ALGEBRAS WITH FEW SUBPOWERS ARE TRACTABLE

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The results presented here are joint work with J. Berman, P. Idziak, R. McKenzie, M. Valeriote and R. Willard.

We introduce some new invariants for finitely generated varieties $\mathcal{V}(\mathbf{A})$. The behavior of these invariants is very closely tied to many classical Maltsev properties.

More precisely, we make the following definitions, for any finite algebra $\mathbf{A}$: For every positive integer n,

- $\mathrm{s}_{\mathbf{A}}(n)$ is the logarithm, base 2, of the cardinality of the lattice of subuniverses of $\mathbf{A}^n$.
- $\mathrm{g}_{\mathbf{A}}(n)$ is the least integer κ such that every subuniverse of $\mathbf{A}^n$ has an at most κ-element generating set.
- $\mathrm{i}_{\mathbf{A}}(n)$ is the least integer κ such that every independent subset of $\mathbf{A}^n$ has at most κ elements (where $X \subseteq A^n$ is independent iff no proper subset of X generates the same subalgebra of $\mathbf{A}^n$ as does X).

We prove that the property that function $\mathrm{i}_{\mathbf{A}}(n)$ has growth 'similar' to n^k is a strong Maltsev property. The other two functions related to subpowers satisfy

(1) $\mathrm{g}_{\mathbf{A}}(n) \leq \mathrm{i}_{\mathbf{A}}(n) \leq \mathrm{s}_{\mathbf{A}}(n) \leq C \cdot n \cdot \mathrm{g}_{\mathbf{A}}(n)$, for $C = \log_2 |A|$.

These three functions exhibit a strong dichotomy, namely either all three have polynomial growth, or all three are at least singly exponential. We will say that the finite algebras for which the first holds have *few subpowers*. If the growth of the three functions is polynomial, then $\mathcal{V}(\mathbf{A})$ is a congruence modular variety.

We prove that the condition that $\mathbf{A}$ has few subpowers implies the existence of terms $d(x, y)$, $p(x, y, z)$ and $s(x_0, x_1, \ldots, x_k)$ satisfying

$$\begin{aligned}
p(x,y,y) &\approx x\\
p(x,x,y) &\approx d(x,y)\\
d(x,d(x,y)) &\approx d(x,y)\\
s(x,y,y,y,\ldots,y,y) &\approx d(y,x)\\
s(y,x,y,y,\ldots,y,y) &\approx y\\
s(y,y,x,y,\ldots,y,y) &\approx y\\
&\vdots\\
s(y,y,y,y,\ldots,y,x) &\approx y.
\end{aligned}$$

Now, a pair $(a, b) \in A^2$ is a *minority* pair if $d(a, b) = b$. The pair $(f, g) \in (A^n)^2$ is a minority splitting with index (i, a, b) when $f(j) = g(j)$ for all $j < i$, $f(i) = a$, $g(i) = b$ and (a, b) is a minority pair.

Let $\mathbf{A}$ be such that $\mathrm{i}_{\mathbf{A}}(n)$ has growth 'similar' to n^k. We prove that a subalgebra $\mathbf{B} \leq \mathbf{A}^n$ is characterized by its projections $proj_I(B)$ to all subsets $I \subseteq n$ with $|I| \leq k$ and with the set of indices of all the minority splittings in B^2.

This enables us to slightly modify V. Dalmau's algorithm from [1] and generalize his result to say that finite algebras with few subalgebras of powers are tractable.

References

[1] V. Dalmau, *Generalized majority-minority operations are tractable,* in P. Panangaden, editor, Proceedings of the Twentieth Annual IEEE Symp. on Logic in Computer Science, LICS 2005, 438–447. IEEE Computer Society Press, June 2005.