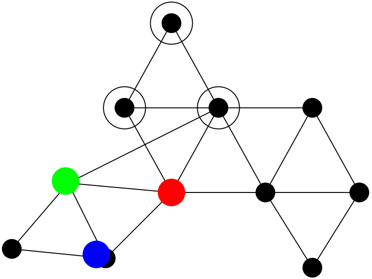

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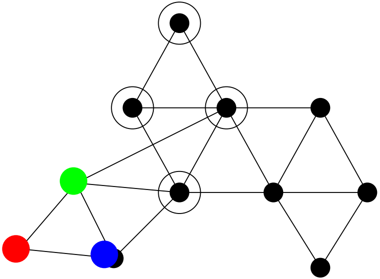


Necessary Conditions?

Solving the k-width problem for $\equiv^{\mathrm{FO}^k}$ looks like an extraordinarily difficult question (undecidable?)

But, perhaps surprisingly, for $\leq^{\exists\mathrm{FO}^{+,k}}$ it is doable.

Theorem (A..., Bulatov, and Dalmau)

The DKV condition is also necessary.

Corollary

The following are equivalent:

1. *the treewidth of the Gaifman graph of the core of* **A** *is less than k*
2. $\mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B} \Longrightarrow \mathbf{A} \to \mathbf{B}$ *for every* **B**.

Part III

INDUCTIVE DEFINITIONS AND DATALOG

Inductive Definitions: Example

There is a path from x to y:

$$\begin{aligned} P^{(0)}(x,y) &:= \text{false} \\ P^{(n+1)}(x,y) &:= x = y \vee (\exists z)(E(x,z) \wedge P^{(n)}(z,y)). \end{aligned}$$

$$P(x,y) \equiv \bigvee_n P^{(n)}(x,y)$$

Inductive Definitions: General Form

Let $\varphi(\mathbf{x}, X)$ be a formula with r variables and an r-ary second order variable X that appears *positively*. We form the iterates:

$$\begin{array}{rcl} \varphi^{(0)}(\mathbf{x}) & := & \text{false} \\ \varphi^{(n+1)}(\mathbf{x}) & := & \varphi(\mathbf{x}, X/\varphi^{(n)}) \end{array}$$

The union $U = \bigvee_n \varphi^{(n)}$ is a fixed point, in fact the least one:

Theorem (Knaster-Tarski)

On every finite structure,

- $U(\mathbf{x}) \equiv \varphi(\mathbf{x}, X/U)$
- *if* $X(\mathbf{x}) \equiv \varphi(\mathbf{x}, X)$, *then* $U(\mathbf{x}) \subseteq X(\mathbf{x})$.

Fixed-Point Logics

Least Fixed Point Logic (LFP):
closure of FO under inductive definitions.

Existential LFP ($\exists\mathrm{LFP}$):
closure of $\exists\mathrm{FO}$ under inductive definitions.

Existential-Positive LFP ($\exists\mathrm{LFP}^+$):
closure of $\exists\mathrm{FO}^+$ under inductive definitions.

And the k-variable fragments: LFP^k, $\exists\mathrm{LFP}^k$, and $\exists\mathrm{LFP}^{+,k}$

Fixed-Point Logics vs. Datalog

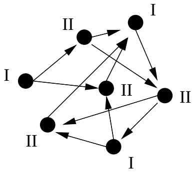
Datalog is just convenient syntax for $\exists\mathrm{LFP}^+$:

$$\begin{aligned} P(x,y) &\ :-\ \ x = y \\ P(x,y) &\ :-\ \ (\exists z)(E(x,z) \wedge P(z,y)) \end{aligned}$$

We can do the same for LFP.

