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HOPF BIFURCATION OF DELAY DIFFERENTIAL EQUATIONS WITH DELAY DEPENDENT PARAMETERS

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ABSTRACT. When the associated characteristic equation for a model involving delay differential equations is a transcendental equation with constant or delay dependent coefficients, a standard requirement for Hopf bifurcation is the existence of a pair of pure imaginary roots for some value of the delay. We develop a method that can be applied to locate intervals where a pair of pure imaginary roots of any such second order transcendental equation with delay dependent coefficients are likely to exist for some value of the delay in the interval, and when they exist, how to find them. An example is given illustrating our approach.

1 Introduction In the study of certain models involving a discrete delay τ, an associated characteristic equation of the linearization evaluated at an equilibrium of interest of the following form

$$P(\lambda) = \lambda^2 + p(\tau)\lambda + (q(\tau)\lambda + c(\tau))e^{-\lambda\tau} + \alpha(\tau) = 0, \tag{1.1}$$

where $p(\tau), q(\tau), c(\tau)$, and $\alpha(\tau)$ are continuous functions of $\tau \geq 0$, is studied. This is a second order transcendental equation in λ in which the coefficients can depend on the delay τ. For an example of such a model, see (3.1) in Section 3, with the coefficients of (1.1) given by (3.2). A standard requirement for a Hopf bifurcation at such an equilibrium with associated characteristic equation of the form (1.1), is that a pair of complex conjugate roots of (1.1) cross the imaginary axis of the complex plane (with nonzero imaginary part) as a parameter in the model is

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varied. Determining whether or not this occurs and for what value of the parameter can be difficult, especially when the coefficients in (1.1) depend on the delay. See, for example, [**3, 4, 6, 7, 8, 9**].

Beretta and Kuang [**1**] consider the more general nth order transcendental equation,

$$P_n(\lambda, \tau) + Q_m(\lambda, \tau)e^{-\lambda\tau} = 0, \tag{1.2}$$

where $n > m$ are nonnegative integers and

$$P_n(\lambda, \tau) = \sum_{k=0}^{n} p_k(\tau)\lambda^k, \qquad Q_m(\lambda, \tau) = \sum_{k=0}^{m} q_k(\tau)\lambda^k.$$

Functions $q_k(\tau)$ and $p_k(\tau)$ are assumed to be continuous and differentiable for $\tau \geqslant 0$. They propose a systematic method of how to find pure imaginary roots of (1.2). Although their method is constructive, it also relies on numerical techniques.

In this paper, we restrict our attention to second order transcendental equation (1.1). We present sufficient conditions that guarantee the existence of pure imaginary roots for (1.1) for some value of the delay τ. As well, we give sufficient conditions that guarantee that no such value of the delay exists. A procedure that can be used in applications is also presented to find the delay values at which pure imaginary roots occur. The method depends in part on numerical techniques, but extra care is taken to define all functions involved explicitly. This can be used in applications in order to determine analytically intervals of values of the delay parameter τ in which critical values of the delay at which the pure imaginary roots are likely to exist. This can be useful, since it restricts the interval of the search where the numerical techniques need to be applied.

In Section 2 we give our main results, and then summarize our method. In Section 3, an example is given illustrating our method for the single patch case in the model studied in Brauer, van den Driessche, and Wang [**2**], where they considered an epidemic model in a patchy environment and assume that the host has a period of immunity of fixed length τ after recovery from disease. We conclude with a discussion.

2 Main results Characteristic equations of the form (1.1) are often studied in order to understand changes in the local stability of equilibria of certain delay differential equations. It is therefore important to determine the values of the delay τ at which there are roots with zero real part. There is a real root equal to zero if and only if $\alpha(\tau) = -c(\tau)$. Hence, we will assume that $\alpha(\tau) \neq -c(\tau)$ and restrict our attention to determining when there are pure imaginary roots. If $q(\tau) = c(\tau) = 0$, then (1.1) is a quadratic equation in λ, and the roots are easily determined. If $q(\tau)$ or $c(\tau)$ is nonzero, (1.1) is a transcendental equation and it is much more difficult to find the roots. Here, we assume that both $c(\tau)$ and $q(\tau)$ are not equal to zero simultaneously and investigate when (1.1) has pure imaginary roots.

Assume that $\lambda = i\omega$ $(\omega > 0)$ is a root of (1.1). Then

$$P(i\omega) = -\omega^2 + ip(\tau)\omega + (iq(\tau)\omega + c(\tau))e^{-i\omega\tau} + \alpha(\tau) = 0.$$

Using Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, in the above equation we obtain

$$\begin{aligned} &-\omega^2 + \alpha(\tau) + q(\tau)\omega\sin(\omega\tau) + c(\tau)\cos(\omega\tau) \\ &\qquad\qquad + i\left(p(\tau)\omega + q(\tau)\omega\cos(\omega\tau) - c(\tau)\sin(\omega\tau)\right) = 0. \end{aligned}$$

Separating the real and imaginary parts yields,

$$\begin{cases} c(\tau)\cos(\omega\tau) + q(\tau)\omega\sin(\omega\tau) = \omega^2 - \alpha(\tau), \\ c(\tau)\sin(\omega\tau) - q(\tau)\omega\cos(\omega\tau) = p(\tau)\omega. \end{cases} \tag{2.1}$$

Solving for $\cos(\omega\tau)$ and $\sin(\omega\tau)$, we obtain

$$\begin{cases} \sin(\omega\tau) = \dfrac{c(\tau)p(\tau)\omega + q(\tau)\omega(\omega^2 - \alpha(\tau))}{c^2(\tau) + q^2(\tau)\omega^2}, \\ \cos(\omega\tau) = \dfrac{c(\tau)(\omega^2 - \alpha(\tau)) - q(\tau)p(\tau)\omega^2}{c^2(\tau) + q^2(\tau)\omega^2}. \end{cases} \tag{2.2}$$

Note that if one can find (τ, ω) satisfying (2.2), then (1.1) will have a pair of pure imaginary roots $\pm i\omega$ at τ. Our goal in the remaining part of this paper is to find solutions of (2.2).

Recalling that $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$, squaring both sides of the equations in (2.2), adding, and rearranging gives

$$\omega^4 + (p^2(\tau) - q^2(\tau) - 2\alpha(\tau))\omega^2 + \alpha^2(\tau) - c^2(\tau) = 0. \tag{2.3}$$

Solving for potential positive roots of (2.3) using the quadratic formula, we obtain,

$$\omega_1(\tau) = \left(\frac{1}{2}\big(q^2(\tau) - p^2(\tau) + 2\alpha(\tau) + \sqrt{(q^2(\tau) - p^2(\tau) + 2\alpha(\tau))^2 - 4\left(\alpha^2(\tau) - c^2(\tau)\right)}\big)\right)^{\frac{1}{2}}. \tag{2.4}$$

and

$$\omega_2(\tau) = \left(\frac{1}{2}\big(q^2(\tau) - p^2(\tau) + 2\alpha(\tau) - \sqrt{(q^2(\tau) - p^2(\tau) + 2\alpha(\tau))^2 - 4\left(\alpha^2(\tau) - c^2(\tau)\right)}\big)\right)^{\frac{1}{2}}. \tag{2.5}$$

Beretta and Kuang [**1**], who were also interested in determining whether (1.1) has pure imaginary roots also obtained (2.3). In their method, they proceed under the assumption that a positive root $\omega(\tau)$ of (2.3) exists. They set $\omega\tau = \theta(\tau)$ in the left hand side of (2.2) and continue under the assumption that a solution $\theta(\tau)$ exists. However, (2.2) may or may not have solutions. Here, we show how to determine whether or not such a solution $\theta(\tau)$ exists based on conditions in terms of $p(\tau)$, $q(\tau)$, $c(\tau)$, and $\alpha(\tau)$ and when it does, we give an explicit expression for $\theta(\tau)$.

Remark. Note that a solution of (2.2) must satisfy (2.3), but the converse need not be true, since (2.3) is obtained by squaring and adding the equations in (2.2).

