CANADIAN APPLIED
MATHEMATICS QUARTERLY
Volume 10, Number 1, Spring 2003

GROWTH AND COMPETITION IN THE NUTRIENT DRIVEN SELF-CYCLING FERMENTATION PROCESS

Based on an invited presentation at the annual meeting of the Canadian Applied and Industrial Mathematics Society/Société canadienne de mathématiques appliquées et industrielles, Victoria, BC, June 2001.

R. J. SMITH AND G. S. K. WOLKOWICZ

1 Introduction. Self-cycling fermentation is a computer-aided biotechnological process developed for environmental cleanup programs such as sewage treatment or the reduction of toxic waste. Waste (nutrient) in a well-stirred tank is consumed by microorganisms, and used for growth and reproduction. A probe inserted in the tank monitors the system until certain conditions are met. The computer then instigates a rapid emptying and refilling process. A fraction of the contents of the tank is released into the environment and then replaced by an equal volume of fresh medium. The process then begins anew with the microorganism consuming the new medium until the conditions for the emptying and refilling process are met again. Under the right conditions, this cycling continues indefinitely, and the process does not require an operator or any estimate of the natural cycle time of the microorganisms in advance.

Since the duration of the emptying and refilling process is extremely short compared to the cycling time, this can be approximated by an instantaneous change in state. Thus, the system can be modelled by a system of impulsive differential equations. Such systems describe processes

---

Accepted for publication on February 20, 2002.

The work of the first author was partially supported by the Natural Sciences and Engineering Research Council of Canada and by a Richard Fuller Science Scholarship. This research is based on part of this author's Ph.D. thesis at McMaster University. The work of the second author was partially supported by the Natural Sciences and Engineering Research Council of Canada.

AMS subject classification: 92D25.

that are continuous under most conditions, but undergo instantaneous changes when certain conditions are satisfied.

The development of the process is described in Sheppard and Cooper [**5**] and a model similar to the one discussed here is given in Wincure, Cooper, and Rey [**6**]. Impulsive differential equations are described in Bainov and Simeonov [**1**], [**2**], [**3**] and Lakshmikantham, Bainov, and Simeonov [**4**].

2 Growth in the nutrient driven self-cycling fermentation process. Let t_n denote the time at which the n-th moment of impulse occurs, i.e., when the nutrient reaches the specified threshold, $\bar{s}$. In accordance with the theory of impulsive differential equations, we define

$$\Delta y \equiv y^+ - y^- \equiv y_n^+ - y_n^- \equiv y(t_n^+) - y(t_n^-) \equiv \lim_{t\to t_n^+} y(t) - \lim_{t\to t_n^-} y(t).$$

The following model of a single species in the self-cycling fermentation process with nutrient concentration as the triggering factor is considered.

$$\begin{aligned}
\frac{ds}{dt} &= -\frac{1}{Y}f(s)x && s \neq \bar{s}\\
\frac{dx}{dt} &= -\bar{d}x + f(s)x && s \neq \bar{s}\\
\Delta s &= -r\bar{s} + rs^i && s = \bar{s}\\
\Delta x &= -rx(t_n^-) && s = \bar{s}\\
& s(0) > \bar{s}, \quad x(0) \geq 0.
\end{aligned} \tag{2.1}$$

In this model, s denotes the concentration of waste (nutrient) in the fermentation tank, assumed to be toxic to the environment, x the biomass of the population of microorganisms that consume the nutrient, Y the cell yield constant, $\bar{d}$ the species specific death rate, $\bar{s}$ the tolerance of the waste in the environment consistent with standards set by the appropriate environmental protection agency, s^i the concentration of the pollutant in the environment added to the tank at the beginning of each new cycle, and r the emptying/refilling fraction. It is assumed that $\bar{d} \geq 0$, $Y > 0$, $s^i > \bar{s} > 0$, and $0 < r < 1$.

