Algebraic Tractability Criteria for Infinite-Domain Constraint Satisfaction Problems

Manuel Bodirsky

Institut für Informatik
Humboldt-Universität zu Berlin

May 2007

Outline

The Constraint Satisfaction Problem

Let $\Gamma = (V; R_1, \ldots, R_l)$ be a relational structure. V might be infinite!
Let τ be the (finite) signature of Γ.

CSP(Γ)

Input: A finite τ-structure S.

Question: Is there a homomorphism from S to Γ?

The Constraint Satisfaction Problem

Let $\Gamma = (V; R_1, \ldots, R_l)$ be a relational structure. V might be infinite!
Let τ be the (finite) signature of Γ.

CSP(Γ)

Input: A finite τ-structure S.

Question: Is there a homomorphism from S to Γ?

Examples:

CSP($(\mathbb{N}, =, \neq)$): is x and y disconnected wrt $=$ whenever '$x \neq y$' is in S?
CSP($(\mathbb{Q}, <)$): digraph acyclicity
CSP($(\mathbb{Q}, \{(x, y, z) \mid x < y < z \ \vee \ z < y < x\})$): the betweenness problem

Acyclic H-colorings

Fix digraph H.

Acyclic H-coloring: (Feder+Hell+Mohar)

Input: A digraph G

Question: Can we H-color G such that each color class is acylic?

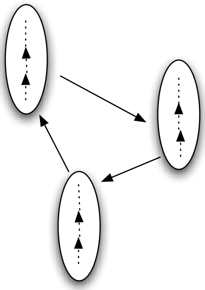
Acyclic H-colorings

Fix digraph H.

Acyclic H-coloring: (Feder+Hell+Mohar)

Input: A digraph G

Question: Can we H-color G such that each color class is acylic?



Formulation as CSP(Γ): choose $\Gamma = H[(\mathbb{Q}, <)]$.

CSPs in Temporal Reasoning

Betweenness:

CSP$((\mathbb{Q}, \{(x, y, z) \mid x < y < z \ \vee \ z < y < x\}))$

NP-complete (Garey+Johnson)

Min-Ordering:

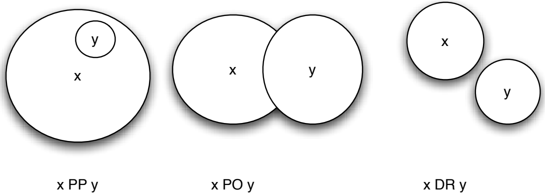
CSP$((\mathbb{Q}, \{(x, y, z) \mid x > y \ \vee \ x > z\}))$

Simple linear time algorithm

Neither Datalog nor Maltsev-like

Spatial Reasoning

Formalism 'RCC-5' in Artificial Intelligence



Consistency Problem for Basic Relations:

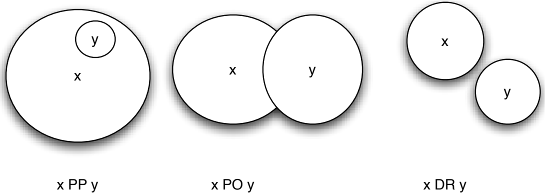
Input: A relational structure (V, DR, PO, PP) where DR, PO, PP are binary relations

Question: Can we assign non-empty regions satisfying all the constraints?

Formulation as CSP(Γ):

Spatial Reasoning

Formalism 'RCC-5' in Artificial Intelligence



Consistency Problem for Basic Relations:

Input: A relational structure (V, DR, PO, PP) where DR, PO, PP are binary relations

Question: Can we assign non-empty regions satisfying all the constraints?

Formulation as CSP(Γ):
choose $\Gamma = (2^X \setminus \emptyset, DR, PO, PP)$ for an infinite set X

A Fundamental Lemma

Let $\mathcal{C}$ be a set of τ-structures.

Definition 1.

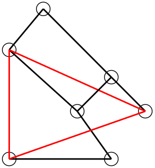
$\mathcal{C}$ is closed under disjoint unions if whenever $A, B \in \mathcal{C}$ then $A + B \in \mathcal{C}$.
$\mathcal{C}$ is closed under inverse homomorphisms if $B \in \mathcal{C}$ and $A \rightarrow^h B$ implies $A \in \mathcal{C}$.