Define conditions (H_1) and (H_2) as follows:

$$(H_1) \qquad \begin{cases} q^2(\tau) - p^2(\tau) + 2\alpha(\tau) > 0, \\ \alpha^2(\tau) - c^2(\tau) > 0, \\ (q^2(\tau) - p^2(\tau) + 2\alpha(\tau))^2 - 4\left(\alpha^2(\tau) - c^2(\tau)\right) \geqslant 0. \end{cases}$$

$$(H_2) \qquad \begin{array}{l} \alpha^2(\tau) - c^2(\tau) < 0, \quad \text{or} \\ \alpha^2(\tau) - c^2(\tau) = 0 \quad \& \quad q^2(\tau) - p^2(\tau) + 2\alpha(\tau) > 0. \end{array}$$

Lemma 1. *If* (H_1) *holds for all* τ *in some interval* I*, then* (2.3) *has two positive roots* $\omega_1(\tau) \geqslant \omega_2(\tau)$ *for all* $\tau \in I$ *with* $\omega_1(\tau) > \omega_2(\tau)$ *when all the inequalities in* (H_1) *are strict. If* (H_2) *holds for all* τ *in some interval* I*, then* (2.3) *has only one positive root,* $\omega_1(\tau)$ *for all* $\tau \in I$*. If no interval exists where either* (H_1) *or* (H_2) *holds, then there are no positive real roots of* (2.3)*.*

Corollary 1. *A standard requirement for Hopf bifurcations is that there exists some interval* I *such that for all* $\tau \in I$*,* $(q^2(\tau) - p^2(\tau) + 2\alpha(\tau))^2 - 4\left(\alpha^2(\tau) - c^2(\tau)\right) \geqslant 0$*, and if* $\alpha^2(\tau) - c^2(\tau) \leqslant 0$*, then* $q^2(\tau) - p^2(\tau) + 2\alpha(\tau) > 0$*.*

To simplify notation, denote the right hand sides of (2.2) as follows

$$h_1(\omega, \tau) = \frac{c(\tau)p(\tau)\omega + q(\tau)\omega(\omega^2 - \alpha(\tau))}{c^2(\tau) + q^2(\tau)\omega^2}, \quad \omega \geqslant 0, \tag{2.6}$$

$$h_2(\omega, \tau) = \frac{c(\tau)(\omega^2 - \alpha(\tau)) - q(\tau)p(\tau)\omega^2}{c^2(\tau) + q^2(\tau)\omega^2}, \quad \omega \geqslant 0. \tag{2.7}$$

In most cases, it is a challenge to seek solutions of (2.2) directly. Instead, in what follows, we consider the associated systems of the form

$$\begin{cases} \sin(\theta(\tau) + 2k\pi) = h_1(\omega_i(\tau), \tau), \\ \cos(\theta(\tau) + 2k\pi) = h_2(\omega_i(\tau), \tau), \end{cases} \qquad i = 1, 2, \quad k = 0, 1, 2, \dots \tag{2.8}$$

where $\omega_i(\tau)$ is given by (2.4) for $i = 1$ or (2.5) for $i = 2$.

Theorem 1. *If* $\theta(\tau)$ *satisfies* (2.8) *and* $\theta(\tau) + 2k\pi$ *(k nonnegative integer) intersects* $\tau\omega_i(\tau)$ *at some* $\bar{\tau}_i$*, then* $(\bar{\tau}_i, \omega_i(\bar{\tau}_i))$ *will be a solution of* (2.2)*, and hence,* (1.1) *has a pair of pure imaginary roots* $\pm i\omega_i(\bar{\tau}_i)$*.*

We begin by investigating when (2.8) has a solution $\theta(\tau)$. To avoid zero denominators in (2.6) and (2.7) we consider the two cases $c(\tau) \neq 0$, and $c(\tau) = 0$ but $q(\tau) \neq 0$ separately.

2.1 The case $c(\tau) \neq 0$**.** For a fixed τ, assume that $\omega_i(\tau)$ is positive ($i = 1, 2$). By (2.6) and (2.7) noting that $\omega_i(\tau)$ is a root of (2.3), we obtain $h_1^2(\omega_i(\tau), \tau) + h_2^2(\omega_i(\tau), \tau) = 1$. Define functions

$$\theta_i(\tau) = \arccos(h_2(\omega_i(\tau), \tau)), \quad i = 1, 2. \tag{2.9}$$

It follows that $\theta_i(\tau) \in [0, \pi]$. If $h_1(\omega_i(\tau), \tau) \geq 0$, then $\sin(\theta_i(\tau) + 2k\pi) = h_1(\omega_i(\tau), \tau)$. Hence $\theta_i(\tau)$ satisfies (2.8). If $h_1(\omega_i(\tau), \tau) < 0$, then

$$\cos(2\pi - \theta_i(\tau) + 2k\pi) = \cos(\theta_i(\tau)) = h_2(\omega_i(\tau), \tau)$$

and

$$\sin(2\pi - \theta_i(\tau) + 2k\pi) = -\sqrt{1 - \cos^2(2\pi - \theta_i(\tau) + 2k\pi)} = h_1(\omega_i(\tau), \tau).$$

Therefore $\theta(\tau) = 2\pi - \theta_i(\tau)$ satisfies (2.8).

Theorem 2. *Assume that $\omega_i(\tau)$ is positive ($i = 1, 2$). Consider the following conditions:*

(i) $q(\tau) > 0$ *and* $\alpha(\tau)q(\tau) - c(\tau)p(\tau) < 0$.
(ii) $q(\tau) < 0$, $\alpha(\tau)q(\tau) - c(\tau)p(\tau) < 0$, *and* $\omega_i^2(\tau) < \frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{q(\tau)}$.
(iii) $q(\tau) > 0$, $\alpha(\tau)q(\tau) - c(\tau)p(\tau) > 0$, *and* $\omega_i^2(\tau) > \frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{q(\tau)}$.
(iv.a) $q(\tau) = 0$ *and* $\frac{p(\tau)}{c(\tau)} \geqslant 0$.
(iv.b) $q(\tau) = 0$ *and* $\frac{p(\tau)}{c(\tau)} < 0$.

If one of (i), (ii), (iii) *or* (iv.a) *holds, then $\theta_i(\tau) \in [0, \pi]$ and satisfies* (2.8). *If* (iv.b) *holds, then $2\pi - \theta_i(\tau) \in [\pi, 2\pi]$ and satisfies* (2.8).

Proof. For any fixed τ, assume $\omega_i(\tau)$ is positive. From (2.6) and (2.7)

$$\begin{aligned} &h_1^2(\omega, \tau) + h_2^2(\omega, \tau) \\ &= \frac{c^2(\tau)p^2(\tau)\omega^2 + q^2(\tau)\omega^2(\omega^2 - \alpha(\tau))^2 + 2c(\tau)p(\tau)\omega q(\tau)\omega(\omega^2 - \alpha(\tau))}{(c^2(\tau) + q^2(\tau)\omega^2)^2} \\ &+ \frac{c^2(\tau)(\omega^2 - \alpha(\tau))^2 + q^2(\tau)\omega^2 p^2(\tau)\omega^2 - 2c(\tau)p(\tau)\omega q(\tau)\omega(\omega^2 - \alpha(\tau))}{(c^2(\tau) + q^2(\tau)\omega^2)^2} \\ &= \frac{(c^2(\tau) + q^2(\tau)\omega^2)(p^2(\tau)\omega^2 + (\omega^2 - \alpha(\tau))^2)}{(c^2(\tau) + q^2(\tau)\omega^2)^2} \\ &= \frac{p^2(\tau)\omega^2 + (\omega^2 - \alpha(\tau))^2}{c^2(\tau) + q^2(\tau)\omega^2}. \end{aligned}$$

A rearrangement of (2.3), noting that $\omega_i(\tau)$ is a root gives

$$h_1^2(\omega_i(\tau), \tau) + h_2^2(\omega_i(\tau), \tau) = 1. \tag{2.10}$$

We now consider each case (i), (ii), (iii), (iv.a) and (iv.b) separately.

(i) By (2.6), $h_1(0,\tau) = 0$ and $\lim_{\omega\to+\infty} h_1(\omega,\tau) = +\infty$. From a straightforward calculation using the assumption that $q(\tau) > 0$ and $\alpha(\tau)q(\tau) - c(\tau)p(\tau) < 0$, it follows that $\omega = 0$ is the only root of $h_1(\omega,\tau) = 0$. Hence, $h_1(\omega,\tau) > 0$ for any $\omega > 0$. For $\omega_i(\tau) > 0$, we obtain $h_1(\omega_i(\tau),\tau) > 0$. Therefore, $\theta_i(\tau)$ satisfies (2.8).

(ii) By (2.6), $h_1(0,\tau) = 0$ and $\lim_{\omega\to+\infty} h_1(\omega,\tau) = -\infty$. By assumption (ii)

$$\frac{\partial h_1(\omega,\tau)}{\partial\omega}\Big|_{\omega=0} = -\frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{c^2(\tau)} > 0.$$

Solving $h_1(z(\tau),\tau) = 0$ for $z(\tau)$, we obtain the unique positive root

$$z(\tau) = \sqrt{\frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{q(\tau)}}. \tag{2.11}$$

Therefore $h_1(\omega,\tau) > 0$ for $\omega \in (0, z(\tau))$ and $h_1(\omega,\tau) < 0$ for $\omega \in (z(\tau), +\infty)$ (see Figure 1(a)).

