Bounded width problems and algebras

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CSP and algebras

- Let A be finite relational structure of finite type. Let $CSP(A)$ denote the *constraint satisfaction problem* over A.
- To each problem $CSP(A)$ is associated an *algebra* $\mathbf{A}$:
  base set of $\mathbf{A}$ = base set of A
  operations of $\mathbf{A}$ = operations preserving the relations of A.
- This talk is focused on finite algebras that arise from so-called *bounded width* CSP's; problems of the form $CSP(A)$ for which a particular local algorithm decides the problem in polynomial time.

Structure of Talk

- Definition of bounded width
- Bounded strict width
- (l, k)-tree duality
- The width 1 case
- Examples of width 2 and of no bounded width
- Bounded width and the Hobby-McKenzie types
- Related notions of width
- Results in the congruence distributive case

Definition of bounded width

- In a 1998 paper Feder and Vardi studied a special type of CSP's termed *problems of bounded width*.
- Their original definition of these problems involves a logical programming language called Datalog, or comes equivalently via certain two-player games.
- Both of their definitions are proved to be equivalent to what follows.

Definition of bounded width

- Let k be a positive integer. The subsets of size at most k of a set are called *k-subsets*.
- Fix a structure A and integers $0 \leq l < k$.

(l, k)-algorithm

Input: Structure I similar to A.

Initial step: To every k-subset K of I assign the relation $\rho_K = Hom(K, A) \leq \mathbf{A}^{\mathbf{K}}$.

Iteration step:

- Choose, provided they exist, two k-subsets H and K of I such that $|H \cap K| \leq l$ and there is a map $\varphi \in \rho_H$ with the property that $\varphi|_{H \cap K}$ does not extend to any map in ρ_K.
- Then throw out all such maps from ρ_H.
- If no such H and K are found then stop and output the current relations assigned to the k-subsets of I.

Definition of bounded width

- The relations given in the initial step are called the *input relations* of the (l, k)-algorithm.
- We refer to the relations ρ_K obtained during the algorithm as *k-relations*.
- The k-relations obtained at the end of the algorithm are called the *output relations*.
- Observe that the k-relations are all subalgebras of a power of $\mathbf{A}$.
- Moreover, the output relations form an *l-consistent* system of relations, i.e., any two of them restricted to a common domain of size at most l are the same.

Definition of bounded width

- Notice that the choice of the pair H and K in each iteration step of the algorithm is arbitrary.
- So the (l, k)-algorithm has several different versions depending on the method of the choice of the pair H and K.
- By using induction one can prove that the output relations produced by the (l, k)-algorithm are the same for all versions of the algorithm.
- Since the number of k-subsets of I is $\mathcal{O}(|I|^k)$, and in each iteration step the sum of the sizes of the k-relations is decreasing, one can make the algorithm stop in polynomial time in the size of the structure I.

Definition of bounded width

- Clearly, if the output relations of the (l,k)-algorithm for I are empty then there is no homomorphism from I to A; however, it might be that the converse does not hold.
- We say that a problem $CSP(A)$ has *width (l,k)* if for any input structure I there exists a homomorphism from I to A whenever the output relations of the (l,k)-algorithm are nonempty.
- $CSP(A)$ has *width l* if it has width (l,k) for some k
- CSP(A) has *bounded width* if it has width l for some l.
- Structure A has *width (l,k)*, *width l*, *bounded width* if the related CSP(A) has the same properties.

Definition of bounded width

- It follows that $CSP(A)$ has bounded width if and only if for some choice of parameters l and k the (l, k)-algorithm correctly decides the problem $CSP(A)$: in particular, we get that $CSP(A) \in \mathbf{P}$.
- Suppose that $(l, k) \leq (l', k')$. It can be easily verified that if $CSP(A)$ has width (l, k) then it has width (l', k').

Bounded strict width

- Let $k \geq 3$. A k-ary operation t satisfying the identities $t(y, x, \ldots, x) = t(x, y, \ldots, x) = \cdots = t(x, \ldots, x, y) = x$ is called a *near-unanimity operation*.
- A structure A is called *k-near-unanimity* if it admits a k-ary near-unanimity operation.
- If A is k-near-unanimity then it is $k+1$-near-unanimity. Indeed, $s(x_1, \ldots, x_k, x_{k+1}) = t(x_1, \ldots, x_k)$ is a $(k+1)$-ary near-unanimity operation if t is a k-ary nu operation.