Example: the set of all triangle-free graphs

Observation (Feder+Vardi).

$\mathcal{C}$ =CSP(Γ) for some relational structure Γ if and only if
$\mathcal{C}$ is closed under disjoint unions and inverse homomorphisms.

Examples of CSPs

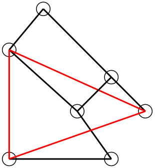


Triangle-Freeness:

Input: A graph G

Question: Is G triangle-free?

Examples of CSPs



Triangle-Freeness:

Input: A graph G

Question: Is G triangle-free?

Vector-space CSP:

Input: A system of linear equations $x + y = z$ and disequations $x \neq y$

Question: Is there a d and an assignment of d-dimensional Boolean vectors to the variables that satisfies all the constraints?

Outline

Outline

Primitive Positive Interpretations

Definition 2 (First-order Interpretation).

A τ-structure Δ has a interpretation in Γ if there is

- first-order formula $\delta(x_1, \ldots, x_d)$,
- for each m-ary $R \in \tau$ a first-order formula $\phi_R(x_1, \ldots, x_{md})$, and
- a surjective map $h : \delta(\Gamma^d) \to \Delta$

such that for all $a_1, \ldots, a_m \in \delta(\Gamma^d)$

$$\Delta \models R(h(a_1), \ldots, h(a_m)) \Leftrightarrow \Gamma \models \phi_R(a_1, \ldots, a_m) .$$

Note: much more powerful than first-order definitions.

An interpretation is primitive positive if δ and the ϕ_R are primitive positive.

Observation .

If Δ has a pp-interpretation in Γ then there is a polynomial-time reduction from CSP(Δ) to CSP(Γ).

ω-categoricity

How can we recognize whether Γ pp-interprets Δ?

Definition 3.

A relational structure Γ is ω-categorical iff every countable model of the first-order theory of Γ is isomorphic to Γ.

Example: $(\mathbb{Q}, <)$ (Kantor)

More: Infinite-dimensional vector-spaces over finite fields
The countable atomless boolean algebra
The countably infinite random graph
The universal homogeneous triangle-free graph

Many more: All Fraisse-limits of amalgamation classes of structures with finite signature are ω-categorical (and homogeneous)

ω-categorical Templates

Every CSP we have seen in this talk so far can be formulated with an ω-categorical Γ!

Get new ω-categorical structures from old:

Observation .

If Δ is first-order interpretable in an ω-categorical structure Γ, then Δ is also ω-categorical.

- Allen's Interval Algebra and all its fragments are ω-categorical
- All CSPs in MMSNP can be formulated with an ω-categorical template (MB+Dalmau'06)

The Basic Galois Connection

Theorem 4 (Engeler,Ryll-Nardzewski,Svenonius).

Tfae:

- Γ is ω-categorical
- $\text{Aut}(\Gamma)$ is oligomorphic, i.e., there are finitely many orbits of *k*-tuples in $\text{Aut}(\Gamma)$, for each *k*
- all orbits of *k*-tuples in $\text{Aut}(\Gamma)$ are first-order definable

Inv-Aut form a Galois connection between structures and permutation groups:

- Inv(Aut(Γ)): expansion by all first-order definable relations
- Aut(Inv(F)): locally closed permutation group

A Preservation Theorem

Definition 5.

A homomorphism f from Γ^k to Γ is called a polymorphism.
We say that f preserves all relations in Γ.

Example: $(x, y) \mapsto \max(x, y)$
is a polymorphism of $(\mathbb{Q}, <)$,
but not of $(\mathbb{Q}, \mathit{Betweenness})$

Theorem 6 (MB+Nesetril'03).

A relation R has a pp definition in an ω-categorical structure Γ if and only if R is preserved by all polymorphisms of Γ.

Homomorphic Equivalence

Definition 7.

Two structures Γ, Δ are homomorphically equivalent if Γ is homomorphic to Δ and vice versa.

Example: $H[(\mathbb{Q}, <)]$ and $(\mathbb{Q}, <)$ are homomorphically equivalent for any finite acyclic digraph H.

Observation .

Two ω-categorical structures Γ and Δ are homomorphically equivalent if and only if CSP(Γ) equals CSP(Δ).

Cores

Let Γ be ω-categorical.

Definition 8.