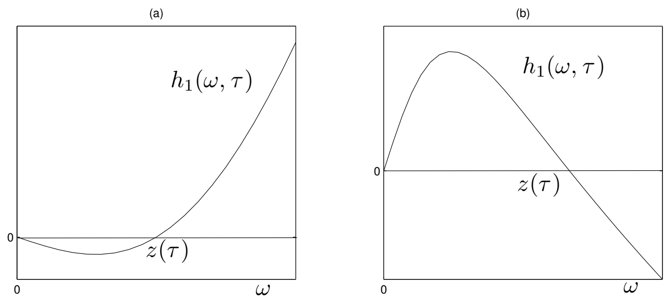

FIGURE 1: A schematic diagram of $h_1(\omega,\tau)$ for fixed τ. (a) The case (ii) holds. (b) The case (iii) holds.

From assumption $\omega_i^2(\tau) < \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}$ and (2.11), we have $\omega_i(\tau) < z(\tau)$. It follows that $h_1(\omega_i(\tau),\tau) > 0$. Therefore, $\theta_i(\tau)$ satisfies (2.8).

(iii) By (2.6), $h_1(0,\tau) = 0$ and $\lim_{\omega\to+\infty} h_1(\omega,\tau) = +\infty$. By assumption

$$\frac{\partial h_1(\omega,\tau)}{\partial\omega}\Big|_{\omega=0} = -\frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{c^2(\tau)} < 0.$$

As in (ii), $z(\tau) = \sqrt{\frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}}$ is the unique positive root of $h_1(z(\tau),\tau) = 0$. Therefore $h_1(\omega,\tau) < 0$ for any $\omega \in (0, z(\tau)$ and $h_1(\omega,\tau) > 0$ for $\omega \in (z(\tau), +\infty)$ (see Figure 1(b)). From assumption $\omega_i^2(\tau) > \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}$ and (2.11), we have $\omega_i(\tau) > z(\tau)$. This implies that $h_1(\omega_i(\tau),\tau) > 0$. Therefore $\theta_i(\tau)$ satisfies (2.8).

(iv.a) Since $q(\tau) = 0$, functions $h_1(\omega,\tau)$ and $h_2(\omega,\tau)$ defined in (2.6) and (2.7) reduce to the following form:

$$h_1(\omega,\tau) = \frac{p(\tau)\omega}{c(\tau)}, \qquad h_2(\omega,\tau) = \frac{\omega^2 - \alpha(\tau)}{c(\tau)}.$$

From $\frac{p(\tau)}{c(\tau)} \geqslant 0$, we have $h_1(\omega,\tau) \geqslant 0$ for any $\omega \geqslant 0$. For $\omega_i(\tau) > 0$, we obtain $h_1(\omega_1(\tau),\tau) \geq 0$. Therefore $\theta_i(\tau)$ satisfies (2.8).

(iv.b) Since $q(\tau) = 0$, functions $h_1(\omega,\tau)$ and $h_2(\omega,\tau)$ have the following form

$$h_1(\omega,\tau) = \frac{p(\tau)\omega}{c(\tau)}, \qquad h_2(\omega,\tau) = \frac{\omega^2 - \alpha(\tau)}{c(\tau)}.$$

From $\frac{p(\tau)}{c(\tau)} < 0$, we have $h_1(\omega,\tau) < 0$ for any $\omega \geqslant 0$. For $\omega_i(\tau) > 0$, we obtain $h_1(\omega_i(\tau),\tau) < 0$. It follows that $2\pi - \theta_i(\tau)$ satisfies (2.8). Noting that $0 \leqslant \theta_i(\tau) \leqslant \pi$, we obtain $2\pi - \theta_i(\tau) \in [\pi, 2\pi]$. □

In Theorem 2 all of the conditions in (ii) and (iii) are easy to check except possibly whether or not $\omega_i^2(\tau) < \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}$. For this reason we introduce conditions (A_2), (A_3), (A_5) and (A_6). These conditions may appear more complicated, but are actually easier to check. If condition (A_1) or (A_7) is satisfied, then (2.8) has no solutions. We also introduce (A_4), so that all possible cases (involving strict inequalities) are considered. However, if (A_4) holds, this approach does not lead to an improvement over applying Theorem 2 directly.

(A_1) $$-\frac{\alpha(\tau)}{c(\tau)} > 1, \quad \frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)} > 1.$$

(A_2) $$-\frac{\alpha(\tau)}{c(\tau)} > 1, \quad \frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)} < 1, \quad \text{and}$$
$$\frac{c(\tau)\alpha(\tau)+c^2(\tau)}{c(\tau)-p(\tau)q(\tau)-q^2(\tau)} > \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}.$$

(A_3) $$-\frac{\alpha(\tau)}{c(\tau)} < 1, \quad \frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)} > 1, \quad \text{and}$$

$$\frac{c(\tau)\alpha(\tau)+c^2(\tau)}{c(\tau)-p(\tau)q(\tau)-q^2(\tau)} < \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}.$$

(A_4) $$-1 < -\frac{\alpha(\tau)}{c(\tau)} < 1, \quad -1 < \frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)} < 1.$$

(A_5) $$-\frac{\alpha(\tau)}{c(\tau)} > -1, \quad \frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)} < -1, \quad \text{and}$$
$$\frac{c(\tau)\alpha(\tau)-c^2(\tau)}{c(\tau)-p(\tau)q(\tau)+q^2(\tau)} < \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}.$$

(A_6) $$-\frac{\alpha(\tau)}{c(\tau)} < -1, \quad \frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)} > -1, \quad \text{and}$$
$$\frac{c(\tau)\alpha(\tau)-c^2(\tau)}{c(\tau)-p(\tau)q(\tau)+q^2(\tau)} > \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}.$$

(A_7) $$-\frac{\alpha(\tau)}{c(\tau)} < -1, \quad \frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)} < -1,$$

When a positive root $l(\tau)$, $L(\tau)$ satisfies $h_2(l(\tau),\tau)=1$ or $h_2(L(\tau),\tau)=-1$ respectively, it is unique and given by

(2.12) $$l(\tau)=\sqrt{\frac{c(\tau)\alpha(\tau)+c^2(\tau)}{c(\tau)-p(\tau)q(\tau)-q^2(\tau)}} \quad \text{and}$$
$$L(\tau)=\sqrt{\frac{c(\tau)\alpha(\tau)-c^2(\tau)}{c(\tau)-p(\tau)q(\tau)+q^2(\tau)}}.$$

Theorem 3. *Assume that $\omega_i(\tau)>0$. If either (A_3) or (A_5) holds, then*
$$\omega_i^2(\tau) < \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}.$$
If either (A_2) or (A_6) holds, then
$$\omega_i^2(\tau) > \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}.$$

Proof. For any fixed τ. From (2.7),

(2.13) $$h_2(0,\tau) = -\frac{\alpha(\tau)}{c(\tau)} \quad \text{and} \quad \lim_{\omega\to\infty} h_2(\omega,\tau) = \frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)}.$$

First assume that (A_3) holds. From (2.13), $h_2(0,\tau) < 1$ and $\lim_{\omega\to\infty} h_2(\omega,\tau) > 1$. There exists a unique $l(\tau) > 0$ (see (2.12)) such that $h_2(l(\tau),\tau) = 1$. Therefore, $h_2(\omega,\tau) \leqslant 1$ for any $\omega \in [0, l(\tau)]$ and $h_2(\omega,\tau) > 1$ for $\omega > l(\tau)$ (see Figure 2(b)). The last inequality in assumption (A_3) implies that $l(\tau) < z(\tau)$ (see (2.11)). By (2.10), we have $-1 \leqslant h_2(\omega_i(\tau),\tau) \leqslant 1$, which implies that $\omega_i(\tau) \leqslant l(\tau)$. Therefore,

$$\omega_i^2(\tau) \leqslant l^2(\tau) < z^2(\tau) = \frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{q(\tau)}.$$

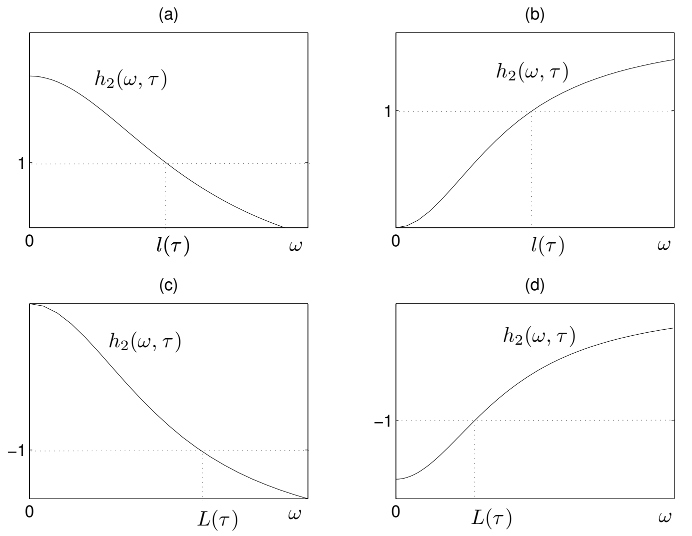

FIGURE 2: A schematic diagram of $h_2(\omega,\tau)$ for fixed τ. (a) The case (A_2) holds. (b) The case (A_3) holds.(c) The case (A_5) holds.(d) The case (A_6) holds.

