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GLOBAL DYNAMICS OF A MATHEMATICAL MODEL OF COMPETITION IN THE CHEMOSTAT: GENERAL RESPONSE FUNCTIONS AND DIFFERENTIAL DEATH RATES*

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Abstract. A model of exploitative competition of n species in a chemostat for a single, essential, nonreproducing, growth-limiting resource is considered. S. B. Hsu [*SIAM J. Appl. Math.*, 34 (1978), pp. 760–763] applies LaSalle's extension theorem of Lyapunov stability theory to study the asymptotic behavior of solutions in the special case that the response functions are modeled by Michaelis–Menten dynamics. G. J. Butler and G. S. K. Wolkowicz [*SIAM J. Appl. Math.*, 45 (1985), pp. 138–151], on the other hand, allow more general response functions (including monotone and nonmonotone functions), but their analysis requires the assumption that the death rates of all the species are negligible in comparison with the washout rate, and hence can be ignored. By means of Lyapunov stability theory, the global dynamics of the model for a large class of response functions are studied, including both monotone and nonmonotone functions (though it is not as general as the class studied by Butler and Wolkowicz) and the results in Hsu are extended for this class, to the differential death-rate case. That is, it is shown that for this class the outcome depends on the relative sizes of the break-even concentrations. Provided that these concentrations are distinct, at most one competitor population avoids extinction, the one with the lowest break-even concentration. All populations approach limiting values.

Key words. chemostat, competition, Lyapunov function, differential death rates, global dynamics

AMS(MOS) subject classifications. 92A15, 92A17, 34C15, 34C35

1. Introduction. In this paper we study the global dynamics of the following model of the chemostat in which n populations of microorganisms compete exploitatively for a single, essential, nonreproducing, growth-limiting substrate:

$$(1.1) \qquad S'(t) = (S^0 - S(t))D - \sum_{i=1}^{n} \frac{x_i(t) p_i(S(t))}{y_i},$$

$$x_i'(t) = x_i(t)(-D_i + p_i(S(t))), \qquad i = 1, 2, \cdots, n,$$

where $S(0) \geq 0$ and $x_i(0) > 0$, $i = 1, 2, \cdots, n$ and S^0, D, D_i, and y_i are positive constants.

In these equations (assuming for convenience that the volume of the culture vessel is one cubic unit), $S(t)$ denotes the concentration of the substrate at time t; $x_i(t)$ denotes the concentration of the ith population of microorganisms at time t; $p_i(S)/y_i$ represents the uptake rate of substrate of the ith population; we assume that $p_i(S)$ represents the per-capita growth rate of the ith population and so y_i is a growth yield constant; S^0 denotes the concentration of substrate in the feed bottle; D denotes the input rate from the feed bottle containing the substrate and the washout rate of substrate, microorganisms, and byproducts from the growth chamber; each $D_i = D + \varepsilon_i$, where ε_i denotes the species specific death rate of species x_i. (It is interesting to note that the analysis of the model requires no assumptions on the signs of the ε_i's, as long as the D_i's all remain positive. This leaves the D_i's open to other interpretations.)

For general background on model (1.1), in the case of both monotone and nonmonotone response functions, the reader is referred to Butler and Wolkowicz [6] and the references given therein. As in [6], keeping in mind that certain substrates are growth-limiting when their concentrations are low, as well as growth-inhibiting when their concentrations are high (see, for example, [1], [2], [4], [5], [11], [16]), whereas other substrates only limit growth when their concentrations are low (see, for example, [11], [14]), we make the following assumptions on the form of the response functions p_i, $i = 1, 2, \cdots, n$ in model (1.1):

(1.2)
$$p_i : \mathbb{R}_+ \to \mathbb{R}_+,$$

(1.3)
$$p_i \text{ is continuously differentiable,}$$

(1.4)
$$p_i(0) = 0,$$

and there exist uniquely defined positive extended real numbers λ_i and μ_i, with $\lambda_i \leq \mu_i$ such that

(1.5)
$$p_i(S) < D_i \quad \text{if } S \notin [\lambda_i, \mu_i],$$
$$p_i(S) > D_i \quad \text{if } S \in (\lambda_i, \mu_i).$$

Here λ_i and μ_i represent the break-even concentrations of the substrate for the ith competitor. In the case where $p_i(S) < D_i$ for all $S > 0$, then $\lambda_i = \mu_i = +\infty$. On the other hand, if $p_i(S)$ is monotone increasing or if $p_i(S) > D_i$ for all $S > \lambda_i$, then $\mu_i = +\infty$.

In particular, we consider the following prototypes of response functions often found in the literature (see, for example, [1], [2], [4], [11], [14], [16]). Three prototypes of monotone response functions are

(1.6)   (i) Lotka-Volterra: $p_i(S) = r_i S$;

By (1.5), $r_i = D_i / \lambda_i$.

(1.7)   (ii) Michaelis–Menten: $p_i(S) = \dfrac{c_i S}{(a_i + S)}$;

For convenience, let $c_i = m_i D_i$. By (1.5), if λ_i is finite, then $m_i > 1$ and $a_i = \lambda_i(m_i - 1)$.

(1.8)   (iii) sigmoidal: $p_i(S) = \dfrac{m_i D_i S^2}{((a_i + S)(b_i + S))}$;

By (1.5), if λ_i is finite, then $m_i = \dfrac{(a_i + \lambda_i)(b_i + \lambda_i)}{\lambda_i^2} > 1$.