The response function is denoted f and satisfies

i) $f\colon \mathbb{R} \to \mathbb{R}$,
ii) f is continuously differentiable,

iii) $f(0) = 0$,
iv) $f'(s) > 0, \quad \text{if } s > 0$.

Let λ denote the break-even concentration of nutrient, that is the concentration of the nutrient so that $f(\lambda) = \bar{d}$. If $f(s) < \bar{d}$ for all $s > 0$, then we assume $\lambda = \infty$.

Also define

$$s_{\text{int}} \equiv Y \int_{\bar{s}}^{(1-r)\bar{s}+rs^i} \left(1 - \frac{\bar{d}}{f(s)}\right) ds.$$

The sign of $\bar{s}$ plays a key role in predicting the dynamics. Note that if $\bar{s} \geq \lambda$, then $s_{\text{int}} > 0$. However, if $\bar{s} < \lambda$, the sign of s_{int} depends on the parameters. For example, if $(1-r)\bar{s} + rs^i \leq \lambda$, then $s_{\text{int}} < 0$.

Theorem 1. *Consider model* (2.1). *There exists a unique nontrivial positive periodic orbit if and only if $s_{int} > 0$. This periodic orbit has exactly one impulse per period and has the property of asymptotic phase.*

Let $T > 0$ denote the period of the periodic orbit. At the impulse points the periodic orbit satisfies

$$s(t_n^-) = \bar{s}, \qquad s(t_n^+) = (1-r)\bar{s} + rs^i,$$
$$x(t_n^-) = \frac{s_{int}}{r}, \qquad x(t_n^+) = (1-r)\frac{s_{int}}{r}.$$

1. *Assume that $s_{int} > 0$.*
   (i) *If $\bar{s} \geq \lambda$, then the periodic orbit attracts all orbits with initial conditions satisfying $x(0) > 0$.*
   (ii) *If $\bar{s} < \lambda$, then the periodic orbit attracts all orbits with initial conditions satisfying $x(0)$ sufficiently large or $\int_{\bar{s}}^{s(0)}(1 - \frac{\bar{d}}{f(s)})\, ds > 0$ and $x(0) > 0$. In particular, if $s(0) \geq (1-r)\bar{s} + rs^i$, then the periodic orbit attracts all orbits with initial conditions satisfying $x(0) > 0$.*
   (iii) *If $\int_{\bar{s}}^{s(0)}(1 - \frac{\bar{d}}{f(s)})\, ds \leq 0$, and $x(0) > 0$ is sufficiently small, then there are no moments of impulse and $x(t) \to 0$, $s(t) \to s^* = s^*\big(s(0), x(0)\big)$, as $t \to \infty$, where $\bar{s} \leq s^* < \lambda$.*

   In cases (i) *and* (ii) *the fermentor cycles indefinitely and so there exists an infinite sequence of times $\{t_n\}_{n=1}^{\infty}$. As $n \to \infty$, $t_n \to \infty$, $t_{n+1} - t_n \to T$, $x(t_n^-) \to \frac{s_{int}}{r}$, and $x(t_n^+) \to (1-r)\frac{s_{int}}{r}$.*

For all positive integers n, solutions satisfy $s(t_n^+) = (1-r)\bar{s} + rs^i$ *and* $s(t_n^-) = \bar{s}$*, and one of the following holds:*

(a) $t_{n+1} - t_n = T$, $x(t_n^-) = \frac{s_{int}}{r}$*, and* $x(t_n^+) = (1-r)\frac{s_{int}}{r}$*; or*

(b) $t_{n+1} - t_n > T$, $x(t_n^-) < x(t_{n+1}^-)$, $x(t_n^+) < x(t_{n+1}^+)$, $x(t_n^-) < \frac{s_{int}}{r}$*, and* $x(t_n^+) < (1-r)\frac{s_{int}}{r}$*; or*

(c) $t_{n+1} - t_n < T$, $x(t_n^-) > x(t_{n+1}^-)$, $x(t_n^+) > x(t_{n+1}^+)$, $x(t_n^-) > \frac{s_{int}}{r}$*, and* $x(t_n^+) > (1-r)\frac{s_{int}}{r}$.