Suppose (A_5) holds. By (2.13), $h_2(0,\tau) > -1$ and $\lim_{\omega\to\infty} h_2(\omega,\tau) < -1$. There exists a unique $L(\tau) > 0$ (see (2.12)) such that $h_2(L(\tau),\tau) = -1$. Hence $h_2(\omega,\tau) \geqslant -1$ for any $\omega \in [0, L(\tau)]$ and $h_2(\omega,\tau) < -1$ for $\omega > L(\tau)$ (see Figure 2(c)). By the last inequality of (A_5), $L(\tau) < z(\tau)$.

By (2.10), we have $-1 \leqslant h_2(\omega_i(\tau), \tau) \leqslant 1$, which implies that $\omega_i(\tau) \leqslant L(\tau)$. Therefore

$$\omega_i^2(\tau) \leqslant L(\tau) < z^2(\tau) = \frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{q(\tau)}.$$

Assume (A_2) holds. By (2.13), $h_2(0,\tau) > 1$ and $\lim_{\omega\to\infty} h_2(\omega,\tau) < 1$. There exists a unique $l(\tau) > 0$ such that $h_2(l(\tau),\tau) = 1$. Hence $h_2(\omega,\tau) \leqslant 1$ for any $\omega \in (l(\tau), +\infty)$ and $h_2(\omega,\tau) > 1$ for $\omega \in (0, l(\tau))$ (see Figure 2(a)). By the last inequality of (A_2), $l(\tau) > z(\tau)$. By (2.10), we have $-1 \leqslant h_2(\omega_i(\tau),\tau) \leqslant 1$, which implies that $\omega_i(\tau) \geqslant l(\tau)$. Therefore

$$\omega_i^2(\tau) \geqslant l^2(\tau) > z^2(\tau) = \frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{q(\tau)}.$$

Suppose (A_6) holds. From (2.13), $h_2(0,\tau) < -1$ and $\lim_{\omega\to\infty} h_2(\omega,\tau) > -1$. There exists a unique $L(\tau) > 0$ such that $h_2(L(\tau),\tau) = -1$. Hence $h_2(\omega,\tau) \geqslant -1$ for any $\omega \in (L(\tau), +\infty)$ and $h_2(\omega,\tau) < -1$ for $\omega \in (0, L(\tau))$ (see Figure 2(d)). By the last inequality of (A_6), $l(\tau) > z(\tau)$. By (2.10), we have $-1 \leqslant h_2(\omega_i(\tau),\tau) \leqslant 1$, which implies that $\omega_i(\tau) \geqslant l(\tau)$. Therefore

$$\omega_i^2(\tau) \geqslant l^2(\tau) > z^2(\tau) = \frac{\alpha(\tau)q(\tau) - c(\tau)p(\tau)}{q(\tau)}.$$

□

Note that if all but the last inequality in any of the assumptions (A_2), (A_3), (A_5), or (A_6) is violated, our method breaks down, because it is then uncertain whether or not $\omega_i^2(\tau) > \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}$. As well, if (A_4) holds, then $-1 < h_2(\omega,\tau) < 1$ and so it is not possible to define $l(\tau)$ and $L(\tau)$, thus again this approach does not help us to determine whether $\omega_i^2(\tau) > \frac{\alpha(\tau)q(\tau)-c(\tau)p(\tau)}{q(\tau)}$. Recall also that if $\alpha(\tau) = -c(\tau)$, then zero is a root of (1.1).

Theorem 4. *If either* (A_1) *or* (A_7) *holds, then system* (2.8) *has no solutions.*

Proof. For any fixed τ, Assume (A_1) holds. Multiplying $c^2(\tau)$ on both sides of $-\frac{\alpha(\tau)}{c(\tau)} > 1$ gives $c(\tau)\alpha(\tau)+c^2(\tau) < 0$. From the second inequality of (A_1), we have $c(\tau) - p(\tau)q(\tau) - q^2(\tau) > 0$. Hence $l(\tau)$ is not real (see

(2.12) for the definition of $l(\tau)$). Then $h_2(\omega, \tau) = 1$ has no positive root. Hence, $h_2(\omega, \tau) > 1$ for any $\omega > 0$. Therefore the equation for $\cos(\theta + 2\pi)$ in (2.8) has no solutions. Assume (A_7) holds. Multiplying $c^2(\tau)$ on both sides of $-\frac{\alpha(\tau)}{c(\tau)} < -1$ gives $c(\tau)\alpha(\tau) - c^2(\tau) > 0$. From $\frac{c(\tau)-p(\tau)q(\tau)}{q^2(\tau)} < -1$, we have $c(\tau) - p(\tau)q(\tau) + q^2(\tau) < 0$. Hence $L(\tau)$ is not real (see (2.12) for the definition of $L(\tau)$). We have $h_2(\omega, \tau) < -1$ for any $\omega > 0$. Again, the equation for $\cos(\theta + 2\pi)$ in (2.8) has no solutions. The conclusion follows. □

Theorem 5. *Assume that* $\omega_i(\tau) > 0$ *and* $\theta_i(\tau)$ *is a solution of* (2.8) *for* $\tau \in I_i$, *where* I_i *is a closed interval including* 0. *Let* $M_i = \max_{\tau\in I_i} \tau\omega_i(\tau)$. *If* $(2n + 1)\pi < M_i < 2(n + 1)\pi$, *then* $\theta_i(\tau) + 2k\pi$ *and* $\tau\omega_i(\tau)$ *have at least one intersection, where* $1 \leqslant k \leqslant n$, *and* $n = 1, 2, \ldots$.

Proof. By (2.9), $\theta_i(\tau) \in [0, \pi]$. Therefore $\theta_i(\tau)+2k\pi \in [2k\pi, (2k+1)\pi]$. Since $0 \in I_i$, we obtain $\min_{\tau\in I_i} \tau\omega_i(\tau) = 0 < \theta_i(\tau)+2k\pi$. If $(2n+1)\pi < M_i < 2(n+1)\pi$, noting $1 \leqslant k \leqslant n$, we have $(2k+1)\pi < M_i$. This implies that $\max_{\tau\in I_i} \tau\omega_i(\tau) > \theta_i(\tau) + 2k\pi$. By the Mean Value Theorem, the conclusion follows. □

Later in Section 3, we provide an application where we apply Theorem 5 and define the interval I_i explicitly, guaranteeing that $\theta_i(\tau) + 2k\pi$ and $\tau\omega_i(\tau)$ have at least one intersection in I_i.

2.2 The case $c(\tau) = 0$ and $q(\tau) \neq 0$ When $c(\tau) = 0$, the proof of Theorem 2 that we gave cannot be used. In cases (ii) and (iii), the proof involved the derivative of $h_1(\omega, \tau)$ at $\omega = 0$, and this denominator is equal to zero if $c(\tau) = 0$. Also in one of the assumptions in each of cases (iv.a) and (iv.b) one of the expressions would have a denominator equal to zero. Similarly, conditions (A_1)–(A_7) are no longer defined. For these reasons, in this section we consider the case that $c(\tau) = 0$. Some of the proofs in this case are actually more straightforward, since certain expressions are less complicated. As well, some parts of the proofs in the proceeding section do not involve division by zero and can still be used.

