MATH 3F03 Assignment 2
Marking Comments

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Preamble

How to use this document: Check your marked assignment solutions for boxed numbers. Each boxed number corresponds to a comment in this document. If any points are deducted from your solution related to a comment, the number of points deducted is included in brackets after that comment.

For example,

7. *Page 555, exercise 6.*

[1] Did not check stability of equilibria **(-2)**

means that in question 7 (which is exercise 6 on page 555 of *Hirsch, Smale, Devaney*), you forgot to check the stability of equilibria, and so I deducted two points.

All questions are marked out of 10.

Note: Only the most commonly applicable comments are included here. Your assignment may have some additional, specific comments.

Comments

1. *Page 57, exercise 2.*
Each part of this question was marked out of five points.

[1] Note that the eigenvectors for the matrix in canonical form ($J = T^{-1}AT$) are always $e_1 = (1, 0)$ and $e_2 = (0, 1)$ (the standard basis vectors for $\mathbb{R}^2$) so you don't actually have to solve for them to write out the general solution to $Y' = JY$ – that's one reason why canonical form is nice!

[2] Similar matrices, like J and A (which are similar by the equation $J = T^{-1}AT$), always have the same eigenvalues, so you don't have to solve for the eigenvalues of J since you've already found the eigenvalues of A.

3. *Page 58, exercise 6.*

[1] The question asks that you sketch regions of the ab-plane where the *canonical form* of the system is different, not where the system has different types of *phase portraits*. The question of which regions in the ab-plane have different canonical forms is simpler – though if you did identify the different phase portraits (which is more work) that's at least good practice!

There are four distinct canonical forms J for the case where A is a 2×2 matrix:

i. Real, distinct eigenvalues λ, μ:

$$J = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

ii. Repeated eigenvalue λ, where $\text{geom}(\lambda) = 2$:

$$J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

iii. Repeated eigenvalue λ, where $\text{geom}(\lambda) = 1$:

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

iv. Complex conjugate eigenvalues $\lambda = \alpha \pm i\beta$:

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Away from $(a, b) = (0, 0)$, we are always in the case i, because we always have real, distinct eigenvalues – even if one of them is 0 – so you don't have to single out the cases where one eigenvalue is 0 as different (see the solutions for more detail). The only point in the ab-plane that has a canonical form that is different than case i is $(a, b) = (0, 0)$, because at this point, the system has the canonical form $J = A = 0$. This is case ii, not iii.

[2] Did not sketch the ab-plane (as the question asks) and did not identify the different canonical forms, but your eigenvalue analysis is correct – see comment [1]. **(-2)**

[3] You did sketch the ab-plane and correctly deduced the various phase portraits in the system, but you did not identify the different canonical forms – see comment [1]. **(-1)**

[4] Did not sketch the ab-plane (as the question asks) but otherwise correctly identified the different canonical forms. **(-1)**

[5] Just because $\lambda = 0$ is a repeated eigenvalue for $(a, b) = (0, 0)$ does not immediately imply that the canonical form is case iii above (there's also case ii for repeated eigenvalues) – see explanation in comment [1] and the solutions. **(-1)**

4. *Page 59, exercise 11.*

[1] Either missing or incorrect phase portrait/phase portrait description – see solutions. **(-3)**

5. *Page 59, exercise 13.*

- [1] The statement you're proving here is actually the Cayley-Hamilton theorem, which actually holds for any $n \times n$ matrix A (and not just in the 2×2 case you've been asked to show in this question).
- [2] You can't assume that A satisfies its own characteristic equation when that's actually what you're trying to prove here (that would be circular reasoning).

6. *Page 59, exercise 14.*

- [1] You should have prefaced your proof with the statement, "Either $V \in \mathbb{R}^2$ is an eigenvector of A with eigenvalue λ, or it is not", to make it clear that your proof works for all $V \in \mathbb{R}^2$.
- [2] Make sure you show that your candidate eigenvector $W = (A - \lambda I)V$ is *non-zero*, as eigenvectors are (by definition) non-zero vectors. **(-1)**