A prototype for a nonmonotone response function is

(1.9)   (iv) inhibition: $p_i(S) = \dfrac{m_i D_i S}{(a_i + S)(b_i + S)}$;

By (1.5), if λ_i is finite, then $m_i = (a_i + \lambda_i)(b_i + \lambda_i)/\lambda_i$ and $b_i = \lambda_i \mu_i / a_i$.

Forms (i)–(iii) are also often referred to as Holling type I, II, and III, respectively.

For a derivation of the model and the underlying assumptions in the special case that the response functions are of Michaelis–Menten form, the reader is referred to Waltman, Hubbel, and Hsu [15]. In this case, model (1.1) is often referred to as the

Monod model. In [9], Hsu, Hubbell, and Waltman give a complete global analysis of (1.1) in the case where all the response functions are assumed to be of Michaelis–Menten form and the species specific death rates are assumed to be negligible in comparison with the washout rate, i.e., $D_i = D$ for all $i = 1, \cdots, n$. Hsu [10] applies a LaSalle extension theorem (see [12]) of Lyapunov stability theory to give a simple, elegant proof of the results in [9]. The Lyapunov function he uses works in the case of differential death rates. However, this Lyapunov function seems to work only for Michaelis–Menten or Lotka–Volterra response functions $p_i(S)$. On the other hand, Butler and Wolkowicz [6] relax the restriction in [9], [10] of Michaelis–Menten kinetics. Their analysis of (1.1) is valid for any *realistic* response functions (including non-monotone response functions). However, the proof in [6] depends critically on the technical assumption that $D_i = D$ for all $i = 1, \cdots, n$. Under this assumption, it can be shown that the substrate concentration is a linear function of the competitor concentrations. Armstrong and McGehee [3] analyze the n-dimensional model that results if the equation for the substrate is eliminated from the model and the substrate variable in the competitor equations is replaced by this linear function. Their analysis is valid for any monotone response functions.

In this paper we generalize [3], [9], [10] by allowing a larger class of response functions, and we generalize [3], [6], [8] for this class of response functions by allowing differential death rates. In particular, we show that for this class of response functions all populations approach limiting values independent of their initial concentrations, but rather dependent on the relative sizes of the break-even concentrations. If all the break-even concentrations exceed the concentration of substrate in the feed vessel, then all populations of competitors die out asymptotically. Otherwise, provided these concentrations are distinct, exactly one competitor population survives, the one with the lowest break-even concentration. Thus, in this situation, the principle of competitive exclusion holds.

For competeness, we include a statement of the extension theorem (invariance principle) of Lyapunov stability theory used in this paper. It is a slightly modified version of the statements given in LaSalle [12] and Hsu [10].

Consider the system of differential equations

$$(1.10) \qquad x' = f(x), \qquad \left(' = \frac{d}{dt}\right),$$

where the vector-valued function $f: \mathscr{G}^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous.

DEFINITION 1.1. We call V a Lyapunov function on $\mathscr{G} \subseteq \mathscr{G}^*$ for (1.10) if
  (i) V is continuous on $\mathscr{G}$,
  (ii) V is not continuous at $\bar{x} \in \bar{\mathscr{G}}$ (the closure of $\mathscr{G}$) implies that

$$\lim_{\substack{x \to \bar{x} \\ x \in \mathscr{G}}} V(x) = +\infty,$$

  (iii) $\dot{V} = \nabla V \cdot f \leq 0$ on $\mathscr{G}$.

THEOREM 1.2. *Assume that V is a Lyapunov function for (1.10) on $\mathscr{G}$. Define $\mathscr{E} = \{x \in \bar{\mathscr{G}} \cap \mathscr{G}^*: \dot{V}(x) = 0\}$. Let $\mathscr{M}$ denote the largest invariant set in $\mathscr{E}$. Then every bounded (for $t \geq 0$) trajectory of (1.10) that remains in $\mathscr{G}$ (for $t \geq 0$) approaches the set $\mathscr{M}$ as $t \to \infty$.*

The proof requires only minor modifications of the proof given in Hale [7, Thm. 1.3, p. 296], where V is assumed to be continuous on $\bar{\mathscr{G}}$, and so (ii) in Definition (1.1) can be omitted.

2. Analysis of the model. Throughout, we identify the set

$$\{(S, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}: S \geq 0 \quad \text{and} \quad x_i \geq 0, i = 1, \cdots, n\}$$

with $\mathbb{R}_+^{n+1}$, and we assume that the populations are labelled so that

$$(2.1) \qquad \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n.$$

For convenience we define

$$(2.2) \qquad \rho_i \equiv \min(S^0, \mu_i), \qquad i = 1, 2, \cdots, n.$$

LEMMA 2.1. *The solutions* $S(t), x_i(t), i = 1, 2, \cdots, n$ *of* (1.1) *are positive and bounded, and if* $\lambda_i < S^0 < \mu_i$ *for some* $i \in \{1, 2, \cdots, n\}$, *then* $S(t) < S^0$ *for all sufficiently large* t.