2. *Assume that* $s_{int} = 0$*. Then* $\liminf_{t\to\infty} x(t) = 0$*. If* $x(0)$ *is sufficiently large or* $s(0) \geq (1-r)\bar{s} + rs^i$ *and* $x(0) > 0$*, then there are an infinite number of impulses, but the time between impulses increases, approaching infinity. If* $\bar{s} < s(0) < (1-r)\bar{s} + rs^i$ *and* $x(0) > 0$ *is sufficiently small, there are no impulses and* $s(t) \to s^* = s^*\big(s(0), x(0)\big)$ *as* $t \to \infty$*, where* $\bar{s} \leq s^* < \lambda$.
3. *Assume that* $s_{int} < 0$*. If* $x(0) > 0$*, there are at most a finite number of impulses, the time between impulses increases, and eventually* $s(t) \to s^* = s^*\big(s(0), x(0)\big)$ *and* $x(t) \to 0$ *as* $t \to \infty$*, where* $\bar{s} \leq s^* < \lambda$.

The proof is given in Smith [**7**].

Thus, if $s_{\text{int}} > 0$, then solutions that undergo impulsive effect once will undergo an infinite number of impulsive effects and will monotonically approach the periodic orbit. If $s_{\text{int}} = 0$, then solutions that undergo impulsive effect once will also undergo impulsive effect an infinite number of times, but the time between impulses will increase without bound and the microorganism will essentially wash out. If $s_{\text{int}} < 0$, then solutions undergo impulsive effect at most a finite number of times, and then the microorganism washes out.

3 Competition.

We consider the self-cycling fermentation process with two species competing for a single nonreproducing nutrient. The model is extended in the obvious way to include more than one species. There are many interesting issues to resolve. For example, can more than one species coexist in the tank and if so, is the result initial condition dependent? Can we predict the outcome in advance?

In this section, we fix $r = \frac{1}{2}$ throughout, since this is the value of r used in the experimental literature (see Wincure, Cooper and Rey [**6**]). We use a specific form for the uptake functions, the Monod form, and we assume the death rates are negligible.

Thus, the model we study for two-species competition in the self-cycling fermentation process is

$$\begin{aligned}
&\frac{ds}{dt} = -\frac{1}{Y_1}xf_1(s) - \frac{1}{Y_2}yf_2(s) && s \neq \bar{s} \\
&\frac{dx}{dt} = xf_1(s) && s \neq \bar{s} \\
&\frac{dy}{dt} = yf_2(s) && s \neq \bar{s} \\
&\Delta s = -\frac{\bar{s}}{2} + \frac{s^i}{2} && s = \bar{s} \\
&\Delta x = -\frac{x(t_n^-)}{2} && s = \bar{s} \\
&\Delta y = -\frac{y_n^-}{2} && s = \bar{s} \\
&s(0) > \bar{s}, \quad x(0) \geq 0, \quad y(0) \geq 0,
\end{aligned} \tag{3.1}$$

where all the parameters are defined as for model (2.1) with the appropriate subscripting where necessary. The Monod form for the response functions is given by

$$f_i(s) = \frac{\mu_i s}{K_i + s}, \quad i = 1, 2.$$

Here, μ_i denotes the maximum specific growth rate and K_i denotes the half saturation constant for each species.

Note that each face where one of the populations is absent is invariant. As well, on each face the model reduces to the one discussed in the previous section and so Theorem 1 applies. Also, since the species specific death rate for each population is assumed to be zero, the break-even concentration for each population equals zero and so the corresponding value of s_{int} is positive on each face. Therefore, there is a nontrivial periodic orbit on each face where one of the populations is absent. We can use impulsive Floquet theory (see [**3**]) to analyse the stability of these planar periodic orbits with respect to the entire s, x, y-space. For model (3.1), the Floquet multiplier giving the stability of the nontrivial periodic orbit on the face where $x \equiv 0$ with respect to the positive cone is given by

$$\mu_{2x} = 2^{\frac{\mu_2(K_1+s^i)}{\mu_1(K_2+s^i)}-1} \cdot \left(\frac{K_2 + \frac{s^i}{2} + \frac{\bar{s}}{2}}{K_2 + \bar{s}} \right)^{\frac{\mu_2(K_1-K_2)}{\mu_1(K_2+s^i)}}. \tag{3.2}$$