Γ is called a core if every endomorphism of Γ is an embedding.
Γ is called model-complete if every embedding of Γ into Γ is elementary.

Theorem 9.

Every ω-categorical structure Γ is homomorphically equivalent to a model-complete core Δ. Moreover,

- Δ is unique up to isomorphism
- orbits of k-tuples are primitive positive definable in Δ
- Δ is ω-categorical.

Consequence: can expand cores Δ by finitely many constants without changing the complexity of CSP(Δ).

The Algebra of a Template

Definition 10.

The algebra $\mathrm{Al}(\Gamma)$ of Γ

- has the same domain as Γ.
- has as functions the polymorphisms of Γ.

Observation: $\mathrm{Al}(\Gamma)$ is a locally closed clone, i.e.,

- contains all projections,
- is closed under compositions, and
- is locally closed: if for all finite subsets S of the domain there is $g \in \mathrm{Al}(\Gamma)$ s.t. $g(a) = f(a)$ for all $a \in S^k$, then $f \in \mathrm{Al}(\Gamma)$.

The Pseudo-Variety of an Algebra

Definition 11.

The smallest class of algebras that contains an algebra $\mathbf{A}$ and is closed under subalgebras, homomorphic images, and finite direct products is called the pseudo-variety $\mathcal{V}(\mathbf{A})$ generated by $\mathbf{A}$.

Let Γ be ω-categorical.

Theorem 12.

A relational structure Δ has a primitive positive interpretation in Γ

if and only if

there is algebra $\mathbf{B}$ in $\mathcal{V}(\text{Al}(\Gamma))$ all of whose operations are polymorphism of Δ.

Hardness

Let Γ be an ω-categorical model-complete core.

Theorem 13.

If there is an expansion Γ' of Γ by finitely many constants such that $\mathcal{V}(AI(\Gamma'))$ contains a 2-element algebra where all operations are essentially permutations, then CSP(Γ) is NP-hard.

All hard ω-categorical CSPs satisfy this condition.

Conjecture.

Assuming P $\neq$ NP, the opposite implication is true as well.

Outline

Outline

Quasi Near-unanimity Operations

A quasi near-unanimity function (qnuf) is an operation satisfying

$$f(x,\ldots,x,x,y) = f(x,\ldots,x,y,x) = \cdots = f(y,x,\ldots,x) = f(x,\ldots,x)$$

Theorem 14 (MB+Dalmau'06).

An ω-categorical model-complete core Γ has a k-ary qnuf if and only if CSP(Γ) has strict bounded width (and hence, CSP(Γ) is tractable).

Remark: AI community says "local k-consistency implies global consistency" if CSP(Γ) has strict width $k-1$.

Examples:

- $(\mathbb{Q},<)$ has a majority.
- $(\mathbb{Q},\leq,\neq)$ has a 5-ary, but no nuf and no 4-ary qnuf (MB+Chen'07 / Koubarakis)
- $(\mathbb{N},\{(x,y,u,v) \mid x\neq y \vee u \neq v\})$ has a 5-ary, but no 4-ary qnuf.

Horn Tractability

If $\Gamma = (D; R_1, \ldots, R_l)$ is a relational structure,
denote by Γ^c the expansion of Γ by $\neg R_1, \ldots, \neg R_l$.

Theorem 15 (MB+Chen+Kara+vonOertzen'07).

Suppose that

- Γ is ω-categorical and admits quantifier-elimination
- Δ is first-order definable in Γ
- $\mathrm{CSP}(\Gamma^c)$ is tractable
- there is an isomorphism $i : \Gamma^2 \to \Gamma$, and
- Δ is preserved by i.

Then $\mathrm{CSP}(\Delta)$ is tractable.

Horn Tractability

If $\Gamma = (D; R_1, \ldots, R_l)$ is a relational structure,
denote by Γ^c the expansion of Γ by $\neg R_1, \ldots, \neg R_l$.

Theorem 15 (MB+Chen+Kara+vonOertzen'07).

Suppose that

- Γ is ω-categorical and admits quantifier-elimination
- Δ is first-order definable in Γ
- $\mathrm{CSP}(\Gamma^c)$ is tractable
- there is an isomorphism $i : \Gamma^2 \to \Gamma$, and
- Δ is preserved by i.

Then $\mathrm{CSP}(\Delta)$ is tractable.