Proof. $S(t)$ is positive for all $t > 0$ since $S(\tau) = 0$ for any $\tau \geq 0$ implies that $S'(\tau) > 0$. Also, $x_i(t) > 0$ for all $t > 0$ since the boundary face where $x_i \equiv 0$ is invariant, and hence, by uniqueness of solutions, it cannot be reached in finite time by trajectories for which $x_i(0) > 0$. It holds that

$$\left(S(t) + \sum_{i=1}^{n} \frac{x_i(t)}{y_i}\right)' \leq (S^0 - S(t))D - \sum_{i=1}^{N} \frac{x_i(t)D_i}{y_i} \leq S^0 D - \bar{D}\left(S(t) + \sum_{i=1}^{n} \frac{x_i(t)}{y_i}\right),$$

where $\bar{D} \equiv \min(D, D_1, D_2, \cdots, D_n)$. Therefore,

$$S(t) + \sum_{i=1}^{n} \frac{x_i(t)}{y_i} \leq \exp(-\bar{D}t)\left(S(0) + \sum_{i=1}^{n} \frac{x_i(0)}{y_i} - \frac{DS^0}{\bar{D}}\right) + \frac{DS^0}{\bar{D}},$$

and, since all solutions are positive, it follows that all solutions are bounded.

Suppose that $\lambda_i < S^0 < \mu_i$. If $S(t) > S^0$ for all sufficiently large t, and hence $S'(t) < 0$ for all sufficiently large t, then $S(t) \downarrow S^* \geq S^0$. However, $S^* > S^0$ implies that $S'(t) \leq (S^0 - S^*)D < 0$ for all t and hence $S(t) \downarrow -\infty$, a contradiction. If $S^* = S^0$, since $\lambda_i < S^0 < \mu_i$, $S(t) \downarrow S^0$, which implies that $S(t) \in (\lambda_i, \mu_i)$ for all sufficiently large t. This implies that $x_i'(t) > 0$ for all sufficiently large t. Therefore $x_i(t) \uparrow x_i^* > 0$. Since $x_i(t)$ and $x_i''(t)$ are bounded for all $t > 0$, using the mean value theorem (see Miller [13]) it can be shown that $x_i'(t) \to 0$ as $t \to \infty$, and hence $S(t) \to \lambda_i$ or μ_i, again a contradiction. Therefore, either $S(t) < S^0$ for all t or there exists a $\tau \geq 0$ such that $S(\tau) = S^0$. Then, however, $S'(\tau) < 0$, and so $S(t) < S^0$ for all $t \geq \tau$. $\square$

LEMMA 2.2. *For all solutions of* (1.1), *if* $\lambda_i \geq S^0$, *then* $\lim_{t \to \infty} x_i(t) = 0$.

Proof. By an argument similar to the one given in the previous lemma, it follows that either $S(t) \downarrow S^0$ as $t \to \infty$ or $S(t) < S^0$ for all sufficiently large t.

If $S(t) < S^0 \leq \lambda_i$ for all sufficiently large t, then $x_i'(t) \leq 0$ and so $x_i(t) \downarrow x_i^* \geq 0$. As in the previous lemma, it can be shown that $x_i'(t) \to 0$ and so either $x_i^* = 0$ or $S(t) \to \lambda_i$ or μ_i. However, $\mu_i > S^0$, so $S(t) \to \mu_i$ is impossible. If $\lambda_i > S^0$, then $S(t) \to \lambda_i$ is also impossible. If $\lambda_i = S^0$ and $S(t) \to \lambda_i$, then

$$S'(t) < \frac{(S^0 - S(t))D - x_i(t)p_i(S(t))}{y_i} \leq -\frac{x_i^* D_i}{(4y_i)} < 0$$

for all sufficiently large t. This implies that $S(t) \downarrow -\infty$, a contradiction.

If $S(t) \downarrow S^0$ as $t \to \infty$, the proof is similar. $\square$

THEOREM 2.3. *Assume that* $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ *and* $\lambda_1 < S^0 < \mu_1$. *If it is possible to find constants* $\alpha_i > 0$, *for each* $i \geq 2$ *satisfying* $\lambda_i < S^0$ *such that*

$$(2.3) \qquad \max_{0 < S < \lambda_1} g_i(S) \leq \alpha_i \leq \min_{\lambda_i < S < \rho_i} g_i(S),$$

where

$$(2.4) \qquad g_i(S) = \frac{p_i(S)(-D_1 + p_1(S))(S^0 - \lambda_1)}{D_1(-D_i + p_i(S))(S^0 - S)} ,$$

then all solutions of (1.1) *satisfy* $\lim_{t \to \infty} S(t) = \lambda_1$, $\lim_{t \to \infty} x_1(t) = y_1 D(S^0 - \lambda_1)/D_1$, *and* $\lim_{t \to \infty} x_i(t) = 0$, $i = 2, 3, \cdots, n$ (*i.e., the critical point* $E_{\lambda_1} = (\lambda_1, y_1 D(S^0 - \lambda_1)/D_1, 0, \cdots, 0)$ *is globally asymptotically stable with respect to the interior of* $\mathbb{R}_+^{n+1}$).