Bounded strict width theorem (Feder and Vardi)

Let $2 \leq l < k$.

1. *Every $(l+1)$-near-unanimity structure whose relations are at most k-ary has width (l, k).*
2. *Every $(l+1)$-near-unanimity structure has width l.*

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Proof of the theorem for $l = 2$ and $k = 3$:

- Let A be a 3-near-unanimity structure with at most ternary relations.

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- Let $j = 4$, $\{1,2,3,4\}$ any four element subset of I and (a,b,c) any tuple in the output relation $\rho_{\{1,2,3\}}$.

$$\begin{array}{c|c|ccc|cl}
1 & a & & a & a & a & \\
2 & b & b & & b & b & \\
3 & c & c & c & & c & \\
4 & & d_1 & d_2 & d_3 & d & = t(d_1, d_2, d_3)
\end{array}$$

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1	a		a	a	a	
2	b	b	b'	b	b	
3	c	c	c	c'	c	
4		d_1	d_2	d_3	d	$= t(d_1, d_2, d_3)$

Bounded strict width

- Then any 3-projection of the 4-tuple (a, b, c, d) is in the related ternary output relation. Hence (a, b, c, d) is a homomorphism from $\{1, 2, 3, 4\}$ to A.

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- Then we replace the ternary ρ relations with the 4-ary relations that correspond to the four element subsets of I and contain the tuples (a, b, c, d) whose any 3-projection is in the related ternary output relation.

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- For $j = 5$ we use these new 4-ary relations and a 4-ary nu operation.

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- Proceeding in this way, finally we get to a nonempty set of homomorphisms from I to A, Q.e.d..

- Actually, the above proof shows that every partial map from I to A which satisfies the output relations extends to a full homomorphism.

(l,k)-tree duality

- A relational structure is an *(l, k)-tree* if it is a union of certain substructures called nodes where the size of each node is at most k and the nodes can be listed in such a way that the intersection of the i-th node and the union of the first $i-1$ nodes has at most l elements and is contained in one of the the first $i-1$ nodes.
- A relational structure A has an *(l, k)-tree duality* if for any I that admits no homomorphism to A there exists an (l, k)-tree T such that T admits a homomorphism to I and admits no homomorphism to A.

Theorem (Feder and Vardi)

A structure A has width (l, k) if and only if it has an (l, k)-tree duality.

The width 1 case

- A relational structure is a *tree* if the tuples of its relations have no multiple component and the tuples can be listed in such a way that the i-th tuple intersects the union of the first $i-1$ tuples in one element. A *forest* is a disjoint union of trees.
- An n-ary operation f is *totally symmetric* if $f(a_1, \dots, a_n) = f(b_1, \dots, b_n)$ whenever $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$.
- We define a relational structure B_A of the same type as A. The base set of B_A is the set of nonempty subsets of A and for each m-ary relational symbol r
  $(A_1, \dots, A_m) \in r_{B_A}$ iff $r_A \cap \prod_{i=1}^{m} A_i$ is a subdirect product of the A_i.

The width 1 case

Width 1 Theorem (Feder and Vardi, Dalmau and Pearson)

TFAE:

1. A *has width 1.*
2. A *has a* $(1, k)$*-tree duality for some* k.
3. A *has a tree duality.*
4. B_A *admits a homomorphism to* A.
5. A *admits a totally symmetric operation of arity the maximum size of the relations of* A.

The width 1 case

We define the notion of *cycles* of I similarly to hypergraphs:

- a tuple with multiple components is a cycle,
- two different tuples without multiple components form a cycle if they share at least two components,
- more than two tuples without multiple components form a cycle if they can be listed in a cyclic way that the consecutive ones share a single component and the nonconsecutive ones share no components.

The girth of I is the length of its shortest cycle. If I is a forest its girth is defined to be the infinity.

The hardest part of the proof of the Width 1 Theorem uses a generalization of a theorem of Erdős:

Big girth lemma (Feder and Vardi)

For any I that admits no homomorphism to $\mathbb{A}$ and any positve integer n there exists a structure J of girth at least n such that J admits a homomorphism to I, but not to $\mathbb{A}$.

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- By the lemma we may assume that the girth of I is at least $k + 1$.

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- Since A has $(1, k)$-tree duality, there is a $(1, k)$-tree T that maps to I under a homomorphism f such that T does not map to A.

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- Let T' be the forest obtained from T by replacing each node of T by its f-image in I in the obvious manner (with the necessary gluing).

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- Clearly, T' maps homorphically into I.