If $c(\tau) = 0$, (2.2) reduces to

$$\begin{cases} \sin(\omega\tau) = \dfrac{\omega^2 - \alpha(\tau)}{q(\tau)\omega}, \\[2ex] \cos(\omega\tau) = \dfrac{-p(\tau)}{q(\tau)}. \end{cases} \quad . \tag{2.14}$$

In this case, $h_1(\omega, \tau)$ and $h_2(\omega, \tau)$ are given by:

$$h_1(\omega,\tau) = \frac{\omega^2 - \alpha(\tau)}{q(\tau)\omega}, \qquad h_2(\tau) = \frac{-p(\tau)}{q(\tau)} \quad \text{for } \omega > 0. \tag{2.15}$$

Note that $h_2(\omega, \tau)$ does not depend explicitly on ω and can be considered as a function of τ alone, $h_2(\tau)$. Since ω is a factor in the denominator of $h_1(\omega, \tau)$, in this section, we consider $h_1(\omega, \tau)$ and $h_2(\tau)$ for $\omega > 0$. Define

$$\theta(\tau) = \arccos(h_2(\tau)) \quad \text{for } h_2(\tau) \in [-1, 1], \tag{2.16}$$

and consider the associated system

$$\begin{cases} \sin(\theta + 2\pi) = \dfrac{\omega^2 - \alpha(\tau)}{q(\tau)\omega}, \\[2ex] \cos(\theta + 2\pi) = \dfrac{-p(\tau)}{q(\tau)}. \end{cases} \tag{2.17}$$

Theorem 6. *Consider system* (2.17)

(i) $\alpha(\tau) = 0$, $q^2(\tau) - p^2(\tau) > 0$, *and* $q(\tau) > 0$.
(ii) $\alpha(\tau) < 0$, $q^2(\tau) - p^2(\tau) + 4\alpha(\tau) > 0$, *and* $q(\tau) > 0$.
(iii) $\alpha(\tau) > 0$, $q^2(\tau) - p^2(\tau) > 0$, $q(\tau) > 0$, *and* $\omega_i(\tau) > \sqrt{\alpha(\tau)}$.
(iv) $\alpha(\tau) > 0$, $q^2(\tau) - p^2(\tau) > 0$, $q(\tau) < 0$, *and* $\omega_i(\tau) < \sqrt{\alpha(\tau)}$.

If (i) *holds, then* (H_2) *holds and* $\theta(\tau) \in (0, \pi]$ *satisfies* (2.17) *with* $\omega = \omega_1(\tau)$. *If one of* (ii), (iii), *or* (iv) *holds, then* (H_1) *holds and* $\theta(\tau) \in (0, \pi]$ *satisfies* (2.17) *with* $\omega = \omega_i(\tau) > 0$ $(i = 1, 2)$.

Proof. Fix τ.

(i) From $c(\tau) = 0$, $\alpha(\tau) = 0$ and $q^2(\tau) - p^2(\tau) > 0$, it follows that (H_2) holds. By Lemma 1, $\omega_1(\tau)$ is positive, but $\omega_2(\tau)$ is not positive. By (2.10), we have $-1 \leqslant h_2(\omega_1(\tau), \tau) \leqslant 1$. Then $\theta(\tau)$ is defined and $0 \leqslant \theta(\tau) \leqslant \pi$. From $\omega_1(\tau) > 0$ and $q(\tau) > 0$, we obtain $h_1(\omega_1(\tau), \tau) = \omega_1(\tau)/q(\tau) > 0$. By (2.10),

$$h_1(\omega_1(\tau), \tau) = \sqrt{1 - h_2^2(\omega_1(\tau), \tau)}.$$

Since $\cos(\theta(\tau) + 2k\pi) = h_2(\tau)$, $\sin(\theta(\tau) + 2k\pi) = h_1(\omega_1(\tau), \tau)$, which implies that $\theta(\tau)$ satisfies (2.17).

(ii) By $c(\tau) = 0$, $\alpha(\tau) < 0$, and $q^2(\tau) - p^2(\tau) + 4\alpha(\tau) > 0$, it follows that (H_1) holds. By Lemma 1, both roots $\omega_i(\tau)$ are positive ($i = 1, 2$). As in (i), $\theta(\tau)$ is defined and $0 \leqslant \theta(\tau) \leqslant \pi$. Since $\alpha(\tau) < 0$ and $q(\tau) > 0$, function

$$h_1(\omega_1(\tau), \tau) = \frac{\omega_1^2(\tau) - \alpha(\tau)}{q(\tau)\omega_1(\tau)} > 0.$$

The rest of the proof is similar to (i).

(iii) From $c(\tau) = 0$, $\alpha(\tau) > 0$, and $q^2(\tau) - p^2(\tau) > 0$, it follows that (H_1) holds and so $\omega_i(\tau)$ is positive ($i = 1, 2$). As in (i), $\theta(\tau)$ is defined and $0 \leqslant \theta(\tau) \leqslant \pi$. Since $q(\tau) > 0$ and $\omega_i(\tau) > \sqrt{\alpha(\tau)}$, $h_1(\omega_i(\tau), \tau) > 0$. The rest of the proof is similar to (i).

(iv) Conditions $c(\tau) = 0$, $\alpha(\tau) > 0$, and $q^2(\tau) - p^2(\tau) > 0$ imply that (H_1) holds and so $\omega_i(\tau)$ is positive. Since $q(\tau) < 0$ and $\omega_i(\tau) < \sqrt{\alpha(\tau)}$, $h_1(\omega_i(\tau), \tau) > 0$. The rest of the proof is similar to (i).

To prove that $\theta(\tau) \neq 0$, we use the method of contradiction. Suppose that $\theta(\tau) = 0$ for some τ. By (2.16), $h_2(\tau) = 1$. From (2.15), we have $q^2(\tau) - p^2(\tau) = 0$. This contradicts $q^2(\tau) - p^2(\tau) > 0$ in (i), (iii), or (iv). For (ii), since $\alpha(\tau) < 0$ and $q^2(\tau) - p^2(\tau) + 4\alpha(\tau) > 0$ it follows that $q^2(\tau) - p^2(\tau) > 0$, another contradiction. Therefore $\theta(\tau) \in (0, \pi]$. □

Theorem 7. *Assume that $\omega_i(\tau) > 0$ and $\theta(\tau)$ is a solution of* (2.17) *for $\tau \in I_i$, where I_i is a closed interval and $0 \in I_i$. Let $M_i = \max_{\tau \in I_i} \tau\omega_i(\tau)$. If $(2n + 1)\pi < M_i < 2(n + 1)\pi$, there is at least one intersection of $\theta(\tau) + 2k\pi$ and $\tau\omega_i(\tau)$, where $1 \leqslant k \leqslant n$, and n is any positive integer.*

2.3 Summary of the method. In many applications, the delay τ is chosen to be the bifurcation parameter, and all other parameters are assumed to be known and fixed. It can be helpful to determine intervals explicitly on which there are potentially values of τ for which there are pure imaginary roots for the characteristic equation (1.1). Then, in order to find these critical values of τ using numerical techniques, one can restrict attention to these intervals, thus simplifying the search.

One can use our results to do this in the following way. First check whether (H_1) or (H_2) holds for τ on any intervals. If not, then there are no values of τ for which there is a pair of pure imaginary roots, and one need not search any further. In particular, if (H_1) holds on some interval I, by Lemma 1 it follows that there are two positive roots $\omega_i(\tau) > 0,\ i = 1, 2$ of (2.3) for all $\tau \in I$. If (H_2) holds then there is only one positive root $\omega_1(\tau)$. Next determine all values of τ on each such interval I where $c(\tau) = 0$, and then subdivide I into subintervals on which either $c(\tau) = 0$ for all τ in the subinterval or $c(\tau) \neq 0$ for any

τ in the subinterval. On those subintervals where $c(\tau) \neq 0$ for any τ one next proceeds by using Theorem 2. But first determine all values of τ on those subintervals where $q(\tau) = 0$ and again consider subintervals of those intervals where either $q(\tau) = 0$ for all τ in the subinterval or $q(\tau) \neq 0$ for any τ in the subinterval. On each such subinterval under investigation, determine whether one of the four conditions in Theorem 2 holds, passing again to subintervals if necessary. If Theorem 2 holds on any of these subintervals, then the function $\theta_i(\tau)$, defined explicitly in (2.9), satisfies (2.8) on this subinterval, and hence by Theorem 1, there will be a value of τ in that subinterval at which there is a pair of pure imaginary roots if the two functions $\tau\omega_i(\tau)$ and $\theta_i(\tau) + 2k\pi$ intersect as τ is varied over that subinterval. One can then plot these two functions as τ varies on these subintervals in order to determine if any such intersections occur, and if so determine these values using standard numerical techniques.