Proof. From Lemma 2.2. it follows that there is no loss of generality if we assume that $\lambda_i < S^0$ for $i = 1, \cdots, n$ since any population with $\lambda_i \geqq S^0$ approaches extinction, independent of whether there is any competition. Define the Lyapunov function $V(S, x_1, \cdots, x_n)$ as follows:

$$V = \int_{\lambda_1}^{S} \frac{(p_1(\xi) - D_1)(S^0 - \lambda_1)}{D_1(S^0 - \xi)} \, d\xi + \frac{1}{y_1}(x_1 - x_1^* - x_1^*(\ln(x_1) - \ln(x_1^*))) + \sum_{i=2}^{n} \frac{\alpha_i}{y_i} x_i$$

on the set $\mathscr{G} = \{(S, x_1, \cdots, x_n): S \in (0, S^0), x_i \in (0, \infty) \ i = 1, 2, \cdots, n\}$, where $\alpha_i \ i = 2, \cdots, n$ are positive constants to be determined and $x_1^* \equiv y_1 D(S^0 - \lambda_1)/D_1$. Then the time derivative of V computed along solutions of the differential equation is

$$\dot{V} = \frac{x_1}{y_1}(p_1(S) - D_1)\left[1 - \frac{(S^0 - \lambda_1)p_1(S)}{(S^0 - S)D_1}\right]$$
$$+ \sum_{i=2}^{n} \frac{x_i}{y_i}\left[\alpha_i(-D_i + p_i(S)) - \frac{p_i(S)(-D_1 + p_1(S))(S^0 - \lambda_1)}{D_1(S^0 - S)}\right].$$

First, note that the first term in the above sum is always nonpositive for $0 < S < S^0$ and equals zero for $S \in [0, S^0)$ if and only if $S = \lambda_1$ or $x_1 = 0$. For each $i = 2, \cdots, n$, define

$$(2.5) \qquad h_i(S) = \left[\alpha_i(-D_i + p_i(S)) - \frac{p_i(S)(-D_1 + p_1(S))(S^0 - \lambda_1)}{D_1(S^0 - S)}\right].$$

If $S \in [\lambda_1, \lambda_i]$, or if $\mu_i < S^0$ and $S \in [u_i, S^0)$, then $h_i(S) < 0$ for any choice of $\alpha_i > 0$. Thus $h_i(S) < 0$ for every $S \in (0, S^0)$, provided that it is possible to find $\alpha_i > 0$ satisfying (2.3). By the LaSalle extension theorem [12] every bounded solution of (1.1) contained in $\mathscr{G}$, and hence by Lemma 2.1 every solution of (1.1), approaches the set $\mathscr{M}$, the largest invariant subset of $\mathscr{E} = \{(S, x_1, \cdots, x_n) \in \bar{\mathscr{G}}: \dot{V} = 0\}$. $\mathscr{E}$ is made up of points of the following forms:

$$(2.6) \qquad (S, 0, \cdots, 0), \quad \text{where } S \in [0, S^0],$$

$$(2.7) \qquad (\lambda_1, x_1, 0, \cdots, 0), \quad \text{where } x_1 \in [0, \infty).$$

If any point of the form $(\lambda_1, x_1, 0, \cdots, 0)$, where $x_1 > 0$, is in the omega limit set Ω of any solution initiating in the interior of $\mathbb{R}_+^{n+1}$, it would follow that $E_{\lambda_1} \in \Omega$ since E_{λ_1} is globally asymptotically stable on the face $\{(S, x, 0, \cdots, 0): S \geqq 0, x > 0\}$. However, E_{λ_1} is a locally asymptotically stable equilibrium point of (1.1), and so if $E_{\lambda_1} \in \Omega$, it would follow that $\Omega = \{E_{\lambda_1}\}$. No point of the form (2.6) is in Ω, if $S \neq S^0$, because the entire trajectory through any point in Ω must lie in Ω. This would imply that Ω is not compact, yielding a contradiction. If $S = S^0$, since $E^0 = (S^0, 0, \cdots, 0)$ is unstable and its stable manifold does not intersect the interior of $\mathbb{R}_+^{n+1}$, this implies that $\Omega \neq \{E^0\}$, and hence some other point of $\mathscr{E}$ must lie in Ω. However, we have just shown that this is impossible, unless $\Omega = \{E_{\lambda_1}\}$. $\square$

It is of interest to identify classes of response functions where α_i can always be found, and hence Theorem 2.3 can be applied. The general form of $g_i(S)$ (see Fig. 1)
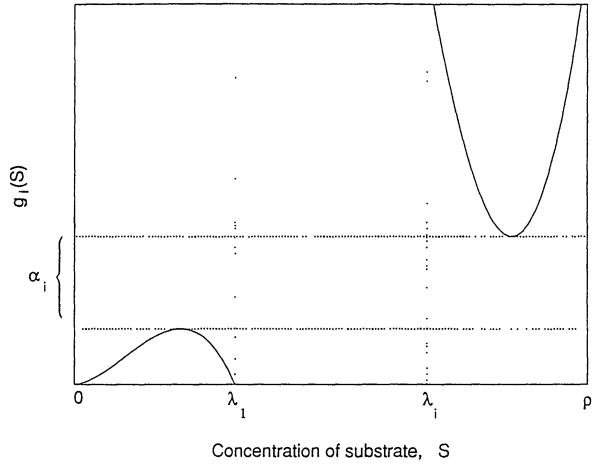
FIG. 1. *A graphical depiction of criterion (2.3). In general, $g_i(S)$ is continuous on $[0, \lambda_1]$ and on (λ_i, ρ_i); $g_i(S) \geqq 0$ for all $S \in [0, \lambda_1] \cup (\lambda_i, \rho_i)$; $g_i(0) = 0 = g_i(\lambda_1)$; $\lim_{S \to \lambda_i^+} g_i(S) = +\infty = \lim_{S \to \rho_i^-} g_i(S)$. Criterion (2.3) requires that for each $i \geqq 2$ for which $\lambda_i < S^0$, there exist $\alpha_i > 0$ such that $\max_{0 \leqq S \leqq \lambda_1} g_i(S) \leqq \alpha_i \leqq \min_{\lambda_i < S < \rho_i} g_i(S)$.*

seems to indicate that it may be possible to find such α_i for a very general class of response functions.