Similarly, the Floquet multiplier giving the stability of the nontrivial periodic orbit on the face where $y \equiv 0$ with respect to the positive cone is given by

$$\mu_{2y} = 2^{\frac{\mu_1(K_2+s^i)}{\mu_2(K_1+s^i)}-1} \cdot \left(\frac{K_1 + \frac{s^i}{2} + \frac{\bar{s}}{2}}{K_1 + \bar{s}} \right)^{\frac{\mu_1(K_2-K_1)}{\mu_2(K_1+s^i)}} . \tag{3.3}$$

It can be shown that the periodic orbit on the face $y \equiv 0$ is asymptotically stable if $\mu_{2x} < 1$ and unstable if $\mu_{2x} > 1$. Similarly, the periodic orbit on the face $x \equiv 0$ is asymptotically stable if $\mu_{2y} < 1$ and unstable if $\mu_{2y} > 1$. In the region of parameter space where $\mu_{2x} > 1$ and $\mu_{2y} > 1$, we proved that both species persist.

Theorem 2. *Consider system* (3.1). *Suppose that the nontrivial Floquet multiplier for the periodic orbit on each face, where $x \equiv 0$ or $y \equiv 0$, given by* (3.2) *and* (3.3) *respectively, satisfies $\mu_{2x} > 1$ and $\mu_{2y} > 1$. Then solutions of* (3.1) *with initial conditions $x(0) > 0$ and $y(0) > 0$ satisfy $\liminf_{t\to\infty} x(t) > 0$, and $\liminf_{t\to\infty} y(t) > 0$.*

The proof is given in Smith [**7**].

The above theorem gives conditions under which there is coexistence of the two species, independent of initial conditions (provided both species are present to begin with). However, it says nothing about the nature of that coexistence. All of the numerical simulations in the appropriate region of parameter space seem to indicate that coexistence is in the form of an attracting impulsive periodic orbit with one impulse per period.

We also investigated the obvious generalisation of (3.1) to three competitors using numerical simulations. We were able to find paramaters and initial conditions where the simulations seemed to indicate that coexistence of three species competing for a single nonreproducing nutrient in the self-cycling fermentation process is also possible in the form of an impulsive periodic orbit with one impulse per period. It is interesting to note that in one of the simulations two populations x and y were able to coexist on an impulsive periodic orbit with population x dominating. When a third competitor was introduced, all three species coexisted on an impuslive periodic orbit, but now population y dominated.

REFERENCES

1. D. D. Bainov and P. S. Simeonov, *Systems with Impulsive Effect*, Ellis Horwood Ltd, Chichester, 1989.
2. ______, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, Burnt Mill, 1993.
3. ______, *Impulsive Differential Equations: Asymptotic Properties of the Solutions*, World Scientific, Singapore, 1995.
4. V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
5. J. D. Sheppard and D. G. Cooper, *Development of computerized feedback control for the continuous phasing of Bacillus Subtilis*, J. Biotechnology and Bioengineering **36** (1990), 539–545.
6. B. M. Wincure, D. G. Cooper and A. Rey, *Mathematical model of self-cycling fermentation*, J. Biotechnology and Bioengineering **46** (1995), 180–183.
7. R. J. Smith, *Impulsive Differential Equations with Applications to Self-Cycling Fermentation*, Ph.D. thesis, McMaster University, Hamilton, 2001.

Department of Applied Mathematics, University of Western Ontario,
London, Ontario, N6A 5B7

Department of Mathematics and Statistics, McMaster University,
1280 Main St West, Hamilton, Ontario, L8S 4K1