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- Note that the f-image of each node of T in I is a forest because I has large girth.
- Let T' be the forest obtained from T by replacing each node of T by its f-image in I in the obvious manner (with the necessary gluing).
- Clearly, T' maps homorphically into I.
- Moreover T maps into T', hence T' cannot map into A.
- Thus, some tree component of T' maps to I but does not map to A, Q.e.d.

Example of a structure of width 2 but not of width 1

- Let $A = (\{0,1\};\ \{0,1\}^2 \setminus \{(0,0)\},\ \{0,1\}^2 \setminus \{(1,1)\})$.
- The clone of A is generated by the ternary nu operation.
- By the Bounded Strict Width Theorem A has width $(2,3)$.
- The only binary operations in the clone of A are the projections.
- There is no totally symmetric operation f in the clone for any arity. For otherwise $g(x,y) = f(x,y,\ldots,y)$ would be a binary commutative operation in the clone.
- Hence by the Width 1 Theorem A is not a structure of width 1.

Open questions

- Let $l < k$. Is it decidable that a finite structure of finite type has width (l, k)?
- Let $l \geq 2$. Is it decidable that a finite structure of finite type has width l?
- Is it decidable that a finite structure of finite type has bounded width?
- Is it decidable that a finite structure of finite type is near unanimity (has bounded strict width)?
- Does there exist a structure for every i that has width $i + 1$ but not width i?

Structures of no bounded width

- The first examples of structures of no bounded width are due to Feder and Vardi.
- They introduced the *structures with the ability to count* and proved that they do not have bounded width.
- Example:
  Let $(A, +)$ be an Abelian group, $a \in A, a \neq 0$. Then the structure $(A; \{0\}, \{(x, y, z) : \ x + y + z = a\})$ has the ability to count and so it does not have bounded width.

Bounded width and the Hobby-McKenzie types

- We say that a finite algebra **A** has *bounded width* if for every relational structure B (of finite type) whose base set coincides with the universe of **A** and whose relations are subalgebras of finite powers of **A**, the structure B has bounded width.
- If a relational structure A has bounded width then the related algebra **A** has bounded width.

Lemma (Larose and Zádori)

Every finite algebra in the variety generated by a bounded width algebra has bounded width.

- The variety $\mathcal{V}(\mathbf{A})$ *interprets* in the variety $\mathcal{V}(\mathbf{B})$ if there exists a clone homomorphism from the clone of term operations of **A** to the clone of term operations of **B**.
- Equivalently: $\mathcal{V}(\mathbf{A})$ interprets in $\mathcal{V}(\mathbf{B})$ if there is an algebra in $\mathcal{V}(\mathbf{A})$ with the same universe as **B**, all of whose term operations are term operations of **B**.

Bounded width and the Hobby-McKenzie types

Theorem (Larose and Zádori)

If **A** *and* **B** *are finite algebras such that* $\mathcal{V}(\mathbf{A})$ *interprets in* $\mathcal{V}(\mathbf{B})$ *and* **A** *has bounded width then* **B** *also has bounded width.*

Lemma

For a locally finite idempotent variety $\mathcal{V}$ *the following are equivalent:*

1. $\mathcal{V}$ *omits types* 1 *and* 2.
2. $\mathcal{V}$ *does not interpret in any variety generated by an affine algebra.*

Theorem (Larose and Zádori)

If **A** *is a finite idempotent algebra of bounded width then* $\mathcal{V}(\mathbf{A})$ *omits types* 1 *and* 2.

Bounded width and the Hobby-McKenzie types

Proof:

- Let **A** be any finite idempotent algebra such that $\mathcal{V}(\mathbf{A})$ admits type 1 or 2.

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- Let **A** be any finite idempotent algebra such that $\mathcal{V}(\mathbf{A})$ admits type 1 or 2.
- Then by the preceding lemma $\mathcal{V}(\mathbf{A})$ interprets in the variety generated by an affine algebra **C**.
- Since $\mathcal{V}(\mathbf{A})$ is idempotent, it interprets in $\mathcal{V}(\mathbf{B})$ where **B** is an algebra on the base set of **C** and the clone of term operations of **B** coincides with the clone of idempotent term operations of **C**.

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- Let us consider the structure $B' = (B; \{0\}, \{(x, y, z) : \ x + y + z = a\})$ where B is the base set of **B** and a is a fixed non-zero element of B.

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- The relations of B' are preserved by all operations of **B** and B' is a structure which has the ability to count.