Example:

Game reachability.



$$\begin{aligned} P(x,y) &\ :-\ \ I(x) \wedge E(x,y) \\ P(x,y) &\ :-\ \ I(x) \wedge (\exists z)(E(x,z) \wedge P(z,y)) \\ P(x,y) &\ :-\ \ II(x) \wedge (\forall z)(E(x,z) \rightarrow P(z,y)) \end{aligned}$$

Infinitary Logics

Infinitary Logic ($\mathrm{L}_{\infty\omega}$):
closure of FO under infinitary conjunctions and disjunctions

Existential $\mathrm{L}_{\infty\omega}$ ($\exists\mathrm{L}_{\infty\omega}$):
closure of $\exists\mathrm{FO}$ under infinitary conjunctions and disjunctions

Existential-Positive $\mathrm{L}_{\infty\omega}$ ($\exists\mathrm{L}^{+}_{\infty\omega}$):
closure of $\exists\mathrm{FO}^{+}$ under infinitary conjunctions and disjunctions

And the *k*-variable fragments: $\mathrm{L}^{k}_{\infty\omega}$, $\exists\mathrm{L}^{k}_{\infty\omega}$, and $\exists\mathrm{L}^{+,k}_{\infty\omega}$

Fixed-Point Logics and Infinitary Logics

Carefully reusing variables we have [Barwise, Kolaitis and Vardi]:

$$\mathrm{FO}^k \subseteq \mathrm{LFP}^k \subseteq \mathrm{L}^k_{\infty\omega}$$

Also for fragments [Kolaitis and Vardi]:

$$\exists\mathrm{FO}^k \subseteq \exists\mathrm{LFP}^k \subseteq \exists\mathrm{L}^k_{\infty\omega}$$

$$\exists\mathrm{FO}^{+,k} \subseteq \exists\mathrm{LFP}^{+,k} \subseteq \exists\mathrm{L}^{+,k}_{\infty\omega}$$

Theorem (Kolaitis and Vardi)

For finite **A** *and* **B***, the following are equivalent:*

1. $\mathbf{A} \equiv^{\mathrm{FO}^k} \mathbf{B}$ *(resp.* $\mathbf{A} \leq^{\exists\mathrm{FO}^k} \mathbf{B}$ *and* $\mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B}$*)*
2. $\mathbf{A} \equiv^{\mathrm{L}^k_{\infty\omega}} \mathbf{B}$ *(resp.* $\mathbf{A} \leq^{\exists\mathrm{L}^k_{\infty\omega}} \mathbf{B}$ *and* $\mathbf{A} \leq^{\exists\mathrm{L}^{+,k}_{\infty\omega}} \mathbf{B}$*)*

Proof sketch (only for $\exists\mathrm{FO}^{+,k}$ vs $\exists\mathrm{L}^{+,k}_{\infty\omega}$):

- The appropriate game for $\exists\mathrm{L}^{+,k}_{\infty\omega}$ goes on for infinitely (ω) rounds.
- But after $|A|^k|B|^k + 1$ rounds, some configuration must repeat, so if Spoiler has not won yet, Duplicator can survive forever. Q.E.D.

Part IV

ON THE WIDTH-k PROBLEM

Structures Having Width-$\exists\mathrm{FO}^{+,k}$

Theorem (Kolaitis and Vardi, Feder and Vardi)

The following are equivalent:

1. $\mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B}$ *implies* $\mathbf{A} \rightarrow \mathbf{B}$ *for every* $\mathbf{A}$
2. $\neg\mathrm{CSP}(\mathbf{B})$ *is* $\exists\mathrm{LFP}^{+,k}$*-definable*
3. $\neg\mathrm{CSP}(\mathbf{B})$ *is* $\exists\mathrm{L}^{+,k}_{\infty\omega}$*-definable*

Where "$\neg\mathrm{CSP}(\mathbf{B})$ is definable in L" means that there exists a sentence φ in L such that $\mathbf{A} \models \varphi$ iff $\mathbf{A} \not\rightarrow \mathbf{B}$.

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4. $\neg\mathrm{CSP}(\mathbf{B})$ *is* $\exists\mathrm{LFP}^{k}$*-definable*
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Proof of $\neg$ (1) $\Rightarrow$ $\neg$ (5)

Suppose there exist $\mathbf{A} \leq^{\exists \mathrm{FO}^{+,k}} \mathbf{B}$ such that $\mathbf{A} \not\to \mathbf{B}$.