On any subintervals of I on which $c(\tau) = 0$ for all $\tau \in I$, proceed as described in the previous case, using Theorem 6 instead of Theorem 2. If in this case any subintervals are identified where in addition $q(\tau) = 0$ for all τ, the characteristic polynomial (1.1) reduces to a quadratic equation in λ, and there are pure imaginary roots on such subintervals if and only if there are values of τ where $p(\tau) = 0$ and $\alpha(\tau) > 0$.

This method is illustrated in the application in the next section.

3 Application In this section, we apply our analytical results to the single patch case of the model studied in Brauer, van den Driessche and Wang [**2**]:

$$(3.1)\qquad \begin{cases} \dot{S}(t) = A - dS(t) - \beta S(t)I(t) + \gamma e^{-d\tau} I(t-\tau), \\ \dot{I}(t) = \beta S(t)I(t) - (\gamma + \epsilon + d)I(t), \\ \dot{R}(t) = \gamma I(t) - \gamma e^{-d\tau} I(t-\tau) - dR(t). \end{cases}$$

In [**2**], it was shown that if $R_0 = \frac{A\beta}{d(\gamma+\epsilon+d)} > 1$, (3.1) has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$ given by

$$S^* = \frac{A}{dR_0}, \quad I^* = \frac{A(1 - \frac{1}{R_0})}{(1 - e^{-d\tau})\gamma + \epsilon + d}, \quad R^* = \frac{\gamma}{d}(1 - e^{-d\tau})I^*.$$

The characteristic equation of system (3.1) at E^* has the same form as (1.1) with

$$(3.2)\qquad \begin{aligned} p(\tau) &= d + \beta I^*, & q(\tau) &= 0, \\ \alpha(\tau) &= \beta(\gamma + \epsilon + d)I^*, & c(\tau) &= -\gamma\beta I^* e^{-d\tau}. \end{aligned}$$

Since $\alpha^2(\tau) - c^2(\tau) > 0$ for any τ, only condition (H_1) is possible. Note that $c(\tau) \neq 0$ and $q(\tau) = 0$, was considered in Section 2.1.

Applying Lemma 1, Theorem 1, and Theorem 2.(iv.b) to model (3.1), we obtain the following theorem.

Theorem 8. *Assume that* $R_0 = \frac{A\beta}{d(\gamma+\epsilon+d)} > 1$ *and the coefficients of* (1.1) *satisfy* (3.2). *Consider hypotheses*

i) $A\beta - 2d(\gamma+\epsilon+d) \geqslant 0, \quad \dfrac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\gamma^2(\epsilon+d)} < 1,$

$\tau \in \left[0, \dfrac{-1}{2d} \ln \dfrac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\gamma^2(\epsilon+d)}\right],$

ii) $A\beta - d(\gamma+2\epsilon+2d) \leqslant 0, \quad \dfrac{4d^2(\gamma+\epsilon+d)^2}{\gamma^2(A\beta - d(\gamma+\epsilon+d))} < 1,$

$\tau \in \left[0, \dfrac{-1}{2d} \ln \frac{4d^2(\gamma+\epsilon+d)^2}{\gamma^2(A\beta - d(\gamma+\epsilon+d))}\right].$

If either i) *or* ii) *holds, condition* (H_1) *holds and so both* $\omega_1(\tau)$ *and* $\omega_2(\tau)$ *are positive. Moreover,* $2\pi - \theta_i(\tau) \in [\pi, 2\pi]$ *and satisfies* (2.8). *If there exists an integer* $k \geqslant 0$ *such that* $2\pi - \theta_i(\tau) + 2k\pi$ *intersects* $\tau\omega_i(\tau)$ *at some* $\bar{\tau}_i$, *then* (1.1) *has a pair of pure imaginary roots* $\pm\omega_i(\bar{\tau}_i)$ *where* $i = 1,\ 2$.

Proof. See **Appendix A**. □

Lemma 3.1 in [**2**] can be rephrased as follows: If $q^2(\tau) - p^2(\tau) + 2\alpha(\tau) \geqslant 2\sqrt{\alpha^2(\tau) - c^2(\tau)}$, there exists an interval (τ_l, τ_u) such that $q^2(\tau) - p^2(\tau) + 2\alpha(\tau) > 0$ and $\Delta(\tau) \equiv (q^2(\tau) - p^2(\tau) + 2\alpha(\tau))^2 - 4\left(\alpha^2(\tau) - c^2(\tau)\right) \geqslant 0$, where either $\tau_l = 0$ and $\Delta(\tau_u) = 0$, or $\Delta(\tau_l) = \Delta(\tau_u) = 0$. This lemma gives the existence of the interval $[\tau_l, \tau_u]$ on which pure imaginary roots are possible. In Theorem 8, we define this interval explicitly in terms of the original parameters of the model. We have $\tau_l = 0$, and in case i),

$$\tau_u > \frac{-1}{2d} \ln \frac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\gamma^2(\epsilon+d)},$$

but in case ii),

$$\tau_u > \frac{-1}{2d} \ln \frac{4d^2(\gamma+\epsilon+d)^2}{\gamma^2(A\beta - d(\gamma+\epsilon+d))}.$$

Lemma 3.2 in [**2**] states that (1.1) with coefficients given by (3.2) has a pair of pure imaginary roots $\pm\omega_i(\bar{\tau}_i)$ at $\tau = \bar{\tau}_i$ if and only if there exists an integer k such that the graph of $\delta_k(\tau)$ intersects $\tau\omega_i(\tau)$ at some $\bar{\tau}_i$, where $\delta_k(\tau) = \eta$ is implicitly defined as the unique solution of $\cot(\eta) = \frac{h_2(\frac{\eta}{\tau},\tau)}{h_1(\frac{\eta}{\tau},\tau)}$ for η in the interval $[2(k-1)\pi, 2k\pi]$ $(k = 1, 2, 3, \dots)$. In Theorem 8, although our results also depend on the assumption that $2\pi - \theta_i(\tau) + 2k\pi$ and $\tau\omega_i(\tau)$ intersect, the function $\theta_i(\tau)$ is explicitly defined by (2.9). If condition i) or ii) holds, since $2\pi - \theta_i(\tau) \in [\pi, 2\pi]$, it follows that $2\pi - \theta_i(\tau) + 2k\pi \in [2k\pi + \pi, 2(k+1)\pi]$. This implies that pure imaginary roots can only take values for τ in $[2k\pi + \pi, 2(k+1)\pi]$. This is consistent with the conclusion in [**2**] that $\delta_k(\tau) \in [2(k-1)\pi, 2k\pi]$.

Zero is not a root of (1.1) with coefficients satisfying (3.2), since $\alpha(\tau) > -c(\tau)$. As pointed out in [**2**], since $e^{-d\tau} \to 0$ as $\tau \to \infty$, in the limit the characteristic equation is given by, $P(\lambda) = \lambda^2 + (d+\beta I^*) + \beta(\gamma + \epsilon + d)I^* = 0$ where $I^* = \frac{A(1-1/R_0)}{\gamma+\epsilon+d}$ and this characteristic equation has all roots with negative real parts. Therefore, complex roots with positive real parts of (1.1) with coefficients satisfying (3.2) cannot enter the right half of the complex plane from infinity. Therefore, the only way that a pair of complex roots with positive real part can appear is by a pair of roots crossing the imaginary axis. If $\frac{dRe(\lambda(\tau))}{d\tau}|_{\tau=\bar{\tau}_i} \neq 0$, by the Hopf Bifurcation Theorem (see Kuang [**5**], pp.60), system (3.1) has a Hopf bifurcation at $\tau = \bar{\tau}_i$.

For numerical simulations, we chose $A = 0.045$, $d = 0.001$, $\epsilon = 0.01$, $\gamma = 0.5$, $\beta = 0.032$, and hence $R_0 = 2.64 > 1$. For such parameters, Theorem 8.*i*) can be used, since $A\beta - 2d(\gamma + \epsilon + d) \approx 0.0003 > 0$,

$$\frac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\gamma^2(\epsilon+d)} = 0.62 < 1,$$

and

$$\tau \in \left[0, \frac{-1}{2d} \ln \frac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\gamma^2(\epsilon+d)}\right] \approx [0, 236].$$

Plotting $\tau\omega_1(\tau)$ and $2\pi - \theta_1(\tau)$ in one figure (see Figure 3 (Left)), there is one intersection at $\bar{\tau}_1 \approx 16$. Figure 3 (Right) indicates that $2\pi - \theta_1(\tau)$ intersects $\tau\omega_1(\tau)$ at $\bar{\tau}_2 \approx 212$.