Idea: All relations in Δ have a quantifier-free Horn definition in Γ.

Remarks

"Let i be an isomorphism between Γ^2 and Γ."

1. Has no analogon in the finite!
2. Not so rare for infinite structures
3. There is automorphism α of Γ such that i satisfies

$$i(x, y) = \alpha(i(y, x))$$

4. All relations with a fo-definition in Γ that are preserved by i form a maximal constraint language.

Applications

Equality Constraints:

$\Gamma := (\mathbb{N}, =)$.

$\Gamma^c = (\mathbb{N}, =, \neq)$ clearly tractable.

i: any bijection between $\mathbb{N}^2$ and $\mathbb{N}$.

$\Delta = (\mathbb{N}, \{(x, y, u, v) \mid x = y \rightarrow u = v)\}$ tractable!

Horn Vector-Space Equations:

$\Gamma := (\mathbb{V}, \{(x, y, z) \mid x + y = z\})$ the infinite-dimensional vector space over a finite field

$\Gamma^c = (\mathbb{V}, \{(x, y, z) \mid x + y = z\}), \{(x, y, z) \mid x + y \neq z\})$
is tractable essentially by Gaussian elimination

i: an isomorphism between $\mathbb{V}^2$ and $\mathbb{V}$.

Hence: can solve Horn equations over $\mathbb{V}$.

More Applications

Spatial Constraints:

$\Gamma := (\mathbb{B}, PP, DR)$ the countable atomless boolean algebra without zero, $PP = \{(x, y) \mid xy = y\}$, $DR = \{(x, y) \mid (x + y)x = x\}$

Γ^c tractable (Renz+Nebel: Datalog)

i: an isomorphism between $\mathbb{B}^2$ and $\mathbb{B}$

Δ: the maximal tractable tractable language that appeared in Drakengren+Jonsson and Renz+Nebel.

Similar applications for

- the universal triangle-free graph,
- “partially-ordered time”,
- “set-constraints”, ...

Outline

Outline

Equality Constraint Languages

An equality constraint language is a relational structure Γ with a first-order definition in $(\mathbb{N}, =)$.

Examples:

- $\Gamma = (\mathbb{N}, \neq, =)$
- $\Gamma = (\mathbb{N}, \{x = y \vee u = v\}, \{x \neq y \vee u \neq v\})$
- $\Gamma = (\mathbb{N}, \{x = y \rightarrow u = v\})$

Equality Constraint Languages

An equality constraint language is a relational structure Γ with a first-order definition in $(\mathbb{N}, =)$.

Examples:

- $\Gamma = (\mathbb{N}, \neq, =)$
- $\Gamma = (\mathbb{N}, \{x = y \vee u = v\}, \{x \neq y \vee u \neq v\})$
- $\Gamma = (\mathbb{N}, \{x = y \rightarrow u = v\})$

Theorem 16 (MB+Kara'07).

Let Γ be an equality constraint language. Then either

- $\mathcal{V}(\mathrm{Al}(\Gamma))$ contains a two-element algebra where all ops. are projections (and CSP(Γ) is NP-complete),
- Γ has a polymorphism f, α satisfying $f(x, y) = \alpha(f(y, x))$ (and CSP(Γ) is in P).

Equality Constraint Languages

An equality constraint language is a relational structure Γ with a first-order definition in $(\mathbb{N}, =)$.

Examples:

- $\Gamma = (\mathbb{N}, \neq, =)$
- $\Gamma = (\mathbb{N}, \{x = y \vee u = v\}, \{x \neq y \vee u \neq v\})$
- $\Gamma = (\mathbb{N}, \{x = y \rightarrow u = v\})$

Theorem 16 (MB+Kara'07).

Let Γ be an equality constraint language. Then either

- $\mathcal{V}(\text{Al}(\Gamma))$ contains a two-element algebra where all ops. are projections (and CSP(Γ) is NP-complete),
- Γ has a polymorphism f, α satisfying $f(x, y) = \alpha(f(y, x))$ (and CSP(Γ) is in P).

Proof uses polymorphisms and a Ramsey-like argument.

Conclusion

1. Allowing countable (ω-categorical) templates greatly expands the scope of (non-uniform) CSPs
2. Polymorphisms are very useful to study their complexity
3. Many more concepts from universal algebra might have generalizations to the oligomorphic setting