So $\neg\mathrm{CSP}(\mathbf{B})$ is not $\exists \mathrm{L}^{k}_{\infty\omega}$-definable. Q.E.D.

Proof of $\neg$ (1) $\Rightarrow$ $\neg$ (5)

Suppose there exist $\mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B}$ such that $\mathbf{A} \not\to \mathbf{B}$.

Claim:

$$\begin{array}{ccc} \mathbf{A} & \leq^{\exists\mathrm{FO}^{+,k}} & \mathbf{B} \\ \downarrow & & \uparrow \\ \mathbf{A} & \leq^{\exists\mathrm{FO}^{k}} & \mathbf{A} \times \mathbf{B} \end{array}$$

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Strategy for Duplicator:

- copy move on $\mathbf{A}$ on first component,
- use the h from the strategy for $\mathbf{A} \leq^{\exists\mathrm{FO}^{+,k}} \mathbf{B}$ for the second.

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Why does it work?

- $a \in R^{\mathbf{A}}$ implies $(a, h(a)) \in R^{\mathbf{A}\times\mathbf{B}}$ because $h :\subset \mathbf{A} \to \mathbf{B}$,
- $a \notin R^{\mathbf{A}}$ implies $(a, h(a)) \notin R^{\mathbf{A}\times\mathbf{B}}$ by the definition of $\mathbf{A} \times \mathbf{B}$.

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But then any $\exists\mathrm{L}^k_{\infty\omega}$ formula that holds on $\mathbf{A}$ also holds on $\mathbf{A} \times \mathbf{B}$.

So $\neg\mathrm{CSP}(\mathbf{B})$ is not $\exists\mathrm{L}^k_{\infty\omega}$-definable. Q.E.D.

How far can we take this?

Questions:

- Can we add LFP to the list?
- Can we add $\mathrm{L}^k_{\infty\omega}$ to the list?
- What are the LFP-definable $\mathrm{CSP}(\mathbf{B})$'s? (resp. $\mathrm{L}^k_{\infty\omega}$)

A partial answer:

Theorem (A..., Bulatov, and Dawar)

If $\mathrm{CSP}(\mathbf{B})$ is definable in $\mathrm{L}^k_{\infty\omega}$, then the variety of the algebra of $\mathbf{B}$ omits types 1 and 2.

This strengthens a result of Larose and Zádori (who had bounded width instead).

Roadmap of the Proof: I

Part 1: Systems of equations in any non-trivial Abelian group is not $\mathrm{L}^k_{\infty\omega}$-definable.

To do that, we construct two systems of equations $\mathbf{A}_1$ and $\mathbf{A}_2$ such that

1. $\mathbf{A}_1$ is satisfiable
2. $\mathbf{A}_2$ is unsatisfiable
3. $\mathbf{A}_1 \equiv^{\mathrm{FO}^k} \mathbf{A}_2$

Ideas borrowed from:

- A result of Cai, Fürer and Immerman in finite model theory.
- A construction of Tseitin in propositional proof complexity.
- Treewidth and the robber cop games of Thomas and Seymour in structural graph theory.

Roadmap of the Proof: II

Part 2: Definability of $\neg\mathrm{CSP}(\mathbf{B})$ in fragments that are closed under Datalog-reductions (such as LFP and beyond) implies that the CSPs with an algebra having a reduct in the variety of the algebra of $\mathbf{B}$ are definable.

This required formalizing the appropriate reductions as Datalog-reductions: homomorphic images, powers, subalgebras. See also [Larose and Zádori, Larose and Tesson].

Roadmap of the Proof: III

Part 3: Algebraic: if the variety does not omit type 1 or 2, then it has the reduct of a module.

Part V

CLOSING REMARKS

Further Directions

- Are there digraphs with width-k, for some $k > 3$ but not width-3?
- Prove that the width-k problem for $\equiv^{\mathrm{FO}^k}$ is undecidable.
- Does $\mathrm{LFP} \cap \mathrm{HOM} = \exists\mathrm{LFP}^+$?