COROLLARY 2.4. *Assume that $\lambda_1 < \lambda_2 \leqq \cdots \leqq \lambda_n$ and $\lambda_1 < S^0 < \mu_1$. For each $i \geqq 2$ for which $\lambda_i < S^0$, if $w_i(S)$ satisfies*

(2.8)
$$w_i(S) \leqq w_i(\lambda_1), \qquad S \in [0, \lambda_1],$$
$$w_i(S) \geqq w_i(\lambda_1), \qquad S \in (\lambda_i, \rho_i),$$

where

(2.9)
$$w_i(S) = g_i(S) \frac{(S - \lambda_i)}{(S - \lambda_1)}$$

(e.g., if $w_i(S)$ is increasing for all $S \in (0, \rho_i)$), then the critical point $E_{\lambda_1} = (\lambda_1, y_1 D(S^0 - \lambda_1)/D_1, 0, \cdots, 0)$ is globally asymptotically stable with respect to the interior of $\mathbb{R}_+^{n+1}$ for model (1.1).

Proof (See Fig. 2). The function $(S - \lambda_1)/(S - \lambda_i)$ is decreasing on $[0, \lambda_1]$ and on (λ_i, ρ_i). Also, $\lambda_1/\lambda_i < (\rho_i - \lambda_1)/(\rho_i - \lambda_i)$. Therefore,

$$\max_{0 < S < \lambda_1} \frac{(S - \lambda_1)}{(S - \lambda_i)} < \min_{\lambda_i < S < \rho_i} \frac{(S - \lambda_1)}{(S - \lambda_i)}.$$

If $w_i(S)$ satisfies (2.8), then

$$\max_{0 < S < \lambda_1} w_i(S) \leqq \min_{\lambda_i < S < \rho_i} w_i(S).$$

However, $g_i(S) = w_i(S)(S - \lambda_1)/(S - \lambda_i)$. Hence, since all terms are nonnegative,

$$\max_{0 < S < \lambda_1} g_i(S) < \min_{\lambda_i < S < \rho_i} g_i(S),$$

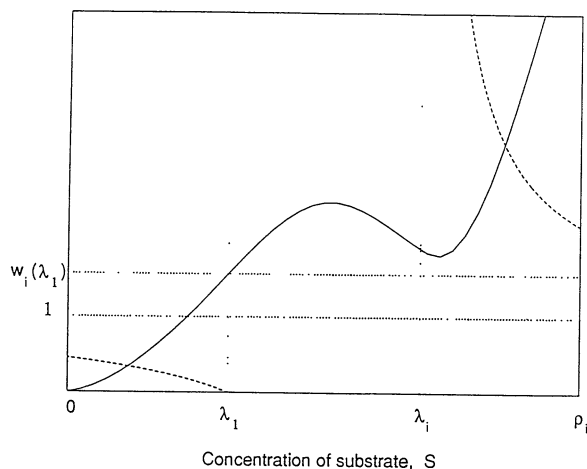and hence the result follows by Theorem 2.3. $\square$

Concentration of substrate, S

FIG. 2. *Graphical multiplication to determine* $g_i(S)$ *and thus verify the hypotheses of Corollary* 2.4. $g_i(S) = w_i(S)\,((S-\lambda_1)/(S-\lambda_i))$; – – – *indicates* $(S-\lambda_1)/(S-\lambda_i)$, —— *indicates* $w_i(S)$.

Before stating the next corollary, we introduce the following notation. For each $i = 1, \cdots, n$ let κ_i denote a positive real number (provided one exists) satisfying

$$(2.10) \qquad \frac{p_i(S)}{S^{\kappa_i}} \text{ is continuous on } [0, S^0]$$

and

$$(2.11) \qquad \lim_{S \to 0^+} \frac{p_i(S)}{S^{\kappa_i}} > 0.$$

Then define

$$(2.12) \qquad f_i(S) = \frac{S^{\kappa_i}(-D_i + p_i(S))}{p_i(S)(S-\lambda_i)}, \qquad S \in [0, S^0]$$

and

$$(2.13) \qquad f(S) = \frac{p_1(S)f_1(S)}{S^0 - S}, \qquad S \in [0, S^0).$$

Note that $f_i(S)$ and $f(S)$ are continuous and nonnegative on their respective domains. In particular, for the four prototypes (1.6)–(1.9), a unique κ_i exists:

(i) Lotka–Volterra: $f_i(S) = 1$,

(ii) Michaelis–Menten: $f_i(S) = \dfrac{m_i - 1}{m_i}$,

(iii) sigmoidal: $f_i(S) = \dfrac{(\lambda_i a_i b_i + S(\lambda_i(a_i + b_i) + a_i b_i))}{(a_i + \lambda_i)(b_i + \lambda_i)}$,