To demonstrate the occurrence of Hopf bifurcation at $\bar{\tau}_1$, we chose constant initial data $S(t) = 15$, $I(t) = 2.5$, and $R(t) = 20$ for $t \in [-\tau, 0]$. For $\tau = 15 < \bar{\tau}_1$, the solution converges to the endemic equilibrium E_* (see Figure 4). For $\tau = 17 > \bar{\tau}_1$, numerical simulation indicates there is a stable periodic solution (see Figure 5). This confirms that Hopf bifurcation of the endemic equilibrium occurs at $\bar{\tau}_1$. This equilibrium

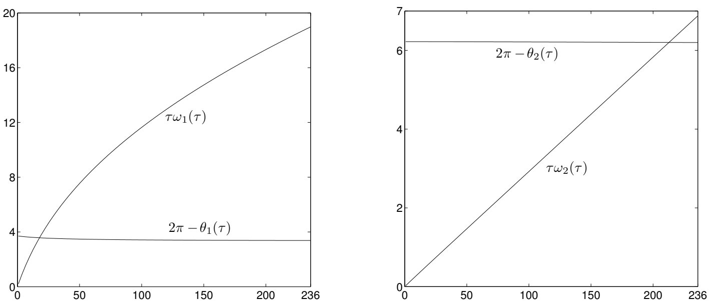

FIGURE 3: Intersection of the functions $2\pi - \theta_i(\tau)$ and $\tau\omega_i(\tau)$ for $i = 1,\ 2$, as τ varies.

loses stability as τ is increased through $\bar{\tau}_1$ and a stable periodic solution appears. For the bifurcation at $\tau = \bar{\tau}_2$, simulations (not shown) confirm that a secondary Hopf bifurcation occurs resulting in the disappearance of this periodic solution.

4 Discussion Beretta and Kuang [**1**] considered the general characteristic equation with delay dependent coefficients given by (1.2). Under the assumption that a positive root $\omega(\tau)$ of

$$|P_n(i\omega,\tau)|^2 - |Q_m(i\omega,\tau)|^2 = 0, \tag{4.1}$$

exists, they define $\theta(\tau) \in [0, 2\pi]$ as the solution of

$$\begin{cases} \sin(\theta(\tau)) = \mathrm{Im}\left(\dfrac{P_n(i\omega,\tau)}{Q_m(i\omega,\tau)}\right), \\ \cos(\theta(\tau)) = -\mathrm{Re}\left(\dfrac{P_n(i\omega,\tau)}{Q_m(i\omega,\tau)}\right). \end{cases} \tag{4.2}$$

and pointed out that in his case, $\omega(\tau)\tau = \theta(\tau)+2\ell\pi$ for some nonnegative integer ℓ. They then defined

$$S_\ell(\tau) = \tau - \frac{\theta(\tau) + 2l\pi}{\omega(\tau)}, \qquad \ell = 0, 1, 2\cdots,$$

and claimed that if there exists $\tau^* > 0$ such that $S_\ell(\tau_*) = 0$ for some ℓ, then a simple pair of pure imaginary roots $\pm i\omega(\tau^*)$ of (1.2) exists.

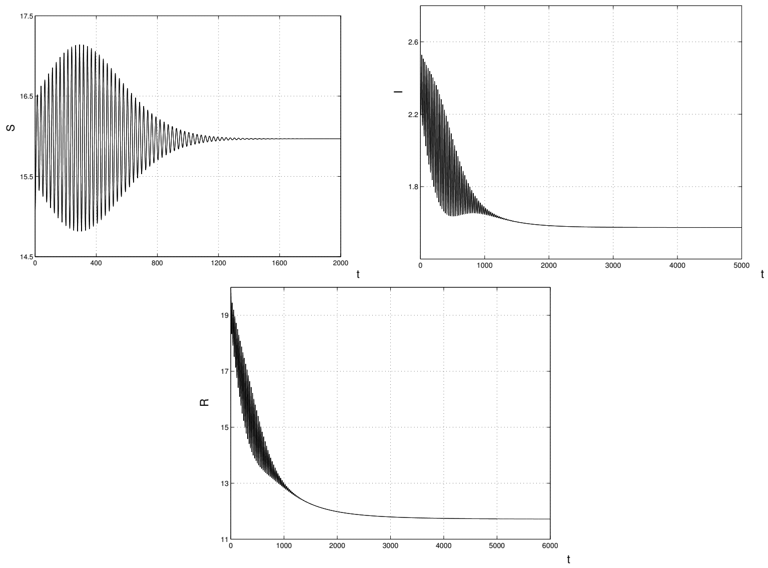

FIGURE 4: Time series illustrating that the endemic equilibrium is stable for $\tau = 15$.

However, the definition of $S_\ell(\tau)$ involved functions $\omega(\tau)$ and $\theta(\tau)$, where $\omega(\tau)$ was given by an analytical expression, but $\theta(\tau)$ was given implicitly, defined as a solution of (4.2). They did not provide analytic criteria to determine whether or not a positive solution $\omega(\tau)$ of (4.1) exists or whether a solution of (4.2) exists.

In this paper we considered (1.2) in the special case that $n = 2$ and $m = 1$, i.e., the second order transcendental equation given by (1.1). In system (4.2),

$$\text{(4.3)} \quad \operatorname{Im}\left(\frac{P_2(i\omega,\tau)}{Q_1(i\omega,\tau)}\right) = h_1(\omega,\tau), \qquad -\operatorname{Re}\left(\frac{P_2(i\omega,\tau)}{Q_1(i\omega,\tau)}\right) = h_2(\omega,\tau)$$

where $h_1(\omega,\tau)$ and $h_2(\omega,\tau)$ are defined in (2.6) and (2.7), respectively. In this case, to distinguish between two potential positive roots of (4.1), we denote them by $\omega_i(\tau)$ $(i = 1, 2)$, defined in (2.4) or (2.5). Condition (H_1) or (H_2) guarantees that either $\omega_1(\tau)$, or $\omega_2(\tau)$, or both are positive (see Lemma 1). We define $\theta_i(\tau)$ explicitly in (2.9) in terms of $\omega_i(\tau)$. We obtain conditions in terms of the coefficients of (1.2) that

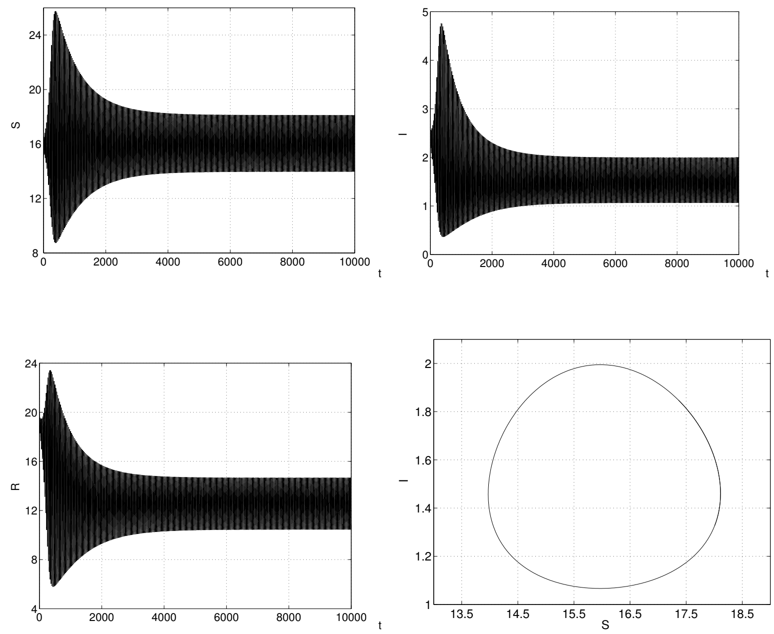

FIGURE 5: Time series (both top and bottom left) and a phase portrait (bottom right) indicating that there is a stable periodic solution for $\tau = 17$.

tell when a solution $\theta_i(\tau)$ of (4.2) exists with $\omega = \omega_i(\tau)$ and show that $\theta_i(\tau) \in [0, \pi]$ provided it is a solution. These conditions were given in Theorems 2, 6, and Corollary 3. If the functions $\theta_i(\tau) + 2k\pi$ and $\tau\omega_i(\tau)$ have intersections as τ varies, then (1.2) has a pair of pure imaginary roots (see Theorem 1). But determining whether or not there are intersections is also of importance in applications. We showed that $\theta_i(\tau) \in [0, \pi]$ and so $\theta_i(\tau) + 2k\pi \in [2k\pi, 2k\pi + \pi]$. If the maximum of $\tau\omega_i(\tau)$ is greater than $2k\pi + \pi$ and the minimum is less than $2k\pi$, then it follows that there are intersections of the functions $\theta_i(\tau) + 2k\pi$ and $\tau\omega_i(\tau)$ as τ varies. This is summarized in Theorem 5.