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- The relations of B' are preserved by all operations of **B** and B' is a structure which has the ability to count.
- So B' has no bounded width.

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- Hence **B** does not have bounded width.

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- The relations of B' are preserved by all operations of **B** and B' is a structure which has the ability to count.
- So B' has no bounded width.
- Hence **B** does not have bounded width.
- Now, the preceding theorem implies that **A** does not have bounded width either, Q.e.d..

Related notions of width

- The notion of relational width is due to Bulatov.
- An algebra $\mathbf{A}$ has relational width k, if for all I and $H \subseteq 2^I$ every k-consistent system of nonempty relations $\rho_L \leq \mathbf{A}^L$, $L \in H$ admits a solution, i.e., there exists a map $\varphi : I \to A$ such that $\varphi|_L \in \rho_L$ for all $L \in H$.
- $\mathbf{A}$ has bounded relational width if it has relational width k for some k.

Theorem (Bulatov)

If $\mathbf{A}$ is a finite idempotent algebra of bounded relational width then $\mathcal{V}(\mathbf{A})$ omits types 1 and 2.

Fact

If an algebra $\mathbf{A}$ has bounded relational width then it has bounded width.

Related notions of width

- The intersection property of algebras was introduced by Valeriote.
- Let **A** be an algebra. Two subalgebras of $\mathbf{A}^I$ are *k-equal* if their restrictions to any k-subset of I agree.
- **A** has the *k-intersection property* if for every finite I and subalgebra **B** of $\mathbf{A}^I$ the intersection of the subalgebras of $\mathbf{A}^I$ that are k-equal to **B** is nonempty.
- We say that **A** has *the intersection property* if it has the k-intersection property for some k.

Fact (Valeriote)

If a finite idempotent algebra **A** *has bounded relational width then it has the intersection property.*

Theorem (Valeriote)

If a finite idempotent algebra **A** *has the intersection property then* $\mathcal{V}(\mathbf{A})$ *omits types* 1 *and* 2. .

Related notions of width

By the previous results the following implications hold for a finite idempotent algebra **A**:

A has bounded relational width	$\Longrightarrow$	**A** has the intersection property
$\Downarrow$	$\searrow$ Bulatov	$\Downarrow$ Valeriote
A has bounded width	$\Longrightarrow$ L & Z	$\mathcal{V}(\mathbf{A})$ omits the types 1 and 2

- None of the reverse implications are known to hold.
- A reasonable goal is to test them in special cases.

Results in the congruence distributive case

- A nontrivial case occurs when $\mathcal{V}(\mathbf{A})$ is a *congruence distributive variety*, i.e. the congruence lattices of the algebras in $\mathcal{V}(\mathbf{A})$ are distributive.
- It is well known that if $\mathcal{V}(\mathbf{A})$ is CD then $\mathcal{V}(\mathbf{A})$ omits types 1 and 2.
- The property that $\mathcal{V}(\mathbf{A})$ is CD is characterized by the existence of a nontrivial idempotent Malcev condition.
- This Malcev condition is thought to be a sequence of sets of identities indexed by $n = 1, 2, 3, \ldots$.
- For each n the terms satisfying the n-th set of identities are called the *n-th Jónsson terms*.

Results in the congruence distributive case

n-th Jónsson terms:

$$
\begin{aligned}
n = 1 :& \quad x = y \\
n = 2 :& \quad p(x,x,y) = p(x,y,x) = p(y,x,x) = x \\
n = 3 :& \quad p_1(x,y,x) = p_1(x,x,y) = p_2(x,y,x) = p_2(y,y,x) = x, \\
& \quad p_1(x,y,y) = p_2(x,y,y)
\end{aligned}
$$

Theorem (Kiss and Valeriote)

If a finite algebra **A** *admits 3rd Jónsson terms then it has bounded relational width.*

Results in the congruence distributive case

- Recall: **A** has the k-intersection property if for every finite I and subalgebra **B** of $\mathbf{A}^I$ the intersection of the subalgebras of $\mathbf{A}^I$ that are k-equal to **B** is nonempty.
- A weaker property: **A** has the *k-complete intersection property* if for every finite I the intersection of the subalgebras of $\mathbf{A}^I$ that are k-equal to $\mathbf{A}^I$ is nonempty.

Theorem (Valeriote)

If a finite algebra **A** *admits Jónsson terms (or equivalently $\mathcal{V}(\mathbf{A})$ is CD) then it has the 2-complete intersection property.*