(iv) inhibition: $f_i(S) = \dfrac{a_i(\mu_i - S)}{(a_i + \lambda_i)(\mu_i + a_i)}.$

With $f_i(S)$ and $f(S)$ defined by (2.12) and (2.13), it follows that

$$(2.14) \qquad g_i(S) = \frac{S^{\kappa_i}(S-\lambda_1)f(S)(S^0-\lambda_1)}{S^{\kappa_1}(S-\lambda_i)f_i(S)D_1} = u_i(S)v_i(S)\frac{(S^0-\lambda_1)}{D_1},$$

where

$$(2.15) \qquad u_i(S) = S^{\kappa_i-\kappa_1}f(S) \quad \text{and} \quad v_i(S) = \frac{S-\lambda_1}{(S-\lambda_i)f_i(S)}.$$

COROLLARY 2.5. *Assume that* $\lambda_1 < \lambda_2 \leqq \cdots \leqq \lambda_n$, $\lambda_1 < S^0$, *and* $\mu_i > S^0$ *for all* $i = 1, \cdots, n$. *For each* $i \geqq 2$ *satisfying* $\lambda_i < S^0$, *if* $u_i(S)$ *is increasing on* $[0, S^0)$ *and* $v_i(S)$ *is decreasing on* $[0, \lambda_1]$ *and on* $(\lambda_i, S^0]$, *then there exist unique points* $\gamma \in (0, \lambda_1)$ *and* $\eta \in (\lambda_i, S^0)$ *such that*

$$(2.16) \qquad u_i(\gamma) = v_i(\gamma) \quad \text{and} \quad u_i(\eta) = v_i(\eta).$$

If

$$(2.17) \qquad v_i(0) \leqq u_i(\eta) \quad \text{and} \quad v_i(S^0) \geqq u_i(\gamma),$$

then the critical point $E_{\lambda_1} = (\lambda_1, y_1 D(S^0-\lambda_1)/D_1, 0, \cdots, 0)$ *is globally asymptotically stable with respect to the interior of* $\mathbb{R}_+^{n+1}$ *for model* (1.1). *In particular, if either*

$$(2.18) \qquad v_i(0) \leqq u_i(\lambda_i) \quad \text{and} \quad v_i(S^0) \geqq u_i(\lambda_1)$$

or

$$(2.19) \qquad v_i(0) \leqq v_i(S^0),$$

then (2.17) *holds, and hence the critical point* $E_{\lambda_1} = (\lambda_1, y_1 D(S^0-\lambda_1)/D_1, 0, \cdots, 0)$ *is globally asymptotically stable with respect to the interior of* $\mathbb{R}_+^{n+1}$ *for model* (1.1).

Proof (See Fig. 3). Since $u_i(0) = 0$, $v_i(0) > 0$, $u_i(\lambda_1) > 0$, and $v_i(\lambda_1) = 0$, and on $(0, \lambda_1)$, $u_i(S)$ is increasing, $v_i(S)$ is decreasing, and both functions are continuous, it follows that the two functions have a unique intersection point $\gamma \in (0, \lambda_1)$. Since $v_i(S) \to \infty$ as $S \to \lambda_i^+$, $u_i(S) \to \infty$ as $S \to S^{0-}$, $u_i(\lambda_i) > 0$ (finite), and $v_i(S^0) > 0$ (finite),
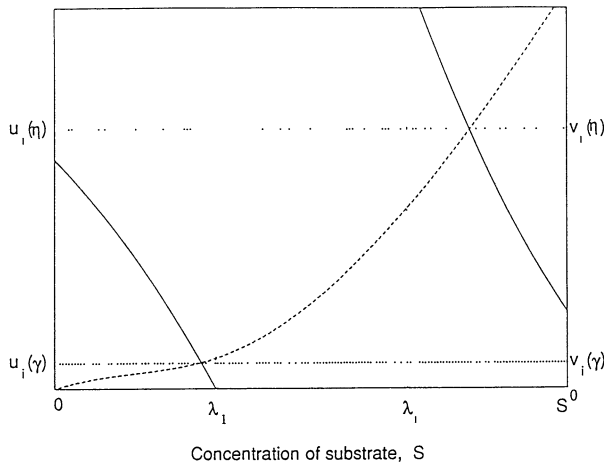

FIG. 3. *Graphical multiplication to determine* $g_i(S)$ *and thus verify the hypotheses of Corollary* 2.5. $g_i(S) = u_i(S)v_i(S)((S^0-\lambda_1)/(D_1))$; – – – *indicates* $u_i(S)$, —— *indicates* $v_i(S)$.

and on (λ_i, S^0), $u_i(S)$ is increasing, $v_i(S)$ is decreasing, and both functions are continuous, it follows that the two functions have a unique intersection point $\eta \in (\lambda_i, S^0)$. Define

$$X_1(S) = \begin{cases} v_i(S) & 0 \le S \le \gamma \\ u_i(S) & \gamma \le S \le \lambda_1 \end{cases}, \qquad X_2(S) = \begin{cases} u_i(S) & 0 \le S \le \gamma \\ v_i(S) & \gamma \le S \le \lambda_1 \end{cases},$$

$$Y_1(S) = \begin{cases} v_i(S) & \lambda_i < S \le \eta \\ u_i(S) & \eta \le S < S^0 \end{cases}, \qquad Y_2(S) = \begin{cases} u_i(S) & \lambda_i \le S \le \eta \\ v_i(S) & \eta \le S \le S^0 \end{cases}.$$