We applied our method to the single patch case of the model studied in [**2**] and showed that there is a Hopf bifurcations for appropriate parameters. In applications, if all parameters are fixed except for the delay τ, it is useful to be able to determine whether either of conditions (H_1) or (H_2) holds, for τ in some interval. We have shown that in most cases one can find such an interval explicitly, and if not, one can at least find an approximation to that interval. How one can then restrict the interval further usig our results is then summarized in Section 2.3. Once these intervals are determined, it is easier to search for values of the delay at which Hopf bifurcations can occur using numerical simulations, since one can restrict ones attention to values of the delay in these intervals.

A Proof of Theorem 8 First assume that i) holds. From $A\beta - 2d(\gamma+\epsilon+d) \geqslant 0$,

$$A\beta - 2d(\gamma+\epsilon+d) > -d\gamma e^{-d\tau}.$$

Adding $d(\gamma+\epsilon+d)$ to both sides and noting $\gamma - \gamma e^{-d\tau} > 0$ gives

$$A\beta - d(\gamma+\epsilon+d) > d(\gamma+\epsilon+d-\gamma e^{-d\tau}) > 0.$$

Therefore,

$$\frac{A\beta - d(\gamma+\epsilon+d)}{\gamma+\epsilon+d-\gamma e^{-d\tau}} > d,$$

which is equivalent to

$$\beta I^\star > d. \tag{A.1}$$

From

$$\tau \leqslant -\frac{1}{2d}\ln\left(\frac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\gamma^2(\epsilon+d)}\right),$$

we have

$$\frac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\epsilon+d} \leqslant \gamma^2 e^{-2d\tau}.$$

Since $\gamma+\epsilon+d-\gamma e^{-d\tau} > \epsilon+d$,

$$\frac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\gamma+\epsilon+d-\gamma e^{-d\tau}} < \frac{4(\gamma+\epsilon+d)(A\beta - d(\gamma+\epsilon+d))}{\epsilon+d} \leqslant \gamma^2 e^{-2d\tau}.$$

Noting $\beta I^\star = \frac{A\beta - d(\gamma+\epsilon+d)}{\gamma+\epsilon+d-\gamma e^{-d\tau}}$, we have

$$4\beta I^\star \leqslant \frac{\gamma^2 e^{-2d\tau}}{\gamma+\epsilon+d}.$$

Multiplying $\beta I^\star$ on both sides and rearranging gives

$$(2\beta I^\star)^2 \leqslant \frac{(\gamma\beta I^\star e^{-d\tau})^2}{\beta I^\star(\gamma+\epsilon+d)}.$$

From (A.1),

$$p^2(\tau) = (d+\beta I^\star)^2 \leqslant (2\beta I^\star)^2 \leqslant \frac{(\gamma\beta I^\star e^{-d\tau})^2}{\beta I^\star(\gamma+\epsilon+d)} = \frac{c^2(\tau)}{\alpha(\tau)}. \tag{A.2}$$

It follows that $-4p^2(\tau)\alpha(\tau)+4c^2(\tau) \geqslant 0$. Therefore,

$$\begin{aligned}(q^2(\tau) - p^2(\tau) + 2\alpha(\tau))^2 &- 4\left(\alpha^2(\tau) - c^2(\tau)\right)\\ &= (-p^2(\tau)+2\alpha(\tau))^2 - 4\left(\alpha^2(\tau)-c^2(\tau)\right)\\ &= p^4(\tau) - 4p^2(\tau)\alpha(\tau) + 4c^2(\tau)\\ &\geqslant -4p^2(\tau)\alpha(\tau) + 4c^2(\tau) \geqslant 0.\end{aligned}$$

By (3.2), $\alpha^2(\tau) - c^2(\tau) > 0$. By (A.2),

$$p^2(\tau) < \frac{c^2(\tau)}{\alpha(\tau)} < \frac{\alpha^2(\tau)}{\alpha(\tau)} = \alpha(\tau) < 2\alpha(\tau).$$

Hence

$$q^2(\tau) - p^2(\tau) + 2\alpha(\tau) = 2\alpha(\tau) - p^2(\tau) > 0.$$

Therefore, condition (H_1) holds.

Now assume that ii) holds. From $A\beta - d(\gamma+2\epsilon+2d) \leqslant 0$,

$$A\beta - 2d(\gamma+\epsilon+d) \leqslant -d\gamma e^{-d\tau}.$$

Adding $d(\gamma+\epsilon+d)$ to both sides and noting that $R_0 = \frac{A\beta}{d(\gamma+\epsilon+d)} > 1$, we obtain

$$d(\gamma+\epsilon+d-\gamma e^{-d\tau}) \geqslant A\beta - d(\gamma+\epsilon+d) > 0.$$

Therefore,

$$d \geqslant \frac{A\beta - d(\gamma+\epsilon+d)}{\gamma+\epsilon+d-\gamma e^{-d\tau}} > 0,$$

which is equivalent to

$$d \geqslant \beta I^\star. \tag{A.3}$$

From

$$\tau \leqslant -\frac{1}{2d}\ln\left(\frac{4d^2(\gamma+\epsilon+d)^2}{\gamma^2(A\beta-d(\gamma+\epsilon+d))}\right),$$

we have

$$\frac{4d^2(\gamma+\epsilon+d)^2}{A\beta-d(\gamma+\epsilon+d)} \leqslant \gamma^2 e^{-2d\tau}.$$

Since $\gamma+\epsilon+d > \gamma+\epsilon+d-\gamma e^{-d\tau}$,

$$\begin{aligned}\gamma^2 e^{-2d\tau} &\geqslant \frac{4d^2(\gamma+\epsilon+d)^2}{A\beta-d(\gamma+\epsilon+d)}\\ &\geqslant \frac{4d^2(\gamma+\epsilon+d)(\gamma+\epsilon+d-\gamma e^{-d\tau})}{A\beta-d(\gamma+\epsilon+d)}.\end{aligned}$$

Noting $\beta I^\star = \frac{A\beta-d(\gamma+\epsilon+d)}{\gamma+\epsilon+d-\gamma e^{-d\tau}}$, we have

$$\beta I^\star \gamma^2 e^{-2d\tau} \geqslant 4d^2(\gamma+\epsilon+d).$$

Multiplying $\beta I^\star$ on both sides and rearranging gives

$$\frac{(\gamma\beta I^\star e^{-d\tau})^2}{\beta I^\star(\gamma+\epsilon+d)} \geqslant 4d^2.$$

From (A.3),

$$p^2(\tau) = (d+\beta I^\star)^2 \leqslant 4d^2 \leqslant \frac{(\gamma\beta I^\star e^{-d\tau})^2}{\beta I^\star(\gamma+\epsilon+d)} = \frac{c^2(\tau)}{\alpha(\tau)}. \quad \text{(A.4)}$$

It follows that $-4p^2(\tau)\alpha(\tau)+4c^2(\tau) \geqslant 0$. Therefore,

$$\begin{aligned}&(q^2(\tau)-p^2(\tau)+2\alpha(\tau))^2 - 4\left(\alpha^2(\tau)-c^2(\tau)\right)\\ &\qquad = (-p^2(\tau)+2\alpha(\tau))^2 - 4\left(\alpha^2(\tau)-c^2(\tau)\right)\\ &\qquad = p^4(\tau) - 4p^2(\tau)\alpha(\tau) + 4c^2(\tau)\\ &\qquad \geqslant -4p^2(\tau)\alpha(\tau) + 4c^2(\tau) \geqslant 0.\end{aligned}$$

By (3.2), $\alpha^2(\tau)-c^2(\tau) > 0$. By (A.4),

$$p^2(\tau) < \frac{c^2(\tau)}{\alpha(\tau)} < \frac{\alpha^2(\tau)}{\alpha(\tau)} = \alpha(\tau) < 2\alpha(\tau).$$

Hence
$$q^2(\tau) - p^2(\tau) + 2\alpha(\tau) = 2\alpha(\tau) - p^2(\tau) > 0.$$
Therefore condition (H_1) holds.

In either case, (H_1) holds. By Lemma 1, both $\omega_1(\tau)$ and $\omega_2(\tau)$ are positive. By Theorem 2 (iv.b), $2\pi - \theta_i(\tau) \in [\pi, 2\pi]$ satisfies (2.8).

If $2\pi - \theta_i(\tau) + 2k\pi$ intersects $\tau\omega_i(\tau)$ at some $\bar{\tau}_i$, by Theorem 1, (1.1) has a pair of pure imaginary roots $\pm\omega_i(\bar{\tau}_i)$. □

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