If (2.17) holds, then

$$(2.20) \qquad \max_{0 \le S \le \lambda_1} X_1(S) \le \min_{\lambda_i < S < S^0} Y_1(S) \quad \text{and} \quad \max_{0 \le S \le \lambda_1} X_2(S) \le \min_{\lambda_i < S < S^0} Y_2(S).$$

Therefore

$$\max_{0 \le S \le \lambda_1} X_1(S) X_2(S) \frac{(S^0 - \lambda_1)}{D_1} \le \min_{\lambda_i < S < S^0} Y_1(S) Y_2(S) \frac{(S^0 - \lambda_1)}{D_1}.$$

This implies that (2.3) holds, and hence Theorem 2.3 can be applied. In particular, if (2.18) holds, then

$$u_i(\gamma) = v_i(\gamma) < v_i(0) \le u_i(\lambda_i) < u_i(\eta)$$

and

$$v_i(S^0) \ge u_i(\lambda_1) > u_i(\gamma),$$

and so (2.17) holds. Similarly, if (2.19) holds, then

$$u_i(\gamma) = v_i(\gamma) \le v_i(0) \le v_i(S^0) < v_i(\eta) = u_i(\eta),$$

and again (2.17) holds. □

Remark 2.6. If we take any combination of the four prototypes (1.6)–(1.9) as the response functions $p_i(S)$, then under the hypotheses of Theorem 2.3 it is always possible to find appropriate constants α_i, and hence in model (1.1) the critical point E_{λ_1} is globally asymptotically stable with respect to solutions initiating in the positive cone. See Table 1 for an indication of how this is proved, i.e., which corollaries apply. In the case where $p_1(S)$ and $p_i(S)$ are both sigmoidal response functions of form (1.8), note that the symbolic manipulation package MAPLE was used to perform the computations, as they were particularly tedious.

TABLE 1.

For prototypes (1.6)–(1.9): an indication of which corollary applies. (a) *indicates that $w_i(S)$ is increasing for* $S \in (0, \rho_i)$ *and so Corollary (2.4) applies;* (b) *indicates that $w_i(S)$ is increasing for $S \in (0, \lambda_1)$ and $w_i(S) > w_i(\lambda_1)$ for $S \in (\lambda_i, \rho_i)$ and so Corollary (2.4) applies, ((a)⇒(b));* (c) *indicates that $u_i(S)$ is increasing for $S \in (0, S^0)$, $v_i(S)$ is decreasing for $S \in [0, \lambda_1]$ and $S \in (\lambda_1, S^0)$, and $v_i(0) > v_i(S^0)$. Therefore Corollary (2.5) applies.*

$p_i(S)$ $p_1(S)$	Lotka–Volterra	Michaelis–Menten	Sigmoidal	Inhibition
Lotka–Volterra	(a), (c)	(a), (c)	(a)	(a)
Michaelis–Menten	(a), (c)	(a), (c)	(a)	(a)
Sigmoidal	(c)	(c)	(a)	(a)
Inhibition	(b)	(b)	(b)	(b)

3. Discussion. In this paper we considered a mathematical model of exploitative competition of n populations in a chemostat for a single, essential, nonreproducing, growth-limiting resource. Using a Lyapunov stability theory approach, we generalized the results in [3], [9], [10] by allowing a larger class of response functions, and we generalized the results in [3], [6] for this class of response functions by allowing differential death rates. This class includes all functions that satisfy the usual conditions assumed to be satisfied by most realistic response functions (conditions (1.2)–(1.5)), along with the technical hypothesis of Theorem 2.3. We showed that this class is quite large, containing any combination of all four prototypes discussed in the paper: the three prototypes for monotone response functions—Lotka–Volterra, Michaelis–Menten, and sigmoidal—and the prototype for a nonmonotone response function describing inhibition by the substrate at large concentrations.

For this class of response functions, we proved that the outcome of competition for the resource depends on the relative values of the break-even concentrations. Each population asymptotically approaches an equilibrium concentration, independent of the initial concentrations of the populations. If the smallest break-even concentration λ_i of a population x_i is larger than S^0, the concentration of resource in the feed vessel, then that population dies out whether or not there are other competitors. Otherwise, assuming that there is only one population x_1 with the lowest break-even concentration λ_1, provided that $\lambda_1 < S^0 < \mu_1$ (where $\mu_1 = +\infty$ is possible, as in the monotone response function case), we showed that x_1 is the only population that avoids extinction, and the limiting concentration of the resource $S(t)$ is λ_1, and that of competitor x_1 is $y_1 D(S^0 - \lambda_1)/D_1$.

In [6], assuming $D_i = D$ for all i, the same conclusions were obtained for a larger class of response functions. Population x_1 was shown to be the sole survivor, provided that the set $Q = \cup_{i \in N} (\lambda_i, \mu_i)$ is connected and $S^0 \in Q$, where $N = \{i: \lambda_i < S^0\}$. We conjecture that these results are also true in the differential death-rate case. However, if $\mu_1 < S^0$, it is not possible to show this with the Lyapunov function that we used. In fact, finding a characterization of the class of response functions for which this Lyapunov function works remains an open question.

In [6] it was also shown that if either $S^0 \notin Q$, or if Q is not connected, though all population concentrations still approach limiting values and at most one population avoids extinction, the outcome of the competition depends on the initial concentrations of the competitors since, in this case, there is more than one locally asymptotically stable equilibrium. Only those populations x_i with their corresponding λ_i equal to the endpoint of one of the connected components of Q have a chance of winning the competition. If $S^0 \notin Q$, there will be an open set of initial concentrations for which all populations will wash out, even if $\lambda_i < S^0$ for several populations.

There is an advantage to using a resource-based approach rather than the seemingly simpler and more general classical approach for modeling competition. In the classical models of competition, parameters describing the *carrying capacity* of the environment as well as *competition coefficients* appear. It would be very difficult, if not impossible, to measure these competition coefficients without growing the competitor populations together. Thus these models tend not to be predictive. On the other hand, the parameters in model (1.1) can be measured by growing each species alone either in batch or continuous culture and, based on the relative sizes of the parameters describing break-even concentrations, the model predicts the qualitative outcome of mixed-growth competition in advance of the actual competition.

The predictions of this model could easily be tested in the laboratory. In fact, the predictions in [9], [10] were tested by Hansen and Hubbell [8]. They considered several

auxotrophic bacterial strains competing for limiting tryptophan. The response functions were modeled by Michaelis–Menten dynamics (see (1.7)), and so to determine the break-even concentrations, they first measured the maximum-per-cell division (birth) rate c_i and the half-saturation constant a_i for each bacterial strain by growing each strain alone in batch culture on tryptophan. From these measurements they calculated λ_i. They then carried out several experiments to rigorously test whether they could predict the outcome based on the break-even concentrations. Their experiments confirmed the mathematical predictions of qualitative outcome. Similar experiments could be performed to test the predictions using microorganisms known to have response functions of other forms, including those described by forms (1.8) and (1.9). See [1], [4], [11], [15] for examples of such microorganisms.

The results in this paper and in [6] suggest another approach for determining the break-even concentrations and, hence, testing the predictions. It seems that it might be possible to predict the outcome of competition of n competitors in a chemostat without knowing the actual form of the response functions. Instead, if each population x_i is grown alone in the chemostat with the appropriate feed concentration S^0 and dilution rate D, then it should be possible to determine the break-even concentration λ_i experimentally by allowing enough time for the concentrations to reach steady-state and then measuring the equilibrium concentration of $S(t)$. If it is known that all the response functions are monotone increasing for concentrations of substrate less than S^0, then each steady-state concentration of the substrate should be λ_i if $\lambda_i < S^0$ or S^0 if $\lambda_i \geqq S^0$. Once the population with the smallest steady-state concentration is determined, this population is predicted to be the winner and sole survivor (unless, of course, the smallest steady-state concentration is S^0, in which case all populations wash out). If nothing is known about the monotonicity properties of a response function for a population x_i, and the steady-state concentration of substrate equals S^0, this means that either $\lambda_i \geqq S^0$ or $\mu_i < S^0$. From the phase portraits in [2], [5], it seems that by repeating the experiment with the initial concentration of the microorganism population relatively high, if $\mu_i < S^0$, it should be possible to avoid the basin of attraction of the washout equilibrium, and hence determine λ_i. Once the population with the smallest steady-state concentration of substrate, say x_1, is found, it remains to determine whether $\mu_1 > S^0$. This can be done by starting up the chemostat with the concentration of substrate in the culture vessel initially greater than S^0. Measurements can then be taken to determine whether the concentration of the microorganism population is increasing or decreasing when the concentration of the substrate is close to S^0. If the concentration of x_1 is increasing, then $\mu_1 > S^0$ and the theory predicts that x_1 will be the only surviving population. If $\mu_1 < S^0$, then in a similar manner (beginning with determining whether $\mu_1 > \lambda_2$), to make predictions, we must determine whether Q is connected.

Recall that $D_i = D + \varepsilon_i$, where D is the flow rate and ε_i is the specific death rate of x_i. The model makes some other predictions that might be useful for industrial applications. First, D changes the λ_i's, and hence can change their relative values. This could result in a reversal of the outcome of competition. Second, changing S^0 can affect whether a species is washed out, independent of competition. However, unless the ε_i are affected by changing S^0, changing S^0 does not affect the relative sizes of the λ_i. Thus, if all the response functions are monotone increasing, changing S^0 should not affect the outcome of the competition unless it is decreased so that $S^0 < \lambda_1$, in which case all populations wash out. On the other hand, if some of the response functions are nonmonotone, changing S^0 could change the outcome of the competition. For example, suppose that $\lambda_1 < \mu_1 < \lambda_2 < \mu_2$. If $\lambda_1 < S^0 < \mu_1$, the theory predicts that x_1 will be the sole survivor, regardless of the initial concentrations. However, if S^0 is

increased so that $\lambda_1 < \mu_1 < \lambda_2 < S^0$, then the outcome will depend on the initial concentrations and, for certain initial conditions, x_2 could be the winner, driving x_1 to extinction.

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